LECTURE 11, 4/27/2022

MORE ON THE NEWTON INTEGRAL. COMPUTING PRIMITIVES OF RATIONAL FUNCTIONS

• *The general Newton integral.* We extend (N) $\int_a^b f$ to functions defined on any nonempty open interval (A, B) with $A < B$ in $\mathbb{R}^*$. These are exactly the intervals $(-\infty, a)$, (a, b), $(a, +\infty)$ and $(-\infty, +\infty) = \mathbb{R}$ with any real numbers $a < b$.

Definition 1 (general Newton integral) *Let $A < B$ be in $\mathbb{R}^*$ and $F, f\colon (A, B) \to \mathbb{R}$ be functions such that F is a primitive of f. We define the Newton integral of f over the interval (A, B) as the difference*

$$(\mathrm{N}) \int_A^B f = F(B) - F(A) := \lim_{x \to B} F(x) - \lim_{x \to A} F(x) \ ,$$

if the last two limits exist and are finite. Then we define the area A_f of the domain D_f under G_f as

$$A_f := (\mathrm{N}) \int_A^B f \ .$$

Like for the earlier Newton integral over (a, b), the value of the present integral does not depend on the choice of F because any two primitives of f differ by a constant shift. If (N) $\int_A^B f$ is defined we say that f is *Newton-integrable over* (A, B) and write that

$$f \in \mathrm{N}(A,\ B) \ .$$

It is not hard to see that if $(A_0, B_0) \subset (A, B)$ are nonempty open intervals then $f \in \mathrm{N}(A, B) \Rightarrow f \,|\, (A_0, B_0)) \in \mathrm{N}(A_0, B_0)$. We will omit restriction symbol in situations like this and write just that $f \in \mathrm{N}(A_0, B_0)$. For instance, $\frac{1}{1+x^2} \in \mathrm{N}(0, +\infty)$ as

$$(\mathrm{N}) \int_0^{+\infty} \frac{1}{1+x^2} = \overbrace{\arctan(+\infty)}^{\lim_{x \to +\infty} \arctan x} - \arctan(0) = \pi/2 - 0 = \pi/2 \; .$$

For $F\colon (A, B) \to \mathbb{R}$ we introduce the notation

$$[F]_A^B := \lim_{x \to B} F(x) - \lim_{x \to A} F(x) \; ,$$

if both limits exist and are finite.

• *Generalizing the general Newton integral.* We extend $(\mathrm{N}) \int_A^B f$ a little more by allowing $B \le A$. We set $(\mathrm{N}) \int_A^A f := 0$ for any function f and have that

$$(\mathrm{N}) \int_B^A f = -\,(\mathrm{N}) \int_A^B f$$

if $f \in \mathrm{N}(A, B)$. We will not prove the next two propositions, the proofs are easy.

Proposition 2 (additivity of integral) *If* $A, B, C \in \mathbb{R}^*$ *and* $f \in \mathrm{N}(\min(A, B, C), \max(A, B, C))$ *then*

$$(\mathrm{N}) \int_A^C f = (\mathrm{N}) \int_A^B f + (\mathrm{N}) \int_B^C f \; ,$$

that is,

$$(\mathrm{N}) \int_A^B f + (\mathrm{N}) \int_B^C f + (\mathrm{N}) \int_C^A f = 0 \; .$$

Proposition 3 (linearity of integral) *If A and B are in $\mathbb{R}^*$, $a, b \in \mathbb{R}$ and $f, g \in \mathrm{N}(A, B)$ then*

$$(\mathrm{N})\int_A^B (af + bg) = a \cdot (\mathrm{N})\int_A^B f + b \cdot (\mathrm{N})\int_A^B g \ .$$

• *Integration by parts.* But we prove the integration by parts formula for the general Newton integral.

Theorem 4 ($(\mathrm{N})\int_A^B$ by parts) *Consider four functions $f, g, F, G\colon (A, B) \to \mathbb{R}$, where $A < B$ are in $\mathbb{R}^*$, such that F, resp. G, is a primitive of f, resp. g. Then the equality*

$$\underbrace{(\mathrm{N})\int_A^B fG}_{T_1} = \underbrace{[FG]_A^B}_{T_2} - \underbrace{(\mathrm{N})\int_A^B Fg}_{T_3}$$

holds whenever two of the three terms T_i are defined.

Proof. 1. Suppose that the first two terms $T_1, T_2 \in \mathbb{R}$ are defined. So fG has on (A, B) a primitive H with $[H]_A^B = T_1$ and $[FG]_A^B = T_2$. Then

$$(FG - H)' = fG + Fg - fG = Fg \ \text{ and } \ [FG - H]_A^B = T_2 - T_1 \ .$$

Thus $FG - H$ is on (A, B) a primitive of Fg and the last equality is a rearrangement of the equality stated in the theorem.

2. Suppose that the first and third term $T_1 \in \mathbb{R}$ and $T_3 \in \mathbb{R}$ are defined. So fG, resp. Fg, has on (A, B) a primitive H_1, resp. H_2, with $[H_1]_A^B = T_1$ and $[H_2]_A^B = T_3$. Then

$$(H_1 + H_2)' = fG + Fg = (FG)' \text{ on } (A, B) \ .$$

By an earlier result (Theorem 9 in Lecture 9) there is a constant c such that $H_1 + H_2 + c = FG$ on (A, B). Hence

$$[FG]_A^B = [H_1 + H_2 + c]_A^B = [H_1]_A^B + [H_2]_A^B = T_1 + T_3$$

which is a rearrangement of the equality stated in the theorem.

3. The case when $T_2, T_3 \in \mathbb{R}$ are defined is similar to Case 1 and is left to the reader as an exercise. $\square$

For example, we set $I_n := (\mathrm{N}) \int_0^{+\infty} x^n \mathrm{e}^{-x}$, $n \in \mathbb{N}_0$. Then $I_0 = [-\mathrm{e}^{-x}]_0^{+\infty} = -\mathrm{e}^{-\infty} - (-\mathrm{e}^{-0}) = -0 - (-1) = 1$. For $n > 0$ we get by the last theorem and induction on n that

$$\begin{aligned} I_n &= (\mathrm{N}) \int_0^{+\infty} x^n \left(-\mathrm{e}^{-x}\right)' \\ &\overset{\text{Thm. 4, } \exists\, T_2 \text{ and } T_3}{=} [-x^n \mathrm{e}^{-x}]_0^{+\infty} + (\mathrm{N}) \int_0^{+\infty} (x^n)' \, \mathrm{e}^{-x} \\ &= -0 + 0 + n \cdot (\mathrm{N}) \int_0^{+\infty} x^{n-1} \mathrm{e}^{-x} \\ &= n \cdot I_{n-1} \,. \end{aligned}$$

Therefore $I_n = n! = \prod_{j=1}^{n} j$ for every $n \in \mathbb{N}_0$. This representation of factorials by integrals can be used to prove the Stirling formula that we mentioned last time.

• *Integration by substitution.* We adapt the two formulae for integration by substitution given in the last lecture for the general Newton integral.

Theorem 5 ((N) $\int_A^B f$ by substitution) *If $A < B$ and $C < D$ are in $\mathbb{R}^*$, $g\colon (A,B) \to (C,D)$, $f\colon (C,D) \to \mathbb{R}$ and g has on (A,B) finite g', then the following two claims are true.*

1. *Suppose that f has on (C,D) a primitive function F. Then the equality*

$$(\mathrm{N})\int_A^B f(g)\cdot g' = (\mathrm{N})\int_{g(A)}^{g(B)} f$$

   holds if the right-hand side is defined.

2. *If g is onto and $g' \neq 0$ on (A,B) then the equality*

$$(\mathrm{N})\int_C^D f = (\mathrm{N})\int_{g^{-1}(C)}^{g^{-1}(D)} f(g)\cdot g'$$

   holds if the right-hand side is defined. Here we have that $\{g^{-1}(C),\, g^{-1}(D)\} = \{A,\, B\}$ (in some order).

Proof. 1. Let A, B, C, D, g, f and F be as stated and let the right-hand side be defined. This means that the limits

$$g(A) := \lim_{x\to A} g(x) \in \mathbb{R}^* \ \text{ and } \ g(B) := \lim_{x\to B} g(x) \in \mathbb{R}^*$$

exist. It follows that $g(A)$ and $g(B)$ are limit points of (C,D). It also means that the right-hand side has the value

$$\lim_{y\to g(B)} F(y) - \lim_{y\to g(A)} F(y)$$

(in particular, the last two limits exist and are finite). We already know that $F(g)$ is on (A,B) a primitive of $f(g)\cdot g'$. Thus

$$\begin{aligned}(\mathrm{N})\int_{g(A)}^{g(B)} f &= \lim_{y\to g(B)} F(y) - \lim_{y\to g(A)} F(y)\\ &= \lim_{x\to B} F(g(x)) - \lim_{x\to A} F(g(x))\\ &= (\mathrm{N})\int_A^B f(g)\cdot g'\ .\end{aligned}$$

Here the first and third equality follow from the definition of the general Newton integral. The crucial middle equality follows by the theorem on limits of composite functions (Theorem 14 in Lecture 5 whose Condition 1 holds as the outer function F is continuous).

2. Let A, B, C, D, g and f be as stated and let the right-hand side be defined. From the proof of part 2 of Theorem 13 in the last lecture we know that g is an increasing or decreasing bijection, and therefore so is the inverse $g^{-1}\colon (C,D)\to(A,B)$ (which is also continuous). Thus the limits

$$g^{-1}(C) := \lim_{y\to C} g^{-1}(y)\in\mathbb{R}^* \ \text{ and } \ g^{-1}(D) := \lim_{y\to D} g^{-1}(y)\in\mathbb{R}^*$$

exist and are equal $\{A,B\}$ (in some order). Since the right-hand side is defined, $f(g)\cdot g'$ has on (A,B) a primitive function G and the right-hand side has the value

$$\lim_{x\to g^{-1}(D)} G(x) - \lim_{x\to g^{-1}(C)} G(x)\ .$$

We already know that $G(g^{-1})$ is on (C,D) a primitive of f. Thus

$$\begin{aligned}(\mathrm{N})\int_{g^{-1}(C)}^{g^{-1}(D)} f(g)\cdot g' &= \lim_{x\to g^{-1}(D)} G(x) - \lim_{x\to g^{-1}(C)} G(x)\\ &= \lim_{y\to D} G(g^{-1}(y)) - \lim_{y\to C} G(g^{-1}(y))\\ &= (\mathrm{N})\int_C^D f\ .\end{aligned}$$

The first and third equality again follow from the definition of the general Newton integral and the second equality again follows in the same way by the theorem on limits of composite functions. $\square$

The two previous formulae show how the general Newton integral relates to the operation of composition of functions. We compute a (N) $\int_A^B f$ by the substitution formulae and integration by parts and by the next table of primitives. We compute the (N) $\int_A^B f$ in two steps. First a PF F of f on (A,B) is found by the mentioned methods. Then, if F exists, it is usually straightforward to determine the limits of F at A and B. For example, last time we computed that

$$\int \sqrt{1-t^2} = \frac{t\sqrt{1-t^2}+\arcsin t}{2} =: F(t) \ \text{ on } (-1,\ 1)\ .$$

(By Proposition 6 in Lecture 8, $F'(-1)=F'(1)=0$, and therefore this relation holds even on $[-1,1]$.) Thus

$$\begin{aligned}(\mathrm{N})\int_{-1}^{1}\sqrt{1-t^2} &= \lim_{t\to 1} F(t) - \lim_{t\to -1} F(t)\\ &= (\arcsin 1)/2 - (\arcsin(-1))/2\\ &= \pi/4 - (-\pi/4) = \pi/2\ .\end{aligned}$$

So for $f(t) = \sqrt{1-t^2} \colon [-1,1] \to \mathbb{R}$ the area of D_f is (defined as) $A_f = \pi/2$. This agrees with the double area π of the unit disc $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ because D_f is its the upper half.

• *A table of antiderivatives of some elementary functions.* This table is obtained completely mechanically by inverting the rules for differentiation in the table of derivatives in Theorem 17 in Lecture 7.

Theorem 6 (a table of primitives) *The following formulas hold.*

1. *On* $\mathbb{R}$*,* $\int \exp(x) = \exp(x)$*,* $\int \sin x = -\cos x$*,* $\int \cos x = \sin x$*,* $\int 1/(1+x^2) = \arctan x$ *(and also* $= -\mathrm{arccot}\, x$*) and* $\int x^n = x^{n+1}/(n+1)$ *for every* $n \in \mathbb{N}_0$*.*
2. *Both on* $(-\infty, 0)$ *and on* $(0, +\infty)$*,* $\int 1/x = \log(|x|)$ *and* $\int x^n = x^{n+1}/(n+1)$ *for every* $n \in \{-2, -3, \dots\}$*.*
3. *On* $(0, +\infty)$*,* $\int x^b = x^{b+1}/(b+1)$ *for every* $b \in \mathbb{R} \setminus \mathbb{Z}$*.*
4. *On every interval* $(k\pi - \pi/2, k\pi + \pi/2)$ *with* $k \in \mathbb{Z}$*,* $\int 1/(\cos x)^2 = \tan x$*.*
5. *On every interval* $(k\pi, (k+1)\pi)$ *with* $k \in \mathbb{Z}$*, one has that* $\int 1/(\sin x)^2 = -\cot x$*.*
6. *On* $(-1, 1)$*,* $\int 1/\sqrt{1-x^2} = \arcsin x$ *(and also* $= -\arccos x$*).*

In connection with the first formula in 2 note that, formally and interestingly, both $(\log x)' = 1/x$ and $(\log(-x))' = (1/(-x)) \cdot (-x)' = 1/x$. This seemingly contradicts the basic result that primitives of the same function only differ by a constant shift. Resolution of this conundrum is simple, the functions $\log x$ and $\log(-x)$ have disjoint definition domains.

• *Computing primitives of rational functions.* This is a large class of functions for which antiderivatives can be explicitly computed. Recall that a rational function $r = r(x)$ is a ratio of two polynomials:

$$r(x) = \frac{p(x)}{q(x)} \colon \underbrace{\mathbb{R} \setminus Z(r)}_{\mathrm{Def}(r)} \to \mathbb{R}\ .$$

Here $p(x), q(x) \in \mathbb{R}[x]$ are polynomials with real coefficients, $q(x)$ is not the zero polynomial and $Z(r) = \{a \in \mathbb{R} \mid q(a) = 0\}$ is the zero set (the set of real roots) of the denominator $q(x)$. It is well known that the cardinality $|Z(r)| \le \deg q$, the *degree* of the polynomial $q = q(x)$. An *irreducible trinomial* $a(x)$ is any real *monic* (= with the leading coefficient 1) quadratic polynomial

$$a(x) := x^2 + bx + c$$

such that $b^2 - 4c < 0$, i.e., $a(x)$ has no real root. Note that then $a(x) > 0$ for every $x \in \mathbb{R}$. For example, x^2+2x+2 is an irreducible trinomial. In the rest of the lecture we prove, modulo the proof of Theorem 8 (the *Fundamental Theorem of Algebra*), the next theorem.

Theorem 7 **($\int r(x)$)** *For any rational function $r = r(x)$ there exists a function $R(x)$ of the form*

$$\begin{aligned} R(x) &= r_0(x) + \sum_{i=1}^{k} s_i \cdot \log(|x - \alpha_i|) + \sum_{i=1}^{l} t_i \cdot \log(a_i(x)) + \\ &\quad + \sum_{i=1}^{m} u_i \cdot \arctan(b_i(x)) \ , \end{aligned}$$

where $r_0(x)$ is a rational function, $k, l, m \in \mathbb{N}_0$, empty sums are defined as 0, $s_i, t_i, u_i \in \mathbb{R}$, $\alpha_i \in Z(r)$, the $a_i(x)$ are irreducible trinomials and the $b_i(x)$ are real non-constant linear polynomials, and such that

$$R(x) = \int r(x)$$

on any nontrivial interval $I \subset \mathrm{Def}(r)$.

It is clear that functions of all four types given above can appear in $R(x)$. For example, we get by linearity of integration, by integration by substitution and by the above table of primitives that

$$\begin{aligned} \textstyle\int r(x) &:= \int \left(\frac{1}{x^4} + \frac{1}{x-1} + \overbrace{\frac{2x+2}{x^2+2x+2}}^{=(\dots)'/(\dots)} + \overbrace{\frac{1}{x^2+2x+2}}^{=1/((x+1)^2+1)} \right) \\ &= -\frac{1}{3x^3} + \log(|x-1|) + \log(x^2+2x+2) + \arctan(x+1) \ , \end{aligned}$$

on any nontrivial interval $I \subset \mathbb{R} \setminus \{0, 1\}$. In the proof of the theorem we describe an algorithm for obtaining primitives of rational functions in the stated form. But before we can start computing

PFs we need to develop the theory of partial fractions.

- *Partial fractions.* We will not prove the next theorem here.

Theorem 8 (FTAlg) *Every non-constant complex polynomial $p(x) \in \mathbb{C}[x]$ has at least one root, a number $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.*

From FTAlg we get irreducible decompositions in $\mathbb{R}[x]$.

Corollary 9 (decompositions of real polynomials) *Every nonzero real polynomial $q(x)$ can be written as*

$$q(x) = c \cdot \underbrace{\prod_{i=1}^{k} (x - \alpha_i)^{m_i}}_{\textit{type 1 r. factors}} \cdot \underbrace{\prod_{i=1}^{l} a_i(x)^{n_i}}_{\textit{type 2 r. factors}}$$

where $c \in \mathbb{R} \setminus \{0\}$ is its leading coefficient, $k, l \in \mathbb{N}_0$, empty products are defined as 1, $m_i, n_i \in \mathbb{N}$, the $\alpha_i \in \mathbb{R}$ are the all distinct real roots of $q(x)$, and the $a_i(x)$ are distinct irreducible trinomials.

Proof. If $\alpha = a + bi \in \mathbb{C}$ is a root of $q(x)$ then also its conjugate $\overline{\alpha} = a - bi$ is a root because $q(x) \in \mathbb{R}[x]$. In more details, the conjugation respects addition and multiplication and fixes real numbers: if $t \in \mathbb{R}$ and $u, v \in \mathbb{C}$ then $\overline{t} = t$, $\overline{u + v} = \overline{u} + \overline{v}$ and $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$. So if $q(x) = \sum_{j=0}^{n} t_j x^j$ then

$$0 = \overline{q(\alpha)} = \overline{\sum_{j=0}^{n} t_j \alpha^j} = \sum_{j=0}^{n} \overline{t_j} \cdot (\overline{\alpha})^j = \sum_{j=0}^{n} t_j \cdot (\overline{\alpha})^j = q(\overline{\alpha}) \ .$$

Also, if $\alpha \in \mathbb{C} \setminus \mathbb{R}$, i.e., if $b \neq 0$ then

$$a_\alpha(x) := (x - \alpha)(x - \overline{\alpha}) = x^2 - 2a \cdot x + (a^2 + b^2) \in \mathbb{R}[x]$$

and is an irreducible trinomial: $(2a)^2 - 4(a^2 + b^2) = -4b^2 < 0$.

If $q(x)$ is a constant polynomial, the corollary holds with the decomposition $q(x) = c$. If $q(x)$ is non-constant, by Theorem 8 it has a root $\alpha \in \mathbb{C}$. We divide $q(x)$ by $x - \alpha$ with remainder and get that $q(x) = (x - \alpha) \cdot q_1(x) + \beta$ for $q_1(x) \in \mathbb{C}[x]$ and $\beta \in \mathbb{C}$. Setting $x = \alpha$ we see that $\beta = 0$ and

$$q(x) = (x - \alpha)q_1(x) \ .$$

If $\alpha \in \mathbb{R}$, the division algorithm for polynomials shows that the polynomial $q_1(x)$ is real. So we have split off one root factor $x - \alpha$ of type 1. If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, we divide $q_1(x)$ by $x - \overline{\alpha}$ with remainder and get that $q_1(x) = (x - \overline{\alpha})s_1(x)$. Then

$$q(x) = (x - \alpha)q_1(x) = (x - \alpha)(x - \overline{\alpha})s_1(x) = a_\alpha(x)s_1(x) \ .$$

Again, $s_1(x)$ is real and we have split off one root factor $a_\alpha(x)$ of type 2. If $q_1(x)$, resp. $s_1(x)$, is non-constant, we apply to it the same procedure and then continue in the same fashion. Eventually splitting off terminates at the constant polynomial c and we get for $q(x)$ the stated decomposition. $\square$

We obtain partial fractions decompositions of rational functions by means of the next identity.

Proposition 10 (Bachet's identity) *Let $p(x)$ and $q(x)$ be two real polynomials with no common complex root, i.e., $p(z) = q(z) = 0$ for no $z \in \mathbb{C}$. Then there exist polynomials $r(x), s(x) \in \mathbb{R}[x]$ such that*

$$r(x) \cdot p(x) + s(x) \cdot q(x) = 1 \ .$$

Proof. For the given polynomials $p(x)$ and $q(x)$ we consider the set of real polynomials

$$S = \{r(x) \cdot p(x) + s(x) \cdot q(x) \mid r(x),\, s(x) \in \mathbb{R}[x]\}$$

and take nonzero $t(x) \in S$ with the minimum degree. We divide any $a(x) \in S$ by $t(x)$ with remainder:

$$a(x) = t(x) \cdot b(x) + c(x)$$

where $b(x), c(x) \in \mathbb{R}[x]$ and $\deg c(x) < \deg t(x)$ or $c(x)$ is the zero polynomial. But $c(x) = a(x) - b(x) \cdot t(x) \in S$ (because S is closed to subtraction and multiples). Thus $c(x)$ is the zero polynomial and $a(x) = b(x) \cdot t(x)$—$t(x)$ divides any element of S. But $p(x), q(x) \in S$ and so $t(x)$ divides both of them. But these polynomials have no common complex root and therefore, by Theorem 8, $t(x)$ is a nonzero constant polynomial. We may assume that $t(x) = 1$ and get the stated identity. $\square$

Theorem 11 (partial fractions) *Every rational function $r(x) = p(x)/q(x) \in \mathbb{R}(x)$, with $q(x)$ decomposed as in Corollary 9, expresses as*

$$r(x) = s(x) + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{\beta_{i,j}}{(x - \alpha_i)^j} + \sum_{i=1}^{l} \sum_{j=1}^{n_i} \frac{\gamma_{i,j} x + \delta_{i,j}}{a_i(x)^j}$$

where $s(x) \in \mathbb{R}[x]$ is a polynomial, k, l, m_i, n_i, α_i and $a_i(x)$ are as in Corollary 9, and $\beta_{i,j}, \gamma_{i,j}, \delta_{i,j} \in \mathbb{R}$.

Proof. After dividing Bachet's identity by $p(x)q(x)$ we have that

$$\frac{1}{p(x)q(x)} = \frac{s(x)}{p(x)} + \frac{r(x)}{q(x)}\,.$$

Iterating this identity we get that for any n real polynomials $q_1(x)$, $\ldots$, $q_n(x)$ such that no $q_i(x)$ and $q_j(x)$ with $i \neq j$ have a common complex root there exist n real polynomials $s_1(x)$, $\ldots$, $s_n(x)$ such that

$$\frac{1}{q_1(x)q_2(x)\ldots q_n(x)} = \sum_{i=1}^{n} \frac{s_i(x)}{q_i(x)} \ .$$

Now let a rational function $r(x) = p(x)/q(x)$ be given and $q(x)$ be decomposed as in Corollary 9. We use the last displayed identity for $n := k + l$, $q_1(x) := (x - \alpha_1)^{m_1}$, $\ldots$, $q_k(x) := (x - \alpha_k)^{m_k}$, $q_{k+1}(x) := a_1(x)^{n_1}$, $\ldots$, $q_{k+l}(x) := a_l(x)^{n_l}$ and get real polynomials $b_1(x)$, $\ldots$, $b_k(x)$, $c_1(x)$, $\ldots$, $c_l(x)$ such that

$$r(x) = \frac{p(x)}{q(x)} = \sum_{i=1}^{k} \frac{b_i(x)}{(x - \alpha_i)^{m_i}} + \sum_{i=1}^{l} \frac{c_i(x)}{a_i(x)^{n_i}} \ .$$

In each of the above $k + l$ fractions we divide numerator by denominator with remainder: $b_i(x) = (x - \alpha_i)^{m_i} \cdot s_i(x) + d_i(x)$ and $c_i(x) = a_i(x)^{n_i} \cdot s_{i+k}(x) + d_{i+k}(x)$ where $d_i(x), s_i(x) \in \mathbb{R}[x]$ and each remainder $d_i(x)$ is either the zero polynomial or has degree less than that of the denominator (which is m_i or $2n_i$). With $s(x) := \sum_{i=1}^{k+l} s_i(x) \in \mathbb{R}[x]$ we rewrite the last displayed equality as

$$r(x) = \frac{p(x)}{q(x)} = s(x) + \sum_{i=1}^{k} \frac{d_i(x)}{(x - \alpha_i)^{m_i}} + \sum_{i=1}^{l} \frac{d_{k+i}(x)}{a_i(x)^{n_i}} \ .$$

For each $i \in \{1, 2, \ldots, k\}$ we repeatedly divide $d_i(x)$ by $x - \alpha_i$ with remainder and express the i-th summand in the first sum in the above stated form. We do the same for each summand in the second sum. In more details, for example $d_{k+1}(x)/a_1(x)^{n_1}$ equals

$$\frac{a_1(x)\cdot e(x)+\gamma_{1,n_1}x+\delta_{1,n_1}}{a_1(x)^{n_1}}=\frac{e(x)}{a_1(x)^{n_1-1}}+\frac{\gamma_{1,n_1}x+\delta_{1,n_1}}{a_1(x)^{n_1}}\ ,$$

then we divide $e(x)$ by $a_1(x)$ with remainder and so on. $\square$

• *A proof of Theorem 7 on the form of* $\int r(x)$. Now we can prove this theorem. We express the given rational function $r(x)$ as a sum of partial fractions as in the previous theorem:

$$r(x)=s(x)+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\beta_{i,j}}{(x-\alpha_i)^j}+\sum_{i=1}^{l}\sum_{j=1}^{n_i}\frac{\gamma_{i,j}x+\delta_{i,j}}{a_i(x)^j}\ .$$

We of course use linearity of antiderivatives and integrate each summand in the expression separately. It is easy to integrate the first two terms: $\int s(x)$ is a polynomial (on any nontrivial real interval I), $\int \beta/(x-\alpha)^j = -\beta/(j-1)(x-\alpha)^{j-1}$ for any $j\ge 2$ and $\int \beta/(x-\alpha)=\beta\log(|x-\alpha|)$, where the last two antiderivatives hold on any nontrivial interval $I\subset\mathbb{R}\setminus\{\alpha\}$. Thus these contributions to $\int r(x)$ are of the first two types given in Theorem 7.

It remains to integrate the third term, which means to compute primitives of the form

$$\int\frac{\gamma x+\delta}{(x^2+bx+c)^j}$$

where $j\in\mathbb{N}$ and $\gamma,\delta,b,c\in\mathbb{R}$ are such that $b^2-4c<0$. With $d:=\sqrt{c-b^2/4}>0$ and $e:=(\delta-\gamma b/2)/d^{2j-1}$ we write the last rational function as

$$\begin{aligned}
\frac{\gamma x+\delta}{(x^2+bx+c)^j} &= \frac{\gamma}{2}\cdot\underbrace{\frac{2x+b}{(x^2+bx+c)^j}}_{T:=(\dots)'/(\dots)^j}+\frac{\delta-\gamma b/2}{(x^2+bx+c)^j} \\
&= \frac{\gamma}{2}\cdot T+e\cdot\underbrace{\frac{1/d}{\big((x/d+b/2d)^2+1\big)^j}}_{U:=(\dots)'/((\dots)^2+1)^j} \\
&= \frac{\gamma}{2}\cdot T+e\cdot U\ .
\end{aligned}$$

By the first integration by substitution formula, $\int T = 1/(j-1)(x^2+bx+c)^{j-1}$ for $j\geq 2$ and $\int T=\log(x^2+bx+c)$ for $j=1$ (on any nontrivial real interval I). Thus we get contributions to $\int r(x)$ of the first and third type given in Theorem 7.

Finally, we compute $\int U$. By the first integration by substitution formula, $\int U = I_j(x/d+b/2d)$ (on any nontrivial real interval I) for

$$I_j = I_j(y) := \int \frac{1}{(y^2+1)^j}\ .$$

For $j\in\mathbb{N}$, integration by parts and differentiation of composite functions lead to the relation

$$\begin{aligned}
I_j &= \int y'\cdot\frac{1}{(y^2+1)^j} = \frac{y}{(y^2+1)^j}+2j\int\frac{(y^2+1)-1}{(y^2+1)^{j+1}} \\
&= \frac{y}{(y^2+1)^j}+2j\cdot I_j-2j\cdot I_{j+1}\ .
\end{aligned}$$

Hence we get the recurrence $I_1=\arctan y$ (by the above table of primitives) and, for $j\in\mathbb{N}$,

$$I_{j+1}=\frac{y}{2j\cdot(y^2+1)^j}-(1-1/2j)\cdot I_j\ .$$

It follows from it that for every $j \in \mathbb{N}$,

$$I_j(y) = u(y) + r \cdot \arctan y$$

where $u(y) \in \mathbb{Q}(y)$ is a rational function and $r \in \mathbb{Q}$. Since $\int U = I_j(x/d + b/2d)$, the last contribution to $\int r(x)$ is of the first and fourth type given in Theorem 7. $\square$

THANK YOU FOR YOUR ATTENTION!