10th International Conference
on Lattice Path Combinatorics
and Applications 2026 (LPC 2026)
Wien, Juli 20–24, 2026

EXTENDING THE SYMBOLIC METHOD IN ENUMERATIVE COMBINATORICS

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based on [MKRH]: MK and RH, Extending the symbolic method in enumerative combinatorics. I, arXiv:2511.00914v9 [math.CO], 45 pp., 2026

• *The symbolic method (in enumerative combinatorics)*

The setup includes a countable set A of structures, a size function $s\colon A \to \mathbb{N}_0$ $(= \{0, 1, 2, \dots\})$, and a weight function $h\colon A \to R$ to a ring $R = \langle R, 0_R, 1_R, +, \cdot \rangle$. The **finiteness axiom** says that

$$|A_n| < \infty \ \forall n \in \mathbb{N}_0, \text{ where } A_n := \{a \in A\colon\ s(a) = n\}\,.$$

Then the main generating function (GF) of $\langle A, s, h \rangle$ is

$$A_h(x) = \textstyle\sum_{n \geq 0} h(A_n)x^n := \sum_{n \geq 0} \big(\sum_{a \in A_n} h(a)\big)x^n \quad (\in R[[x]])\,.$$

Due to the finiteness axiom, every coefficient sum $h(A_h)$ is finite.

• *Two phases of the symbolic method*

1. The formal phase. In it we obtain

- combinatorial relations between structures in A
- recurrences for sums $h(A_n)$ as $n \in \mathbb{N}_0$
- equations $\mathcal{E}$ for $A_h(x)$ and auxiliary GFs $B_h(x)$, $C_h(x)$, $\dots$
- exact formulas = algorithms computing functions $n \mapsto h(A_n)$

Multivariate GFs are often used.

2. The analytic phase. The equations $\mathcal{E} \rightsquigarrow$ asymptotic formulas for the functions $n \mapsto h(A_n)$.

We (MK and RH) are interested in extending the formal phase.

Source books for the formal phase

- L. Comtet, *Advanced Combinatorics*, 1974
`https://kam.mff.cuni.cz/~klazar/Comtet.jpg`
- I. P. Goulden & D. M. Jackson, *Combinatorial Enumeration*, 1983
`https://kam.mff.cuni.cz/~klazar/GouldenJackson.jpg`
- P. Flajolet & R. Sedgewick, *Analytic Combinatorics*, 2009
`https://kam.mff.cuni.cz/~klazar/FlajoletSedgewick.jpg`
- M. H. Albert, Ch. Bean, A. Claesson, É. Nadeau, J. Pantone and H. Ulfarsson, *Combinatorial Exploration: An Algorithmic Framework for Enumeration*, 2026
`https://kam.mff.cuni.cz/~klazar/AlbertEtAl.pdf`

• *Our extension of the symbolic method*

We drop the finiteness axiom – we allow infinite (but countable) sets A_n of structures with size n – and require the following.

1. The ring R in $h\colon A \to R$ is endowed with a complete norm $\|\cdot\|$. For example, $R \in \{\mathbb{R}, \mathbb{C}\}$ and $\|\cdot\|$ is the Archimedean absolute value $|\cdot|$. In part II of [MKRH], we consider R with a non-Archimedean complete norm.
2. Every coefficient series $h(A_n) = \sum_{a \in A_n} h(a)$ absolutely converges in the norm. Absolute convergence means that the sums of series do not depend on the orders of summands.

We exemplify our extension of the symbolic method on generalizations of **Pólya's theorem**.

• *Pólya's 1921 theorem on random walks in G_d*

Let $d \in \mathbb{N}$ $(= \{1, 2, \dots\})$. $G_d = \langle \mathbb{Z}^d, E_d \rangle$, where $\mathbb{Z}$ are the integers, is the nearest neighbor graph on $\mathbb{Z}^d$. Its edges are given by

$$\{\overline{u}, \overline{v}\} \in E_d \iff \textstyle\sum_{i=1}^{d} |u_i - v_i| = 1\,.$$

G_d is $2d$-regular, countable, and (vertex) transitive. We denote the origin $\langle 0, 0, \dots, 0 \rangle \in \mathbb{Z}^d$ by $\overline{0}$. A walk w in G_d is any finite tuple

$$w = \langle \overline{u}_0, \overline{u}_1, \dots, \overline{u}_n \rangle \quad (n \in \mathbb{N}_0)$$

of vertices $\overline{u}_i \in \mathbb{Z}^d$ such that $\{\overline{u}_{i-1}, \overline{u}_i\} \in E_d$ for $i = 1, 2, \dots, n$. We define the length of w by $|w| := n$.

In 1921, George (György) Pólya (1887–1985) asked: *what is the limit, as n goes to ∞, of the probabilities $P_d(\overline{v}, n)$ that a walk w in G_d, starting at $\overline{0}$ and with length n, visits in a step $i > 0$ the given vertex $\overline{v}$?*

He proved that for every $d \in \mathbb{N}$ and $\overline{v} \in \mathbb{Z}^d$,

$$\lim_{n \to \infty} P_d(\overline{v}, n) = \begin{cases} 1 & \dots \quad d \le 2 \ \text{ and} \\ \in (0, 1) & \dots \quad d \ge 3\,. \end{cases} \qquad \text{(PT)}$$

(PT) is usually stated just for $\overline{v} = \overline{0}$.

• *Pólya's theorem via the symbolic method*

We derive (PT) for $\overline{v} = \overline{0}$ by the symbolic method. Then we explain generalizations of (PT) obtained in the extended symbolic method.

So let $d \in \mathbb{N}$ and $\overline{v} \in \mathbb{Z}^d$ be given. We set

$$A_{\overline{v}} := \{\text{walks in } G_d \text{ starting at } \overline{0} \text{ and then visiting } \overline{v}\}\,.$$

The size $s(w) = |w|$ is the length. The weight $h\colon A_{\overline{v}} \to \mathbb{R}_{\ge 0}$ is obtained from $h\colon E_d \to \{\frac{1}{2d}\}$ ($R = \mathbb{R}$) as the extension to walks $w = \langle \overline{u}_0, \overline{u}_1, \dots, \overline{u}_n\rangle$ by

$$h(w) = \textstyle\prod_{i=1}^n h(\{\overline{u}_{i-1},\, \overline{u}_i\}) = (2d)^{-n} = (2d)^{-|w|}\,.$$

Then the probability = the coefficient sum,

$$P_d(\overline{v},\, n) = \textstyle\sum_{w \in A_{\overline{v},\,n}} h(w) =: h(A_{\overline{v},\,n}) = [x^n]A_{\overline{v},\,h}(x)\,.$$

If $\overline{v} = \overline{0}$, we write A for $A_{\overline{0}}$, A_n for $A_{\overline{0},n}$, and $A(x)$ for $A_{\overline{0},h}(x)$. The main GF we are interested in is

$$A(x) = \textstyle\sum_{n\ge 0} h(A_n)x^n = \sum_{n \ge 0} \big(\sum_{w \in A_n} h(w)\big)x^n \quad (\in \mathbb{R}[[x]])\,.$$

The inner sum is always finite.

• *Auxiliary generating functions*

Thus, with a walk $w = \langle \overline{u}_0, \overline{u}_1, \dots, \overline{u}_n\rangle$ in G_d, we have

$$A = \{w\colon\ \overline{u}_0 = \overline{0} \ \&\ \exists\, i > 0\colon\ \overline{u}_i = \overline{0}\}\,.$$

We define another three sets of walks in G_d:

$$\begin{aligned} B &:= \{w\colon\ \overline{u}_0 = \overline{u}_n = \overline{0}\}\,, \\ C &:= \{w\colon\ \overline{u}_0 = \overline{u}_n = \overline{0} \ \&\ n > 0 \ \&\ \overline{u}_i \ne \overline{0} \text{ for } 0 < i < n\}\,, \text{ and} \\ D &:= \{w\colon\ \overline{u}_0 = \overline{0}\}\,. \end{aligned}$$

The corresponding three auxiliary GFs are

$$B(x) = \textstyle\sum_{n\ge 0} h(B_n)x^n,\ C(x) = \sum_{n \ge 0} h(C_n)x^n,$$

and $D(x) = \sum_{n\ge 0} h(D_n)x^n = \sum_{n\ge 0} 1x^n = \frac{1}{1-x}$. We have $h(A_0) = h(C_0) = 0$ and $h(B_0) = h(D_0) = 1$.

• *Two equations between the* GF*s* $A(x)$, $B(x)$, $C(x)$, *and* $D(x)$

eq. 1. $A(x) = C(x)D(x) = C(x)\frac{1}{1-x}$.

eq. 2. $B(x) = \frac{1}{1-C(x)}$, equivalently, $C(x) = 1 - \frac{1}{B(x)}$.

An outline of a proof of eq. 1: split $w \in A$ at the first revisit of $\overline{0}$. An outline of a proof of eq. 2: split $w \in B$ at all visits of $\overline{0}$.

• *A consequence of eq. 1*

By eq. 1,

$$h(A_n) = \textstyle\sum_{j=0}^{n} h(C_j),\ n \in \mathbb{N}_0\,.$$

So the limit probability is the sum of the series $h(C_0) + h(C_1) + \cdots$,

$$\begin{aligned}\lim_{n\to\infty} P_d(\overline{0},\, n) &= \lim_{n\to\infty} h(A_n) = \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} h(C_j) \\ &=: h(C_0) + h(C_1) + \cdots = C(1)\,.\end{aligned}$$

• *A consequence of eq. 2*

This is a relation between the sums

$$B(1) = h(B_0) + h(B_1) + \ldots \ \text{ and } \ C(1) = h(C_0) + h(C_1) + \cdots\,.$$

These sums always exist and belong to $[0,\, +\infty) \cup \{+\infty\}$.

Proposition 1 *If $F(x) = \sum_{n\ge 0} a_n x^n \in \mathbb{R}_{\ge 0}[[x]]$ (absolutely) converges for every $x \in [0, 1)$, then the following limit of a function and sum of a series of coefficients always exist and are equal,*

$$\lim_{x\to 1} F(x) = \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} a_j \quad (\in [0,\, +\infty) \cup \{+\infty\})\,.$$

Using this version of Abel's theorem (see [MKRH] for another two), we deduce the next analytic form of the formal identity $C(x) = 1 - \frac{1}{B(x)}$ in eq. 2.

Proposition 2 *For every $d \in \mathbb{N}$,*

$$C(1) = 1 - \frac{1}{B(1)} \quad (\in [0,\, 1])\,.$$

If $B(1) = +\infty$, then $C(1) = 1$.

• *A proof of* (PT) *for $\overline{v} = \overline{0}$ by the symbolic method*

By counting the closed walks in the set B, one proves the next well-known result. Recall that $B(1)$ is the sum of probabilities of all closed walks in G_d of all lengths.

Proposition 3 $d \le 2 \Rightarrow B(1) = +\infty$ *and* $d \ge 3 \Rightarrow B(1) < +\infty$.

Using Propositions 2 and 3 and $B(1) > 1$, we get (PT) for $\overline{v} = \overline{0}$:

$$\lim_{n\to\infty} P_d(\overline{0}\,, n) = \lim_{n\to\infty} h(A_n) = C(1) = \begin{cases} 1 & \ldots \quad d \le 2 \ \text{ and} \\ \in (0,\, 1) & \ldots \quad d \ge 3\,. \end{cases}$$

• *Generalizing Pólya's theorem*

In these generalizations, we replace $h = h_{\mathrm{Pólya}}\colon E_d \to \{\frac{1}{2d}\}$ with complex weights

$$h = h_{\mathbb{C}}\colon \mathbb{N}_2 := \binom{\mathbb{N}}{2} \to \mathbb{C}$$

on edges of the <u>countable complete graph</u> $K_{\mathbb{N}} = \langle \mathbb{N}, \mathbb{N}_2 \rangle$. It is clear which particular weight $h_{\mathbb{C}}$ (with many values 0) turns into the weight $h_{\mathrm{Pólya}}$ after a bijective renaming of vertices.

A <u>walk</u> w in $K_{\mathbb{N}}$ is any finite tuple

$$w = \langle u_0,\, u_1,\, \dots,\, u_n \rangle \quad (n \in \mathbb{N}_0)$$

of vertices $u_i \in \mathbb{N}$ such that $u_{i-1} \ne u_i$ for $i = 1, 2, \dots, n$. Any weight $h\colon \mathbb{N}_2 \to \mathbb{C}$ extends to walks w with $|w| = n \ge 1$ by

$$h(w) = \textstyle\prod_{i=1}^{n} h(\{u_{i-1},\, u_i\})\,.$$

If $n = 0$, we set $h(w) := 1$. We replace the starting vertex $\overline{0} \in \mathbb{Z}^d$ by $1 \in \mathbb{N}$ and denote the set of all walks in $K_{\mathbb{N}}$ starting at 1 as before,

$$D := \{w\colon\ u_0 = 1\}\,.$$

The size function is again the <u>length</u>, $s(w) = |w| = n$. We have the size n ($\in \mathbb{N}_0$) sets

$$D_n = \{w \in D\colon\ |w| = n\}\,.$$

$D_0 = \{\langle 1 \rangle\}$, but for $n > 0$ every set D_n is countable; we have to say <u>farewell to the finiteness axiom</u>.

• *Set series with complex summands*

We are interested in sums of the series

$$\textstyle\sum_{w \in X_n} h(w) \ \text{ where } \ n \in \mathbb{N}_0 \ \text{ and } \ X \subset D\,.$$

We need a theory of sums of <u>set series</u>, series of the form $\sum_{x \in X} h(x)$ where X is an at most countable set and $h\colon X \to \mathbb{C}$. See [MKRH] for it. Here we have time only for a few results.

Let $S = \sum_{x \in X} h(x)$ be a set series. For finite X, the set series S is just the usual finite sum of complex numbers, with sum 0 if $X = \emptyset$.

If X is countable, we say that the set series S absolutely converges (AK) if either of two equivalent conditions holds.

1. There exists a constant $c > 0$ such that $\sum_{x \in Y} |h(x)| \le c$ for every finite set $Y \subset X$.
2. For every bijection $p\colon \mathbb{N} \to X$, the finite limit

$$s = \lim_{n \to \infty} \sum_{j=1}^{n} h(p(j)) \ \ (\in \mathbb{C})$$

exists.

Exercise: the number s does not depend on the bijection p. Thus we can define the sum of AK set series $S = \sum_{x \in X} h(x)$ as

$$\sum_{x \in X} h(x) = h(S) := s\,.$$

For finite X, the sum $h(S)$ is the usual finite sum.

- *The operations of grouping and product of set series*

We skip the useful operation of linear combination. Recall that a set Y is a partition of a set X if the elements of Y are nonempty and mutually disjoint, and if $\bigcup Y = X$.

Proposition 4 *Let $S = \sum_{x \in X} h(x)$ be an* AK *set series and Y be a partition of X. Then the following holds.*

1. *For every set $Z \in Y$, the set series $\sum_{x \in Z} h(x)$ is* AK *and has sum $s_Z \in \mathbb{C}$.*
2. *The set series $T = \sum_{Z \in Y} s_Z$ is* AK *and has sum s.*
3. *The sum s equals to the sum $h(S)$ of the set series $S = \sum_{x \in X} h(x)$.*

We call the set series T the grouping of S.

Let $S_i = \sum_{x \in X_i} h_i(x)$ for $i = 1, 2$ be two set series. We define their product $S_1 \times S_2$ as the set series

$$S_1 \times S_2 := \sum_{\langle x,\, y \rangle \in X_1 \times X_2} h_1(x) \cdot h_2(y)\,.$$

The expected, but not completely trivial, result holds.

Proposition 5 *If S_1 and S_2 are* AK *and have respective sums s_1 and s_2, then also $S_1 \times S_2$ is* AK *and has sum $s_1 s_2$.*

• *Back to weights and walks in the countable complete graph $K_{\mathbb{N}}$*

We introduce a condition on weights under which we can generalize (PT). Recall that D_n is the set of walks in $K_{\mathbb{N}}$ that start at 1 and have length n.

Definition 6 *We call a weight $h\colon \mathbb{N}_2 \to \mathbb{C}$ $\underline{\text{light}}$ if for every $n \in \mathbb{N}_0$, the set series*

$$\textstyle\sum_{w \in D_n} h(w)$$

absolutely converges.

Let $h\colon \mathbb{N}_2 \to \mathbb{C}$. We define $\underline{V(h)}$ ($\subset \mathbb{N}$) to be the set of vertices $u \in \mathbb{N}$ reachable from 1 by a walk w with $h(w) \neq 0$. Convexity of weights takes the following form.

Proposition 7 *Let $h\colon \mathbb{N}_2 \to \mathbb{C}$ be a light weight and let $u \in V(h)$. Then the series*

$$S_u := \textstyle\sum_{v \in \mathbb{N}\setminus\{u\}} h(\{u,\, v\})$$

absolutely converges.

Let $\xi \in \mathbb{C}$. We say that a light weight h is $\underline{\xi\text{-convex}}$ if every above series S_u, $u \in V(h)$, has sum ξ. This generalizes the usual condition for random walks that the probabilities of transitions from a fixed state to other states sum up to 1. $\underline{\text{0-convex}}$ weights are reminiscent of $\underline{\text{Kirchhoff's second law}}$.

In the case $\overline{v} = \overline{0}$ of (PT), the vertex transitivity of the graphs G_d plays no role. If $\overline{v} \neq \overline{0}$, it is important. This transitivity generalizes to weights $h\colon \mathbb{N}_2 \to \mathbb{C}$ as follows.

Definition 8 *Let $v \in \mathbb{N}$ and $h\colon \mathbb{N}_2 \to \mathbb{C}$. We say that the weight h is $\underline{v\text{-transitive}}$ if there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(1) = v$ and for every edge $e = \{u, u'\} \in \mathbb{N}_2$ we have*

$$h(e) = h(\{f(u),\, f(u')\}).$$

• *What should be a generalization of Pólya's theorem?*

It is not so clear. In (PT), the input is a rather concrete weight $h\colon E_d \to \{\frac{1}{2d}\}$. In the generalizations, we only know that the input weight $h\colon \mathbb{N}_2 \to \mathbb{C}$ is light, and that it is possibly ξ-convex and/or v-transitive for a given vertex $v \neq 1$ (and/or that the range of h is a specific subset of $\mathbb{C}$, as we will see later).

The values $h(e)$ are usually not in $\mathbb{R}_{\geq 0}$, and this means that we have to abandon the probabilistic interpretation of (PT). Our answer to the above question is this.

> Generalizations of (PT) to light weights $h\colon \mathbb{N}_2 \to \mathbb{C}$ generalize the formula
> $$\lim_{n\to\infty} h(A_n)\ (= C(1)) =\ 1 - \frac{1}{B(1)}$$
> in the consequence of eq. 1 and Prop. 2 relating the coefficients of the main GF $A(x)$ to the coefficients of the auxiliary GF of closed walks $B(x)$.

• *Explaining Theorems 3.4, 3.7, and 4.10 of* [MKRH]

The preprint [MKRH] contains 23 such generalizations of (PT) in somewhat fewer theorems; see the summary in the final table of [MKRH]. Here we explain statements of Theorems 3.4, 3.7, and 4.10 in [MKRH]. We need some more notation for it.

• *Conditional and absolute sums of series*

Let $U(x) = \sum_{n\geq 0} u_n x^n \in \mathbb{C}[[x]]$ be GF. What is the sum of the series of coefficients

$$U(1) = u_0 + u_1 + \dots ?$$

If $u_n \in \mathbb{R}_{\geq 0}$, as is the case in (PT), there is a unique answer. For complex coefficients $u_n \in \mathbb{C}$, we have two (or even three) kinds of sums. We denote the conditional sum by $U(1)_1$, so that

$$U(1)_1 := \lim_{n\to\infty} \textstyle\sum_{j=0}^n u_j \quad (\in \mathbb{C})\,,$$

if the limit exists. The absolute sum is denoted by $U(1)_2$ and

$$U(1)_2 := \lim_{n\to\infty} \textstyle\sum_{j=0}^n u_{p(j)} \quad (\in \mathbb{C})\,,$$

if the (unique) limit exists for every bijection $p\colon \mathbb{N} \to \mathbb{N}$. Clearly, if the sum $U(1)_2$ exists, then $U(1)_1$ exists and $U(1)_2 = U(1)_1$. In the well known example

$$U(x) = \textstyle\sum_{n\geq 1} (-1)^{n-1} n^{-1} x^n\,,$$

$U(1)_1 = \log 2$ but $U(1)_2$ does not exist.

If $U(x)=\sum_{n\ge 0}u_nx^n\in\mathbb{R}[[x]]$ and

$$\lim_{n\to\infty}\sum_{j=0}^{n}u_j=+\infty\,,$$

we write that $\underline{U(1)_1=+\infty}$. We similarly define that $U(1)_1=-\infty$.

• *Twists of* GF*s,* sqrt$(\cdot)$*,* $\xi\mathbb{R}_{\ge 0}$*, and sets of walks in* $K_{\mathbb{N}}$

Nonzero numbers $\xi\in\mathbb{C}$ can be normalized as $\xi/|\xi|$ to lie in the set

$$\mathbb{C}_1:=\{z\in\mathbb{C}\colon\ |z|=1\}\,.$$

If $U(x)=\sum_{n\ge 0}u_nx^n\in\mathbb{C}[[x]]$ and $\xi\in\mathbb{C}$, we define the $\underline{\xi\text{-twist}}$ of $U(x)$ by

$$U(\xi x)=U(x)_\xi:=\sum_{n\ge 0}u_n\xi^nx^n\,.$$

For $z\in\mathbb{C}$ we define

$$\mathrm{sqrt}(z):=\{u\in\mathbb{C}\colon\ u^2=z\}\,,$$

so that $|\mathrm{sqrt}(z)|=2$ if $z\ne 0$ and $\mathrm{sqrt}(0)=\{0\}$. For $\xi\in\mathbb{C}_1$, we set

$$\xi\mathbb{R}_{\ge 0}:=\{c\xi\,:\,c\in\mathbb{R}_{\ge 0}\}\,.$$

It is a ray in $\mathbb{C}$ going from the origin in the direction ξ. Note that if $h\colon\mathbb{N}_2\to\xi\mathbb{R}_{\ge 0}$, then for every walk w in $K_{\mathbb{N}}$ we have $h(w)\in\xi^{|w|}\mathbb{R}_{\ge 0}$. Thus $E_h(x)_{1/\xi}\in\mathbb{R}_{\ge 0}[[x]]$ for any $E\subset D$ if h is light.

We consider a walk $w=\langle u_0,u_1,\dots,u_n\rangle$ in $K_{\mathbb{N}}$. Let $v\in\mathbb{N}\setminus\{1\}$. Recall that $D=\{w\colon\ u_0=1\}$. Besides D, we define

$$\begin{aligned}
&A:=\{w\colon\ u_0=1\ \&\ \exists\, i>0\colon\ u_i=1\}\,,\\
&A_v:=\{w\colon\ u_0=1\ \&\ \exists\, i\colon\ u_i=v\}\,,\\
&B:=\{w\colon\ u_0=u_n=1\}\,,\\
&C:=\{w\colon\ u_0=u_n=1\ \&\ n>0\ \&\ 0<i<n\Rightarrow u_i\ne 1\}\,,\\
&B_v:=\{w\colon\ u_0=u_n=1\ \&\ u_i\ne v\},\ \text{and}\\
&C_v:=\{w\colon\ u_0=1\ \&\ u_n=v\ \&\ i<n\Rightarrow u_i\ne v\}\,.
\end{aligned}$$

• *Theorems 3.4, 3.7 and 4.10 in* [MKRH]

We present a sample of three theorems in [MKRH]. They represent each of the three kinds of sums $U(1)$. The following is Theorem 3.4.

Theorem 9 **($U(1)_1=+\infty$)** *Let $\xi\in\mathbb{C}_1$ and $h\colon\mathbb{N}_2\to\xi\mathbb{R}_{\ge 0}$ be a light weight. If $(D_h)_{1/\xi}(1)_1=+\infty$, then*

$$(A_h)_{1/\xi}(1)_1=+\infty\,.$$

The following is Theorem 3.7 of [MKRH].

Theorem 10 ($U(1)_2$) 1. *Let* $\xi \in \mathbb{C}_1$ *and* $h\colon \mathbb{N}_2 \to \mathbb{C}$ *be a* ξ*-convex light weight. If* $\sum_{n\ge 1} \bigl|h(B_n)\bigr| < 1$*, then the sum* $(B_h)_{1/\xi}(1)_2$ *exists and is nonzero, and*

$$\lim_{n\to\infty} \xi^{-n} h(A_n) = 1 - \frac{1}{(B_h)_{1/\xi}(1)_2} \quad (\in \mathbb{C})\,.$$

2. *Let* $h\colon \mathbb{N}_2 \to \mathbb{C}$ *be a* 0*-convex light weight. If* $\sum_{n\ge 1} |h(B_n)| < 1$*, then the sums* $A_h(1)_2$ *and* $B_h(1)_2$ *exist,* $B_h(1)_2 \ne 0$*, and*

$$A_h(1)_2 = 1 - \frac{1}{B_h(1)_2} \quad (\in \mathbb{C})\,.$$

The following is Theorem 4.10 of [MKRH].

Theorem 11 ($U(1)_1$) 1. *Let* $\xi \in \mathbb{C}_1$, $v \in \mathbb{N}\setminus\{1\}$*, and let* $h\colon \mathbb{N}_2 \to \mathbb{C}$ *be a* v*-transitive* ξ*-convex light weight such that* $v \in V(h)$*. If the sums* $(B_{v,h})_{1/\xi}(1)_1$*,* $(B_h)_{1/\xi}(1)_1$*, and* $(C_{v,h})_{1/\xi}(1)_1$ *exist and if* $(B_h)_{1/\xi}(1)_1 \ne 0$*, then*

$$\lim_{n\to\infty} \xi^{-n} h(A_{v,n}) \in \mathrm{sqrt}\Bigl(1 - \frac{(B_{v,h})_{1/\xi}(1)_1}{(B_h)_{1/\xi}(1)_1}\Bigr)\,.$$

2. *Let* $v \in \mathbb{N}\setminus\{1\}$ *and let* $h\colon \mathbb{N}_2 \to \mathbb{C}$ *be a* v*-transitive* 0*-convex light weight such that* $v \in V(h)$*. If the sums* $B_{v,h}(1)_1$*,* $B_h(1)_1$*, and* $C_{v,h}(1)_1$ *exist and if* $B_h(1)_1 \ne 0$*, then the sum* $A_{v,h}(1)_1$ *exists and*

$$A_{v,h}(1)_1 \in \mathrm{sqrt}\Bigl(1 - \frac{B_{v,h}(1)_1}{B_h(1)_1}\Bigr)\,.$$

We see that generalizations of (PT) to light weights $h\colon \mathbb{N}_2 \to \mathbb{C}$ are also explorations of the logical space of assumptions of theorems.

• *Where is the extension of the symbolic method exactly used in the generalizations of* (PT)*, like Theorems 9–11?*

– In the statements of the theorems. For example, $(D_h)_{1/\xi}(1)_1 = \xi^{-0}h(D_0) + \xi^{-1}h(D_1) + \dots$ and each coefficient $h(D_n) = [x^n]D_h(x)$ is the sum of the AK set series $\sum_{w\in D_n} h(w)$.

– In the proofs of the theorems. For example, the analog of the identity $A(x) = C(x)D(x)$ in eq. 1 is the identity

$$A_h(x) = C_h(x)D_h(x)$$

in part 1 of Proposition 3.1 in [MKRH]. The former identity is a standard simple result in the classical symbolic method. The proof of the latter identity requires Propositions 4 and 5 on the operations of grouping and product of set series.

THANK YOU FOR YOUR ATTENTION!