LECTURE 12, 5/4/2022
THE RIEMANN INTEGRAL

• *The Riemann integral after B. Riemann.* We introduced Riemann sums in Lecture 10 and proved there in Corollary 3 that every continuous function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable. In this lecture we develop this theory in full. We consider functions of the type $f\colon [a,b] \to \mathbb{R}$, where $a < b$ are real numbers, partitions $P = (a_0, a_1, \dots, a_k)$ of $[a,b]$, where $k \in \mathbb{N}$ and $a = a_0 < a_1 < \dots < a_k = b$, test points $\overline{t} = (t_1, \dots, t_k)$ of P, where $t_i \in [a_{i-1}, a_i]$, and Riemann sums

$$R(P,\, \overline{t},\, f) = \sum_{i=1}^{k} (a_i - a_{i-1}) \cdot f(t_i)\ .$$

We noted earlier that $R(P, \overline{t}, f)$ is the signed area of the bar graph $B_f = \bigcup_{i=1}^{k} [a_{i-1}, a_i] \times I(0, f(t_i))$ where $I(c,d)$ is the closed real interval with the endpoints c and d. For small norm $\Delta(P) = \max_{1 \le i \le k}(a_i - a_{i-1})$ of P the set B_f closely approximates the domain D_f under the graph G_f of f and one uses limits of Riemann sums (Definition 2 of Lecture 10) to define the area A_f of D_f (part 2 of Definition 5 of Lecture 10). We repeat the definition here in another formulation and introduce by it the Riemann integral. It is a fundamental definition in mathematical analysis, alongside with those of derivative, continuity etc.

Definition 1 (Riemann integral) *We say that a function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, and write that $f \in \mathrm{R}(a,b)$, if there exists a number $L \in \mathbb{R}$ such that for $\forall\, \varepsilon\ \exists\, \delta$ such that for any partition P of $[a,b]$ and any test points $\overline{t}$ of P it holds that*

$$\Delta(P) < \delta \Rightarrow |R(P,\, \overline{t},\, f) - L| < \varepsilon\ .$$

Then we also write

$$(\mathrm{R}) \int_a^b f = L \quad \textit{or} \quad (\mathrm{R}) \int_a^b f(x)\, \mathrm{d}x = L$$

and say that the (Riemann) integral over $[a,b]$ of the function f equals L.

For simplicity of notation we omit the qualification (R) when it is clear that the integral is Riemann one. The latter notation $\int_a^b f(x)\,\mathrm{d}x$, which is due to G. W. Leibniz, comes from Riemann sums: the sign of sum $\sum$ morphed in the integral sign $\int$ and $\mathrm{d}x$ denotes the common length $a_i - a_{i-1}$ of intervals in an *equipartition* P of $[a,b]$. We extend the scope of the notation $\int_a^b f$ slightly by setting $\int_a^a f := 0$ for any $a \in \mathbb{R}$ and any function f, and $\int_b^a f := -\int_a^b f$ if $f \in \mathrm{R}(a,b)$. Since this definition is important, we state two other equivalent forms of it. We leave the proof of the equivalence of all three definitions to the interested reader.

Proposition 2 ($\iff$ definitions of R. integrability) *Let $f\colon [a,b] \to \mathbb{R}$ be any function. The next three claims are logically equivalent.*

1. *$f \in \mathrm{R}(a,b)$.*
2. *(Cauchy's condition) $\forall\,\varepsilon\ \exists\,\delta$ such that for any partitions P and Q of $[a,b]$ with respective test points $\overline{t}$ and $\overline{u}$, if $\Delta(P), \Delta(Q) < \delta$ then $|R(P,\overline{t},f) - R(Q,\overline{u},f)| < \varepsilon$.*
3. *(Heine's definition) For any sequence (P_n) of partitions of $[a,b]$ with test points $\overline{t(n)}$, if $\lim \Delta(P_n) = 0$ then the sequence $\big(R(P_n, \overline{t(n)}, f)\big)$ is convergent.*

If 1 holds then every sequence of Riemann sums in 3 with norms going to 0 has the limit $\lim R(P_n, \overline{t(n)}, f) = \int_a^b f$.

In the next lecture we give yet another equivalent definition of Riemann integrability via approach of J.-G. Darboux.

Finitely many changes in functional values have no influence on the Riemann integral.

Proposition 3 (changing values) *We suppose that f is in $\mathrm{R}(a,b)$ and that $g\colon [a,b] \to \mathbb{R}$ differs from f in only finitely many values. Then $g \in \mathrm{R}(a,b)$ and $\int_a^b g = \int_a^b f$.*

Proof. Let $f \in \mathrm{R}(a,b)$. We suppose that g differs from f in k values on the points $c_1, \dots, c_k \in [a,b]$. Let (P_n) be any sequence of partitions of $[a,b]$ with $\Delta(P_n) \to 0$ and let $\overline{t(n)}$ be test points of P_n. Then

$$\lim\, R(P_n,\, \overline{t(n)},\, f) = \int_a^b f$$

by the previous proposition. But for $n \in \mathbb{N}$,

$$R(P_n,\, \overline{t(n)},\, g) = R(P_n,\, \overline{t(n)},\, f) + O(k \cdot \Delta(P_n))\ .$$

The implicit constant in O can be taken to be $\max_{1 \le i \le k} |g(c_i) - f(c_i)|$. Since $\lim \Delta(P_n) = 0$, also

$$\lim\, R(P_n,\, \overline{t(n)},\, g) = \textstyle\int_a^b f\ .$$

We are done by the previous proposition. $\square$

Of course, if $f, g\colon [a,b] \to \mathbb{R}$, f is not Riemann integrable and g differs from f in only finitely many values then it is not Riemann integrable either (why?). This stability of (R) $\int_a^b f$ is in stark contrast with the fact that (N) $\int_a^b f$ can be destroyed by a single change in functional value (if the Darboux property of f is destroyed). Using the proposition we extend the definition of Riemann integral to any nontrivial bounded interval.

Definition 4 ($\int_a^b f$ for f defined on (a,b)) *Let $a < b$ be real numbers and $f\colon I \to \mathbb{R}$ for an interval of type $I = (a,b)$ or $I = (a,b]$ or $I = [a,b)$. We extend f to $f_0\colon [a,b] \to \mathbb{R}$ by arbitrary values on a and on b and define*

$$\int_a^b f := \int_a^b f_0\ ,$$

if the right-hand side exists.

Like for the Newton integral, restriction preserves Riemann integrability. For simplicity of notation we omit in the next proposition the obvious restriction symbols $f \mid [a,b]$ and $f \mid [b,c]$.

Proposition 5 (on restrictions) *If $a < b < c$ are real numbers and $f\colon [a,c] \to \mathbb{R}$ then*

$$f \in \mathrm{R}(a,\, c) \iff f \in \mathrm{R}(a,\, b) \wedge f \in \mathrm{R}(b,\, c)\ .$$

In the positive case, $\int_a^c f = \int_a^b f + \int_b^c f$.

Proof. The implication $\Rightarrow$. Let $f \in \mathrm{R}(a,c)$ and let an ε be given. We prove for the restriction of f to $[a,b]$ Cauchy's condition of Proposition 2. Let P_0 and Q_0 be two partitions of $[a,b]$ with respective test points $\overline{t(0)}$ and $\overline{u(0)}$ and such that $\Delta(P_0), \Delta(Q_0) < \delta$, where δ guarantees satisfaction of Cauchy's condition for $\mathrm{R}(a,c)$ and ε. We extend P_0 and Q_0 to partitions P and Q of $[a,c]$ arbitrarily but so that $\Delta(P), \Delta(Q) < \delta$ and that the intervals of P and Q contained in $[b,c]$ are identical. We also extend $\overline{t(0)}$ and $\overline{u(0)}$ identically to test points $\overline{t}$ and $\overline{u}$ of, respectively, P and Q. Then indeed

$$\begin{aligned}|R(P_0,\, \overline{t(0)},\, f) - R(Q_0,\, \overline{u(0)},\, f)| \;&=\; |R(P,\, \overline{t},\, f) - R(Q,\, \overline{u},\, f)| \\ &<\; \varepsilon\ .\end{aligned}$$

The proof of Cauchy's condition for the restriction f to $[b,c]$ is similar. The identity $\int_a^c f = \int_a^b f + \int_b^c f$ follows by merging partitions of $[a,b]$ and $[b,c]$ with norms going to 0 in partitions of $[a,c]$ (with norms going to 0) and using the last claim in Proposition 2.

The implication $\Leftarrow$. Let $f \in \mathrm{R}(a,b) \cap \mathrm{R}(b,c)$. It follows that f is bounded and we denote by $d > 0$ the bounding constant. Let P be any partition of $[a,c]$ with test points $\overline{t}$. We split P in the partitions P_1 and P_2 of, respectively, $[a,b]$ and $[b,c]$ and with respective test points $\overline{t(1)}$ and $\overline{t(2)}$ as follows. If $b \in P$ we do the splitting in the obvious way. If $b \notin P$, we obtain P_1 and P_2 by splitting the interval $[a_{i-1}, a_i]$ of P such that $b \in (a_{i-1}, a_i)$ in the intervals $[a_{i-1}, b]$ and $[b, a_i]$, and get $\overline{t(1)}$ and $\overline{t(2)}$ by selecting two arbitrary test points in the two new intervals. Then

$$R(P, \overline{t}, f) = R(P_1, \overline{t(1)}, f) + R(P_2, \overline{t(2)}, f) + O(\Delta(P)d) .$$

Thus satisfaction of Cauchy's condition for $\mathrm{R}(a, c)$ follows from its satisfaction for $\mathrm{R}(a, b)$ and $\mathrm{R}(b, c)$. The identity $\int_a^c f = \int_a^b f + \int_b^c f$ follows by the same argument as for the opposite implication. $\square$

In the last lecture we stated for the Newton integral only the analogue of $\Rightarrow$. Now we state also the opposite implication and leave its proof to the interested reader.

Proposition 6 ($\Leftarrow$ for the Newton $\int$) *Let $A < C < B < D$ be in $\mathbb{R}^*$, $f\colon (A, D) \to \mathbb{R}$ and let $f \in \mathrm{N}(A, B) \cap \mathrm{N}(C, D)$. Then $f \in \mathrm{N}(A, D) \cap \mathrm{N}(C, B)$ and*

$$(\mathrm{N}) \int_A^D f = (\mathrm{N}) \int_A^B f + (\mathrm{N}) \int_C^D f - (\mathrm{N}) \int_C^B f .$$

We give the fourth definition of the area under graph; see Lecture 10 for the definitions of D_f and G_f.

Definition 7 (again A_f) *If $f \in \mathrm{R}(a, b)$ then we define the area A_f of the domain D_f under the graph G_f of the function $f\colon [a, b] \to \mathbb{R}$ (or $f\colon [a, b) \to \mathbb{R}$, ...) as*

$$A_f := \int_a^b f(x)\, \mathrm{d}x .$$

• *Existence and non-existence of the Riemann integral.* We begin with two non-existence results. Recall that for $M \subset \mathbb{R}$ a function $f\colon M \to \mathbb{R}$ is bounded if $\exists\, c\ \forall\, x \in M\colon |f(x)| < c$. Else f is unbounded.

Proposition 8 (unbounded functions are bad) *If the function $f\colon [a,b] \to \mathbb{R}$ is unbounded then $f \notin \mathrm{R}(a,b)$.*

Proof. We suppose that $f\colon [a,b] \to \mathbb{R}$ is unbounded and show that for every n there exists a partition P of $[a,b]$ with test points $\overline{t}$ such that

$$\Delta(P) < 1/n \ \text{ and } \ |R(P,\,\overline{t},\,f)| > n\ .$$

This refutes Cauchy's condition for the Riemann integrability of f.

It follows from the unboundedness of f and from the compactness of $[a,b]$ that there is a convergent sequence $(b_n) \subset [a,b]$ with $\lim b_n = \alpha \in [a,b]$ and with $\lim |f(b_n)| = +\infty$. Let an $n \in \mathbb{N}$ be given. For P we take any partition $P = (a_0,\ldots,a_k)$ of $[a,b]$ with $\Delta(P) < 1/n$ and such that there is a *unique* index $j \in \{1,\ldots,k\}$ for which $\alpha \in [a_{j-1},a_j]$. Then we select arbitrary test points $t_i \in [a_{i-1},a_i]$ for all $i \ne j$ and consider the incomplete Riemann sum

$$s := \sum_{i=1,\,i\ne j}^{k} (a_i - a_{i-1})f(t_i)\ .$$

Now we can select the remaining test point $t_j \in [a_{j-1},a_j]$ so that $|(a_j - a_{j-1})f(t_j)| > |s| + n$ (because $b_n \in [a_{j-1},a_j]$ for every large enough n). We then define $\overline{t}$ as consisting of all these test points and get (by the triangle inequality $|u+v| \ge |u| - |v|$) that

$$|R(P,\,\overline{t},\,f)| \ge |(a_j - a_{j-1})f(t_j)| - |s| > n\ ,$$

as required. $\square$

Proposition 9 (so are too discontinuous functions) *If the function $f\colon [a,b] \to \mathbb{R}$ is discontinuous at every point of some subinterval $[c,d] \subset [a,b]$ with $c < d$, then $f \notin \mathrm{R}(a,b)$.*

For example, since Dirichlet's function $d\colon [0,1] \to \{0,1\}$, given by $d(x) = 0$ for rational x and $d(x) = 1$ for irrational x, is discontinuous everywhere, it is not Riemann integrable. This is easy to see directly, try it as an exercise. To prove Proposition 9 in its generality is harder than to prove Proposition 8 and we need for it the next Theorem 10 which is of an independent interest. Let $a < b$ be real numbers. A set $M \subset [a,b]$ is *sparse (in $[a,b]$)* if for every neighborhood $U(c,\varepsilon)$ with $c \in [a,b]$ there is a neighborhood $U(d,\delta) \subset U(c,\varepsilon) \cap [a,b]$ such that $U(d,\delta) \cap M = \emptyset$.

Theorem 10 (Baire's) *If $a < b$ are real numbers and $[a,b] = \bigcup_{n=1}^{\infty} M_n$ then some of the sets M_n is not sparse.*

Proof. We suppose that in the countable union $[a,b] = \bigcup_{n=1}^{\infty} M_n$ every set M_n is sparse and deduce a contradiction. Since M_1 is sparse, there is a subinterval $[a_1,b_1] \subset [a,b]$ such that $a_1 < b_1$ and $[a_1,b_1] \cap M_1 = \emptyset$. Since M_2 is sparse, there is a subinterval $[a_2,b_2] \subset [a_1,b_1]$ such that $a_2 < b_2$ and $[a_2,b_2] \cap M_2 = \emptyset$. Continuing this way we obtain a sequence of nested intervals

$$[a,\, b] \supset [a_1,\, b_1] \supset [a_2,\, b_2] \supset \cdots \supset [a_n,\, b_n] \supset \ldots$$

such that for every n, $a_n < b_n$ and $[a_n,b_n] \cap M_n = \emptyset$. Let $\alpha := \lim a_n \in [a,b]$. This limit exists and lies in $[a,b]$ because the sequence (a_n) is non-decreasing and is bounded from below by a and from above by b. In fact, $a_n < b_m$ for every n and every m, which implies that $\alpha \in [a_n, b_n]$ for every n. But this means that $\alpha \notin M_n$ for every n, which is a contradiction as $\alpha \in [a, b]$. $\square$

Proof of Proposition 9. Let f, a, b, c and d be as stated (in the hypothesis of the implication). We show that there is an $\varepsilon > 0$ such that for every n there exists a partition P of $[a, b]$ with test points $\overline{t}$ and $\overline{u}$ and such that

$$\Delta(P) < 1/n \;\text{ and }\; R(P,\, \overline{t},\, f) - R(P,\, \overline{u},\, f) > \varepsilon\ .$$

This refutes Cauchy's condition for the Riemann integrability of f.

For $j \in \mathbb{N}$ we define the set $M_j \subset [c, d]$ as

$$\{x \in [c,\, d] \mid \forall\, \delta\ \exists\, y,\, z \in U(x, \delta) \cap [c,\, d]\colon f(y) - f(z) > 1/j\}\ .$$

Since f is discontinuous on $[c, d]$, $\bigcup_{j=1}^{\infty} M_j = [c, d]$. By Baire's theorem there is an $m \in \mathbb{N}$ such that M_m is not sparse in $[c, d]$. This means that there is a subinterval $[c_1, d_1] \subset [c, d]$ such that $c_1 < d_1$ and for every neighborhood $U(e, \delta)$ intersecting $[c_1, d_1]$ the intersection contains a point from M_m.

Let an $n \in \mathbb{N}$ be given. We take for P any partition of $[a, b]$ with $\Delta(P) < 1/n$ and such that the points c_1 and d_1 are among the points of P. For the intervals $[a_{i-1}, a_i]$ of P with interiors disjoint to $[c_1, d_1]$ we select the test points $t_i = u_i \in [a_{i-1}, a_i]$ arbitrarily. If $[a_{i-1}, a_i] \subset [c_1, d_1]$, we can select such points $t_i, u_i \in [a_{i-1}, a_i]$ that $f(t_i) - f(u_i) > 1/m$ (because M_m is dense in $[c_1, d_1]$). Then we define $\overline{t}$, resp. $\overline{u}$, as consisting of all these points t_i, resp. u_i. It follows that the difference $R(P, \overline{t}, f) - R(P, \overline{u}, f)$ equals

$$\sum_{[a_{i-1},\, a_i] \subset [c_1,\, d_1]} (a_i - a_{i-1})(f(t_i) - f(u_i))$$

and this is

$$> \frac{1}{m} \sum_{[a_{i-1}, a_i] \subset [c_1, d_1]} (a_i - a_{i-1}) = \frac{d_1 - c_1}{m} \ .$$

Thus we may select $\varepsilon := (d_1 - c_1)/m$. □

There is a powerful criterion — Lebesgue's theorem below — by which one usually easily determines if the given function is Riemann integrable or not. To state it we need two definitions. For any function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, we define

$$\mathrm{DC}(f) := \{x \in M \mid f \text{ is discontinuous at } x\} \ .$$

We say that a set $M \subset \mathbb{R}$ *has measure* 0 if for every ε there exist intervals $[a_n, b_n]$, $n \in \mathbb{N}$ and $a_n < b_n$, such that

$$M \subset \bigcup_{n=1}^{\infty} [a_n,\, b_n] \ \text{ and } \ \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon \ .$$

It is easy to see that every at most countable set has measure 0, that any countable union of measure 0 sets has measure 0, that any subset of a measure 0 set has measure 0 and that no nontrivial interval has measure 0. We will not prove the next theorem of H. Lebesgue, but in view of Propositions 8 and 9 it is relatively clear why it holds.

Theorem 11 (Lebesgue's) *For any $f\colon [a,b] \to \mathbb{R}$,*

$$f \in \mathrm{R}(a,\, b) \iff f \text{ is bounded and } \mathrm{DC}(f) \text{ has measure } 0 \ .$$

Lebesgue's theorem implies closedness of the class od Riemann integrable functions to several operations.

Corollary 12 (nice operations for $\mathrm{R}(a,b)$) *The following implications hold.*

1. $f, g \in \mathrm{R}(a,b) \Rightarrow cf + dg \in \mathrm{R}(a,b)$ *for any* $c, d \in \mathbb{R}$.
2. $f, g \in \mathrm{R}(a,b) \Rightarrow f \cdot g \in \mathrm{R}(a,b)$.
3. *If* $g\colon [a,b] \to M \subset \mathbb{R}$, $f\colon M \to \mathbb{R}$, $g \in \mathrm{R}(a,b)$ *and* f *is continuous and bounded, then* $f(g) \in \mathrm{R}(a,b)$.
4. *If* $g\colon [c,d] \to [a,b]$, $f\colon [a,b] \to \mathbb{R}$, g *is continuous and* $f \in \mathrm{R}(a,b)$, *then* $f(g) \in \mathrm{R}(c,d)$.

Proof. 1. We suppose that $f, g\colon [a,b] \to \mathbb{R}$ are Riemann integrable. Hence f and g are bounded and so is $cf + dg$. Since $\mathrm{DC}(cf + dg) \subset \mathrm{DC}(f) \cup \mathrm{DC}(g)$ and the latter two sets have measure 0, so has the former set.

2. This proof is similar to the previous one, we only replace the operation of linear combination with multiplication.

3. Since f is bounded, so is the composition $f(g)$. Since the inclusion $\mathrm{DC}(f(g)) \subset \mathrm{DC}(g)$ holds and the latter set has measure 0, so has the former set.

4. This proof is similar to the previous one, the only change is the inclusion $\mathrm{DC}(f(g)) \subset \mathrm{DC}(f)$. $\square$

For example, how do we prove that division preserves Riemann integrability, provided that unbounded functions are avoided? Let $g \in \mathrm{R}(a,b)$ be such that neither $0 \in g[[a,b]]$ nor is 0 a limit point of $g[[a,b]]$. We use part 3 of the corollary for g, $M := g[[a,b]]$ and $f(x) = 1/x$, and get that $1/g \in \mathrm{R}(a,b)$.

Theorem 13 (cont. functions are R. integrable)
If $f\colon [a,b] \to \mathbb{R}$ is continuous then $f \in \mathrm{R}(a,b)$.

Proof. This follows immediately from Theorem 11 because any continuous function defined on a compact set is bounded and has $\mathrm{DC}(f) = \emptyset$. But we also proved it directly already in Corollary 3 of Lecture 10. $\square$

Theorem 14 (monot. functions are R. integrable)
If $f\colon [a,b] \to \mathbb{R}$ is monotone then $f \in \mathrm{R}(a,b)$.

Proof. We assume that f is non-decreasing, the case with non-increasing f is similar. Like for Theorem 13 we first deduce this theorem from Lebesgue's, and then give a direct proof.

The function f is bounded because $f(a) \le f(x) \le f(b)$ for every $x \in [a,b]$. We define an injection $\varphi\colon \mathrm{DC}(f) \to \mathbb{Q}$. This proves that $\mathrm{DC}(f)$ is at most countable, therefore has measure 0 and $f \in \mathrm{R}(a,b)$ by Lebesgue's theorem. If $p \in \mathrm{DC}(f)$ then by the monotonicity of f both one-sided limits

$$l(p) := \lim_{x \to p^-} f(x) \ \text{ and } \ r(p) := \lim_{x \to p^+} f(x)$$

exist, are finite, $l(p) \le f(p) \le r(p)$ and at least one of the two inequalities is strict. We define $\varphi(p)$ to be any fraction in $(l(p), r(p)) \cap \mathbb{Q}$. It is easy to see that $\varphi(p) < \varphi(q)$ for any $p < q$ in $\mathrm{DC}(f)$.

We prove directly that $f \in \mathrm{R}(a,b)$ by proving for f Cauchy's condition of Proposition 2. Let $P = (a_0, \ldots, a_k)$ and $Q = (b_0, \ldots, b_l)$ be two partitions of $[a,b]$ with respective test points $\overline{t}$ and $\overline{u}$ and let an ε be given. We set $\delta := +\infty$ for $f(a) = f(b)$ (when f is a constant function) and else set $\delta := \varepsilon/2(f(b) - f(a))$.

We assume additionally that $P \subset Q$, i.e., that $a_0 = b_{i_0} = a$, $a_1 = b_{i_1}, \ldots, a_k = b_{i_k} = b$ for some indices $i_0 = 0 < i_1 < \cdots < i_k = l$. As earlier, we reduce general partitions P and Q to this case. Let $k = 1$. Then, since f is non-decreasing on $[a, b]$, $R(P, \bar{t}, f) - R(Q, \overline{u}, f)$ is at least

$$(a_1 - a_0)f(a_0) - \sum_{i=1}^{l}(b_i - b_{i-1})f(b_l) = (b - a) \cdot (f(a) - f(b))$$

and, similarly, at most

$$(b - a) \cdot (f(b) - f(a)) \; .$$

So for $k = 1$,

$$|R(P,\, \bar{t},\, f) - R(Q,\, \overline{u},\, f)| \leq (b - a) \cdot (f(b) - f(a)) \; .$$

For general k we use this bound for any partition $a_{r-1} = b_{i_{r-1}} < b_{i_{r-1}+1} < \cdots < b_{i_r} = a_r$ of the interval $[a_{r-1}, a_r]$, $r = 1, 2, \ldots, k$, thus with a replaced by a_{r-1} and b by a_r. If $\Delta(P) < \delta$ (hence $\Delta(Q) < \delta$ too) then by the triangle inequality,

$$\begin{aligned}
&|R(P,\, \bar{t},\, f) - R(Q,\, \overline{u},\, f)| \\
&\leq \sum_{r=1}^{k}(a_r - a_{r-1}) \cdot (f(a_r) - f(a_{r-1})) \\
&\leq \frac{\varepsilon}{2(f(b) - f(a))} \sum_{r=1}^{k}(f(a_r) - f(a_{r-1})) \\
&= \frac{\varepsilon}{2(f(b) - f(a))} \cdot (f(b) - f(a)) = \varepsilon/2 \; .
\end{aligned}$$

If P and Q are general partitions of $[a,b]$ with respective test points $\overline{t}$ and $\overline{u}$ and with $\Delta(P), \Delta(Q) < \delta$, we set $R := P \cup Q$ (then also $\Delta(R) < \delta$) and take arbitrary test points $\overline{v}$ of R. Since $P \subset R$ and $Q \subset R$, we get by the previous case that

$$\begin{aligned}
&|R(P, \overline{t},\, f) - R(Q, \overline{u},\, f)| \le \\
&\le |R(P, \overline{t},\, f) - R(R, \overline{v},\, f)| + |R(R, \overline{v},\, f) - R(Q, \overline{u},\, f)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon\ .
\end{aligned}$$

$\square$

• *Comparison of Riemann and Newton integrals.* We revisit the relation between the Riemann integral and primitive functions that we considered in Lecture 10. We proved there in Corollary 4 that for continuous f, (R) $\int_a^b f =$ (N) $\int_a^b f$. Now we extend it to a more general situation. In the proof of the next theorem, which is known as the *Second Fundamental Theorem of Calculus*, we again rely on Lagrange's mean value theorem.

Theorem 15 (FTC 2) *Let $f\colon (a,b) \to \mathbb{R}$, where $a < b$ are real numbers, have a primitive function $F\colon (a,b) \to \mathbb{R}$ and let $f \in \mathrm{R}(a,b)$ (see Definition 4). Then there exist finite limits $F(a) := \lim_{x\to a} F(x)$ and $F(b) := \lim_{x\to b} F(x)$ and*

$$\text{(R)} \int_a^b f = F(b) - F(a) = \text{(N)} \int_a^b f\ .$$

Proof. We extend f arbitrarily to $f\colon [a,b] \to \mathbb{R}$, assume that $f \in \mathrm{R}(a,b)$ and consider the primitive function F of f on (a,b).

We first prove that the limits

$$F(a) := \lim_{x \to a} F(x) \text{ and } F(b) := \lim_{x \to b} F(x)$$

exist and are finite. For it we show that F is uniformly continuous on (a, b) (in fact, even Lipschitz continuous, see the definition before the next theorem). Then it follows that for any sequence $(a_n) \subset (a, b)$ with $\lim a_n = a$ the sequence $(F(a_n))$ is Cauchy and therefore has a finite limit $F(a)$, which does not depend on the sequence (a_n), and similarly for $F(b)$. Since f is bounded by Proposition 8, we may take a bounding constant $C > 0$. Lagrange's mean value theorem implies that for any subinterval $[c, d] \subset (a, b)$ with $c < d$ there is a point $e \in (c, d)$ such that $F(d) - F(c) = f(e) \cdot (d - c)$. Thus

$$|F(d) - F(c)| = |f(e)| \cdot |d - c| < C|d - c|$$

and F is uniformly continuous on (a, b).

Next we show that $F(b) - F(a) = (\mathrm{R}) \int_a^b f$. Let an ε be given. We may take such numbers $c < d$ in (a, b) that $|F(a) - F(c)|, |F(b) - F(d)| < \varepsilon$, $C|a - c|, C|b - d| < \varepsilon$ and that there is a partition $P = (a_0, \dots, a_k)$ of $[a, b]$ such that $a_1 = c$, $a_{k-1} = d$ and that for any test points $\overline{t}$ of P, $|\int_a^b f - R(P, \overline{t}, f)| < \varepsilon$. From Lecture 10 we know that there exist test points $\overline{e}$ of the restriction of P to $[c, d] = [a_1, a_{k-1}]$ such that

$$F(d) - F(c) = \sum_{i=2}^{k-1} (a_i - a_{i-1}) \cdot f(e_i) \ .$$

We define the test points $\overline{u}$ of P as consisting of $\overline{e}$ and of two arbitrary test points u_1 and u_k in the respective intervals $[a, a_1] =$

$[a, c]$ and $[a_{k-1}, b] = [d, b]$. Then

$$\begin{aligned}
&\left|(\mathrm{R})\int_a^b f - (F(b) - F(a))\right| \\
&\leq \left|(\mathrm{R})\int_a^b f - R(P, \overline{u},\, f)\right| + \big|R(P, \overline{u},\, f) - (F(b) - F(a))\big| \\
&< \varepsilon + \big|R(P, \overline{u},\, f) - (F(d) - F(c))\big| + \big|(F(d) - F(c)) - \\
&- (F(b) - F(a))\big| \\
&\leq \varepsilon + \big|(c - a) \cdot f(u_1) + (b - d) \cdot f(u_k)\big| + \big|F(d) - F(b)\big| + \\
&+ \big|F(a) - F(c)\big| \\
&< 3\varepsilon + C|c - a| + C|b - d| < 5\varepsilon\ .
\end{aligned}$$

But $\varepsilon > 0$ may be arbitrarily small, so (R) $\int_a^b f = F(b) - F(a)$. $\square$

In the literature FTC 2 often appears in the logically more complicated form in which the existence of $F(a)$ and $F(b)$ is included in the assumptions. As we have just seen, it is not necessary.

The *First Fundamental Theorem of Calculus* is as follows. We define for it that a function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, is *Lipschitz continuous* if there is a constant $C > 0$ such that

$$\forall\, x,\, y \in M\colon |f(x) - f(y)| \leq C|x - y|\ .$$

It is a property stronger than continuity or even than uniform continuity; every Lipschitz continuous function is uniformly continuous.

Theorem 16 (FTC 1) *Let $f \in \mathrm{R}(a,b)$. Then $f \in \mathrm{R}(a,x)$ for every $x \in (a,b]$ and the function $F\colon [a,\, b] \to \mathbb{R}$, given by*

$$F(x) := \int_a^x f \ ,$$

is Lipschitz continuous. Moreover, it is such that $F'(x) = f(x)$ for every point $x \in [a,b]$ of continuity of f.

Proof. So let $f \in \mathrm{R}(a,b)$. By Proposition 5, f is Riemann integrable on any subinterval $[a', b']$, $a' < b'$, of $[a,b]$. So F is correctly defined and $F(a) = 0$. Since f is bounded (by Proposition 8), we may take a bounding constant $c > 0$. We set $C := 1 + c$. Let $x < y$ be in $[a,b]$ and, by Definition 1, let P be a partition of $[x,y]$ with test points $\overline{t}$ such that $|\int_x^y f - R(P, \overline{t}, f)| < y - x$. By Proposition 5 and the definition of F,

$$|F(y) - F(x)| = \left| \int_x^y f \right| \le y - x + |R(P,\, \overline{t},\, f)| \le y - x + c(y - x)$$

and $|F(y) - F(x)| \le C|y - x|$. Thus F is Lipschitz continuous.

We prove the second part about the derivative of F. Let x_0 in $[a,b]$ be such that f is continuous at x_0 and let an ε be given. We take a δ such that $x \in U(x_0, \delta) \cap [a,b] \Rightarrow f(x) \in U(f(x_0), \varepsilon)$. Let $x \in P(x_0, \delta) \cap [a,b]$ be arbitrary, say $x > x_0$ (in the case that $x < x_0$ the argument is similar). Then by taking a partition P of $[x, x_0]$ with test points $\overline{t}$ and such that $|\int_{x_0}^x f - R(P, \overline{t}, f)| < \varepsilon(x - x_0)$ we see that

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f - f(x_0)$$

is less than

$$\begin{aligned}&\frac{R(P,\,\bar{t},\,f)+\varepsilon(x-x_0)}{x-x_0}-f(x_0)\\&<\frac{(x-x_0)(f(x_0)+\varepsilon+\varepsilon)}{x-x_0}-f(x_0)=2\varepsilon\ ,\end{aligned}$$

and similarly it is also $> -2\varepsilon$. Thus $F'(x_0) = f(x_0)$. $\square$

Only when I was writing down this proof of FTC 1 I realized (despite that I have been teaching this material since 2004) that the proof yields not only continuity of F but even Lipschitz continuity. As an immediate corollary of FTC 1 we obtain another proof of the last theorem of Lecture 9 that every continuous function has a primitive function.

Corollary 17 (existence of primitives) *Any continuous function $f\colon [a,b] \to \mathbb{R}$ has a primitive function.*

Proof. If $f\colon [a,b] \to \mathbb{R}$ is continuous then $f \in \mathrm{R}(a,b)$ by Theorem 13. By the previous theorem, $\int_a^x f$ is a primitive of $f(x)$ on $[a,b]$. $\square$

We know from Lecture 9 how to glue these primitives in a primitive of a continuous function $f\colon I \to \mathbb{R}$ defined on a general nontrivial real interval I.

It is easy to give examples of functions $f\colon (a,b) \to \mathbb{R}$ that are Riemann integrable but are not Newton integrable, and vice versa. For example, we have that

$$\begin{aligned}&(\mathrm{R})\int_{-1}^{1}\mathrm{sgn}=(\mathrm{R,\ N})\int_{-1}^{0}\mathrm{sgn}+(\mathrm{R,\ N})\int_{0}^{1}\mathrm{sgn}\\&=[-x]_{-1}^{0}+[x]_{0}^{1}=-1+1=0\end{aligned}$$

but

$$(\mathrm{N}) \int_{-1}^{1} \mathrm{sgn}$$

is not defined because $\mathrm{sgn}(x)$ does not have a primitive function on $(-1,1)$, it is not Darboux there.

Now we have to remark that this example can be fixed by using more general primitives. One says that $F\colon I \to \mathbb{R}$ is a *generalized primitive function of* $f\colon I \to \mathbb{R}$, where I is a nontrivial real interval, if F is continuous and $F'(x) = f(x)$ holds for every $x \in I$, up to finitely many exceptions x. One then defines the *extended general Newton integral* of $f\colon (A,B) \to \mathbb{R}$ by setting

$$(\mathrm{N_e}) \int_{A}^{B} f := [F]_A^B$$

for any generalized primitive F of f on (A,B). Now

$$(\mathrm{N_e}) \int_{-1}^{1} \mathrm{sgn}(x) = [\,|x|\,]_{-1}^{1} = 1 - 1 = 0\ .$$

In the second example we have that

$$(\mathrm{N}) \int_{0}^{1} 1/\sqrt{x} = \big[2\sqrt{x}\big]_0^1 = 2 \ \text{ but the integral } \ (\mathrm{R}) \int_{0}^{1} 1/\sqrt{x}$$

does not exist because the integrand is unbounded on the interval $(0,1)$, see Proposition 8. Somebody might say that the latter integral is $+\infty$ because Riemann sums $R(P,\overline{t},1/\sqrt{x}) > 0$ and can be as large as we wish, for partitions P of $[0,1]$ with test points $\overline{t}$ and with $\Delta(P) > 0$ as small as we wish, but this is definitely a wrong value. In the next last lecture we will see how to improve Definition 1 so that the $(\mathrm{R})\int \rightsquigarrow$ the $(\mathrm{R_c})\int$, for which

$$(\mathrm{R_c}) \int_{0}^{1} 1/\sqrt{x} = 2\ ,$$

as it should. So come for the lecture (two weeks from now) or at least read it!

THANK YOU FOR YOUR ATTENTION!