Generalized Davenport–Schinzel sequences: results, problems, and applications

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Abstract

We survey in detail extremal results on Davenport–Schinzel sequences and their generalizations, from the seminal papers of H. Davenport and A. Schinzel in 1965 to present. We discuss geometric and enumerative applications, generalizations to colored trees, and generalizations to hypergraphs. Eleven illustrative examples with proofs are given and nineteen open problems are posed.

1 Introduction

DS sequences. Why a survey on Davenport–Schinzel sequences? (We shall abbreviate this term as DS sequences.) Two combinatorially oriented survey articles have appeared, Stanton and Dirksen [55] and Klazar [29]. Both are now outdated. Sharir and Agarwal [52] and Agarwal and Sharir [3] focus on geometric applications. Survey and historic sections can be found also in [27], [36], and [61], but the main goals of these works lie elsewhere. In this survey we treat the subject in more details and more concisely, pose many open problems, and present several combinatorially interesting and often unexplored generalizations of the original problem. We concentrate on its extremal side but we do discuss related enumerative aspects.

In Section 1 the classical Davenport–Schinzel's extremal functions $\lambda_s(n)$ are introduced and several simple bounds on them are proved. Section 2 surveys the extremal results on DS sequences which were obtained in the early

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period, before the superlinearity of $\lambda_3(n)$ was discovered in [21]. Section 3 explains the superlinear bounds on lengths of DS sequences. Section 4 presents the generalization of DS sequences to any forbidden subsequence, which was introduced in [1]. Section 5 describes various combinatorial situations where DS sequences and their generalizations appear; we discuss geometric graphs, colored trees, 0-1 matrices, ordered bipartite graphs, permutations, and set partitions. In Section 6 we describe a further generalization of DS sequences, or rather of the containment relation that defines them, to ordered hypergraphs; this section surveys some results of [33] and [35].

An *s-DS sequence*, where $s \geq 1$ is an integer, is any finite sequence $u = a_1 a_2 \dots a_l$ over a fixed infinite alphabet A satisfying two conditions:

1. For every $i = 1, 2, \dots, l-1$ we have $a_i \neq a_{i+1}$, which means that u contains no immediate repetition.

2. There do not exist s indices $1 \leq i_1 < i_2 < \cdots < i_s \leq l$ such that $a_{i_1} = a_{i_3} = a_{i_5} = \cdots = a$, $a_{i_2} = a_{i_4} = a_{i_6} = \cdots = b$, and $a \neq b$. That is, u contains no alternating subsequence of length s.

We write DS_s to denote the set of s-DS sequences. What are the elements of the alphabet A is not important. We assume that we have in A all positive integers $1, 2, \dots$, the letters $a, b, c, d, \dots$, and perhaps some other symbols. The set A^* consists of all finite sequences over A. Two sequences from A^* which have the same length and which differ only by an injective renaming of the symbols, for example 121331 and $2c2aa2$, are called *isomorphic*. For our purposes isomorphic sequences are identical. Every element of A^* is isomorphic to a unique *normal sequence*. A sequence u is normal if it is over the alphabet $\{1, 2, \dots, n\}$ for some integer $n > 0$, every $i \in \{1, 2, \dots, n\}$ appears in u, and the first occurrences of $1, 2, \dots, n$ in u, if we scan u from left to right, come in this order.

Example 1. There exist exactly ten normal 4-DS sequences u using at most 3 symbols:

$$u = \emptyset, 1, 12, 121, 1213, 12131, 123, 1231, 1232, \text{and } 12321.$$

□

$\mathbf{N}$ and $\mathbf{N}_0$ denote the sets $\{1, 2, \dots\}$ and $\{0, 1, 2, \dots\}$. We write $[n]$, $n \in \mathbf{N}$, for the set $\{1, 2, \dots, n\}$, and $[a, b]$, $a, b \in \mathbf{N}, a \leq b$, for the set

$\{a, a+1, \ldots, b\}$. For two functions $f, g : \mathbf{N} \to \mathbf{R}$, the asymptotic notation $f \ll g$ is synonymous to the $f = O(g)$ notation and means that $|f(n)| < c|g(n)|$ holds for every $n > n_0$ and a constant $c > 0$. The subscripts, such as $f \ll_k g$, indicate that c depends only on the mentioned parameters. The notation $f = o(g)$ means that $f(n)/g(n) \to 0$ as $n \to \infty$.

Extremal functions $\lambda_{\mathbf{s}}(\mathbf{n})$. For a sequence $u = a_1 a_2 \ldots a_l$ over A, we write $|u|$ to refer to its lentgh l. $S(u) = \{a_1, a_2, \ldots, a_l\}$ is the set of symbols used in u, and $\|u\| = |S(u)|$ is their number. Obviously, always $|u| \ge \|u\|$. We define, for the integers $s, n \ge 1$,

$$\lambda_{s-2}(n) = \max\{|u| : \ u \in \mathrm{DS}_s \ \& \ \|u\| \le n\}. \tag{1}$$

The function $\lambda_{s-2}(n)$ measures the maximum length of s-DS sequences using at most n symbols. It is trivial that, for every $n \ge 1$ and $s \ge -1$, $\lambda_s(n) < \infty$, $\lambda_{-1}(n) = 0$, $\lambda_0(n) = 1$, $\lambda_1(n) = n$, $\lambda_s(1) = 1$ (for $s \ge 0$), and $\lambda_s(2) = s + 1$.

The notation $\lambda_s(n)$ for the maximum lengths of DS sequences was introduced in 1986 by Hart and Sharir [21] and quickly became the standard notation. The shift -2 in the index results from an important application of DS sequences in geometry, which we exaplain in Section 2. All works on DS sequences prior to 1986 use the original notation $N_s(n)$ of Davenport and Schinzel [14]; $N_s(n) = \lambda_{s-1}(n)$. In the survey of these results in Section 2 we use both the original and the modern notation.

Example 2 ([14]). We bound $\lambda_s(n)$ in a rough way, determine $\lambda_2(n)$ precisely, and bound $\lambda_3(n)$ in a finer way.

We begin with the bound

$$\lambda_s(n) \le \binom{n}{2} s + 1 \tag{2}$$

which holds for all $n \ge 1$ and $s \ge 0$. Suppose a sequence $u = a_1 a_2 \ldots a_l$ satisfies condition 1 (no immediate repetition) and $\|u\| \le n$, but its length l exceeds the bound. Then among the $l - 1 \ge \binom{n}{2} s + 1$ two-element sets $\{a_i, a_{i+1}\}$, some $s + 1$ sets must coincide (by the pigeonhole principle), which produces in u an alternating subsequence of length $s + 2$. This proves (2).

Let us prove now that for every $n \ge 1$,

$$\lambda_2(n) = 2n - 1.$$

The sequences $u = 1\ 2\ \ldots\ (n-1)\ n\ (n-1)\ \ldots\ 2\ 1$ show that $\lambda_2(n) \geq 2n-1$. We prove the opposite inequality by induction on n. Certainly $\lambda_2(1) = 1$. In every $u \in \mathrm{DS}_4$ with $\|u\| \leq n$ and $|u| = \lambda_2(n)$ some symbol x appears only once; it is easy to see that any symbol sandwiched in the closest repetition has this property (and since $|u|$ is maximum, there must be a repetition). Delete x and, if necessary, one of its neighbours (to avoid creating an immediate repetition). The sequence v obtained is a 4-DS sequence and $\|v\| \leq n-1$. By induction, $\lambda_2(n) = |u| \leq |v|+2 \leq \lambda_2(n-1)+2 = 2(n-1)-1+2 = 2n-1$.

We prove that

$$\lambda_3(n) \ll n \log n.$$

Let $u \in \mathrm{DS}_5$ with $\|u\| \leq n$ and $|u| = \lambda_3(n)$. Note that the maximum length implies $\|u\| = n$. For every $x \in S(u)$ we set $k(x)$ to be the number of appearances of x in u. Only the first and the last appearance of x in u may have equal neighbours, because equal neighbours of any middle appearance of x would create the forbidden 5-term alternating subsequence. So by deleting at most $k(x)+2$ elements from u we get rid of all appearances of x and create no immediate repetition. The sequence v obtained is a 5-DS sequence and $\|v\| \leq n-1$. Thus $\lambda_3(n) = |u| \leq |v| + k(x) + 2 \leq \lambda_3(n-1) + k(x) + 2$. Summing these inequalities over all $x \in S(u)$, we obtain the inequality $n\lambda_3(n) \leq n\lambda_3(n-1) + \lambda_3(n) + 2n$, which we rewrite as

$$\frac{\lambda_3(n)}{n} - \frac{\lambda_3(n-1)}{n-1} \leq \frac{2}{n-1}.$$

Summing these inequalities for $2, 3, \ldots, n$ leads to the bound $\lambda_3(n) \leq n(1 + 2(1^{-1} + 2^{-1} + \cdots + (n-1)^{-1})) \ll n \log n$. □

2 The early period

The geometric origin of $\lambda_s(n)$. Davenport and Schinzel introduced the sequences, which now bear their names, in 1965 in [14]. They were led to them by the following geometric problem. Suppose $f_1, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ are n continuous functions such that the equation $f_i(x) = f_j(x)$ has for $i \neq j$ at most s solutions $x \in \mathbf{R}$. In other words, the graphs of any two functions intersect in at most s points. The real line then splits uniquely into l nonempty open intervals $I_1 = (-\infty, a_1), I_2 = (a_1, a_2), I_3 = (a_2, a_3), \ldots, I_l = (a_{l-1}, \infty)$ so that the pointwise minimum function $f(x) = \min_{j=1\ldots n} f_j(x)$

coincides on each I_i with a unique function $f_{j(I_i)}$, $1 \le j(I_i) \le n$, and $j(I_i) \ne j(I_{i+1})$. (See Figure 1 in the next section for a very similar situation.) The problem is how large the number l can be. It is easy to prove that the sequence $u = j(I_1)j(I_2)\ldots j(I_l)$ is an $(s+2)$-DS sequence. Thus if every pair f_i and f_j, $i \ne j$, has at most s intersections, the number $|u| = l$ of the distinct portions of the graph of f can be bounded from above by $\lambda_s(n)$. This is the reason for the later -2 shift of s in $\lambda_{s-2}(n)$ compared to DS_s. However, in [14] and all works prior to 1986, $\max\{|u| :\ u \in \mathrm{DS}_s \ \&\ \|u\| \le n\}$ is denoted by $N_{s-1}(n)$ (or by $N(s-1,n)$). For the reader's convenience, in Section 2 we combine both notations. Simply remember that $N_s(n) = \lambda_{s-1}(n)$.

A natural example of a system $\{f_i\}$ with $|\{x \in \mathbf{R} :\ f_i(x) = f_j(x)\}| \le s$ for every fixed $i \ne j$ is any system of distinct polynomials of degree at most s. Or, as was the case in [14], any system of distinct solutions of a given homogeneous linear differential equation with constant coefficients, of order at most $s+1$. The problem to determine or to bound the maximum number l of the portions of the graph of f originated in control theory, and it was communicated to Davenport and Schinzel by K. Malanowski ([14]). They reduced geometry to combinatorics and asked about the values of $N_s(n)$. In [14] they proved that

$$
\begin{aligned}
(\lambda_{s-1}(n) =)\ N_s(n) &\le n(n-1)s+1 && (3)\\
(\lambda_2(n) =)\ N_3(n) &= 2n-1 && (4)\\
(\lambda_3(n) =)\ N_4(n) &< 2n(1+\log n) && (5)\\
(\lambda_{s-1}(n) =)\ N_s(n) &\ll_s n \cdot \exp(10(s\log s)^{1/2}(\log n)^{1/2}). && (6)
\end{aligned}
$$

Their proofs of (3)–(5) are reproduced in Example 2. (We have slightly corrected the proof of (3) to obtain the somewhat better bound (2). Of the two proofs of (4) in [14], we present the second one, based on “an observation, made to us by Mrs. Turan that (...) one of the integers (...) occurs only once”.) They proved further that $N_s(n) \ge (s^2-4s+3)n - C(s)$ ($s > 3$ is odd) and $N_s(n) \ge (s^2-5s+8)n - C(s)$ ($s \ge 4$ is even). Modifying these constructions, they obtained the bound $N_4(n) > 5n - c$.

Davenport's results. In the posthumously published paper [13] (edited by Schinzel), Davenport improved (5) to $N_4(n) \ll n\log n/\log\log n$. He noted that the ratio $N_4(n)/n$ must have a finite limit or it must go to $+\infty$, because $N_4(m+n) \ge N_5(m) + N_5(n)$ for every m and n (easy to see from the definition). He proved more specificly that

$$\lim_{n\to\infty} \frac{N_4(n)}{n} \geq 8.$$

Davenport's third result is the inequality $N_4(lm+1) \geq 6lm - m - 5l + 2$ ($l, m \in \mathbf{N}$), which was "found in collaboration with J. H. Conway". It implies that $N_4(n) \geq 5n - 8$, with the strict inequality for odd $n \geq 13$ and even $n \geq 18$. The note added in proof (apparently by Schinzel) says that Z. Kołba proved that $N_4(2m) \geq 11m - 13$.

The results of Roselle and Stanton. (Recall that $N_s(n) = \lambda_{s-1}(n)$.) Roselle and Stanton proved in [56] that $N_s(3) = 3s - 4$ (for even $s > 3$) and $N_s(3) = 3s - 5$ (for odd $s > 3$). In [49] they proved that $N_s(4) = 6s - 13$ (for even $s > 4$) and $N_s(4) = 6s - 14$ (for odd $s > 4$). Finally, in [48] they proved that $N_s(5) = 10s - 27$ (for even $s > 6$; the case $s = 6$ contains an error) and $N_s(5) = 10s - 29$ (for odd $s > 5$). In [48] also the bound $N_4(n) \geq 5n - 8$ is proved ($n \geq 4$). In [49] Roselle and Stanton gave the general bound ($s > n$)

$$N_s(n) \geq \left\{ \begin{array}{ll} \binom{n}{2}s - F(n) & s \text{ is even} \\ \\ \binom{n}{2}s - F(n) - \lfloor \frac{n-1}{2} \rfloor & s \text{ is odd} \end{array} \right\} \tag{7}$$

where $F(n) = (2n^3 + 9n^2 - 32n + 9)/12$ for odd $n \geq 3$ and $F(n) = (2n^3 + 9n^2 - 32n + 12)/12$ for even n. For $n = 3, 4$, and 5 these bounds are sharp.

If $n = o(s)$, the bounds (2) and (7) yield the asymptotics $N_s(n) = (1 + o(1))\binom{n}{2}s$. But what if n is bigger?

Problem 1. The bounds (2) and (7) give

$$\frac{n^3(1+o(1))}{3} < N_n(n) < \frac{n^3(1+o(1))}{2}.$$

What is the precise asymptotics of $N_n(n)$? □

Further results. Peterkin [44] determined by computer the value $N_5(6) = 29$ and found all 35 longest (normal) 6-DS sequences, corrected the value $N_6(5)$ of Roselle and Stanton to 34 (they had the incorrect value 33), and proved that $N_5(n) \geq 7n - 13$ ($n > 5$) and $N_6(n) \geq 13n - 32$ ($n > 5$).

Burkowski and Ecklund [12] found for small values of n, r, and d the maximum lengths of d-DS sequences over n symbols, in which no symbol appears more than r times. MR reviewer N. G. de Bruijn wrote on [12]: "... The following question was raised by D. J. Newman: Is there a word S in some $\Phi_{n,4}$ [5-DS sequences over n symbols] that contains each symbol at least 5 times? The authors give an affirmative answer (but the proof seems to be incomplete). ..."

Dobson and Macdonald [16] obtained a slight improvement of (7). We state one of their bounds: if n and r are even, then $N_{n+r}(n) \geq \frac{1}{6}(2n^3 + 3n^2(r-2) - 2n(3r-5) + 6r)$. For $n > 2r + 2$ this improves (7). Their other bounds are similar.

Rennie and Dobson [46] derived the inequality

$$\left(n - 2 + \frac{1}{s-3}\right) \cdot N_s(n) \leq n \cdot N_s(n-1) + \frac{2n - s + 2}{s-3}. \tag{8}$$

From it they could obtain good upper bounds on $N_s(n)$ for small values of s and n.

The next table, taken from Rennie and Dobson [46], gives specific bounds for $N_s(n)$ in the range $5 \leq s \leq 12$ and $5 \leq n \leq 12$. The upper bound is obtained from (8). The lower bound is the larger of the lower bounds given by Dobson and Macdonald or (shown in italic) by Roselle and Stanton in (7).

s	5	6	7	8
n				
5	22	34	41	53
6	29	46–47	56–58	72–76
7	36–37	59–62	72–77	96–102
8	43–46	72–78	89–99	120–131
9	50–56	85–96	106–123	145–164
10	57–66	98–115	123–149	170–200
11	64–77	111–136	140–177	195–239
12	71–89	124–158	157–207	220–281

s	9	10	11	12
n				
5	61	73	81	93
6	*85*–88	*102*–105	*115*–117	*132*–135
7	110–119	134–143	*152*–159	*176*–184
8	140–154	170–186	192–207	*223*–240
9	170–193	210–234	236–261	276–303
10	201–236	250–287	284–321	332–373
11	232–283	291–345	332–387	392–450
12	263–334	332–408	381–458	452–534

Mills [40] proved the inequalities $N_4(k^2+5-j) \geq 6k^2-2k+16-6j$ and $N_4(k^2+k+5-j) \geq 6k^2+4k+15-6j$, where $0 \leq j < k$. In [40] and [41] he determined the values of $N_4(n)$ for $n \leq 21$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$N_4(n)$	1	4	8	12	17	22	27	32	37	42	47	53	58	64

n	15	16	17	18	19	20	21
$N_4(n)$	69	75	81	86	92	98	104

The values of $N_4(n)$ form the sequence A002004 of the database [54]. The formula $N_4(n) = 5n-8$, valid for $4 \leq n \leq 11$, breaks down later, as noted already by Davenport.

Szemerédi's general bound. In 1974, Szemerédi [57] published a remarkable result with a difficult proof: for $n \to \infty$,

$$N_s(n) \ll_s n \log^* n. \tag{9}$$

Here $\log^* n$ is the smallest integer $k > 0$ such that $e_k > n$, where $e_1 = \mathrm{e} = 2.71828\ldots$ and $e_{i+1} = \mathrm{e}^{e_i}$. (Nothing changes if we replace e by any other base $b > 1$.) The key part of Szemerédi's proof is a decomposition lemma, which is based on the doubly exponential upper bound in a particular case of the classical Ramsey theorem (triples colored with two colors). The bound (9) improved considerably both (6) and (5).

Mills' article [41] and Stanton and Dirksen's survey [55], both published in 1976, mark the end of the early investigations of $N_s(n) = \lambda_{s-1}(n)$. DS sequences were dormant for the next 10 years.

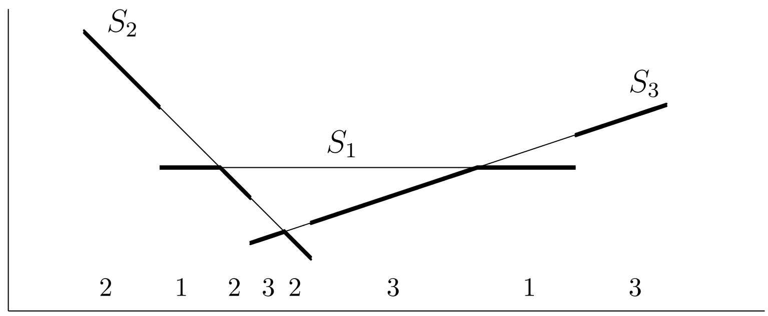


Figure 1: The lower envelope of plane segments.

3 Superlinear growth

New bounds on $\lambda_3(n)$: enigma solved. In the middle of 1980s, the importance of DS sequences for combinatorial and computational geometry was discovered, first by Atallah [7]. Or rather rediscovered, since the geometric motivation was in the background from the very beginning, only computational geometry did not exist in the times of [14]. The functions $\lambda_s(n)$, $s > 2$, remained mysterious. Despite the effort invested in the proofs of (6) and (9), the $O(n)$ upper bounds were not in sight and the correct orders of growth of $\lambda_s(n)$ were unclear. Stanton and Dirksen conjectured in [55] that $\lambda_3(n)/n \to \infty$.

Example 3 ([52]). We illustrate the role of DS sequences in combinatorial geometry by a classical example, which is very similar to the problem in [14] (we discussed it in the beginning of Section 2) but is more recent. Let $S_1, S_2, \ldots, S_n$ be n straight segments in the plane, none of them vertical and no two of them overlapping. We regard them as graphs of n real functions $f_1, \ldots, f_n$ which are now defined only on intervals. We consider the pointwise minimum function $f = \min_i f_i$. As before, f and f_i define a unique splitting of $\mathbf{R}$ into the intervals $I_1, I_2, \ldots, I_l$. The only difference is that now f is undefined on some of the intervals, certainly on I_1 and I_l. Again, for every $i = 1, 2, \ldots, l$ we write down the index j of the segment S_j that forms in I_i the graph of f; the intervals I_i with undefined f are ignored. We obtain a sequence u over $\{1, 2, \ldots, n\}$, $|u| \le l - 2$. In Figure 1, $n = 3$, $l = 10$, and $u = 21232313$. The graph of f is the *lower envelope* of the system $\{S_1, \ldots, S_n\}$, u is the *minimazing sequence*, and the length $|u|$ is the *complexity* of the lower envelope. The fact that every two (nonoverlapping) plane segments intersect in at most one point implies that u is a 5-DS sequence. Thus the complexity of the lower enevelope has the bound $|u| \le \lambda_3(n)$. $\square$

Another geometric connection was recently studied by Alon and Onn [6]. Consider a set X of n points lying on the moment curve in $\mathbf{R}^d$. The partitions of X into p parts with mutually disjoint convex hulls then correspond to the $(d+2)$-DS sequences (immediate repetitions are now allowed) which are over $\{1, 2, \ldots, p\}$ and are of length n. See also Aviran and Onn [8].

New light on $\lambda_3(n)$ was shed by Hart and Sharir in 1986 in their breakthrough article [21]. They proved that

$$n\alpha(n) \ll \lambda_3(n) \ll n\alpha(n), \tag{10}$$

where $\alpha(n)$ is the inverse Ackermann function, which is defined as follows. The Ackermann function $A(n)$ is the diagonal function $A(n) = F_n(n)$ of the hierarchy of functions $F_i : \mathbf{N} \to \mathbf{N}$, $i \in \mathbf{N}$, where $F_1(n) = 2n$ and $F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots))$ with n iterations of F_i. The inverse Ackermann function is then defined by $\alpha(n) = \min\{m \in \mathbf{N} : \; A(m) \ge n\}$. Alternative definitions of the hierarchy and of $A(n)$ can be found in the literature, but these hardly affect the values of $\alpha(n)$. The function $\alpha(n)$ grows to infinity much more slowly than $\log^* n$. (For further information on the role of very fast and very slow functions in combinatorics and computer science, see Loebl and Nešetřil [39].) The asymptotics (10) was not only an improvement upon (9) for $s = 4$, but it settled almost completely the 20 years old riddle of Davenport and Schinzel about the growth rate of $\lambda_3(n)$.

In [21], Hart and Sharir first translated 5-DS sequences to certain tree objects called (generalized path) compression schemes; these are motivated by data structures algorithms. They derived the new upper and lower bounds for the compression schemes, and then translated the bounds back to 5-DS sequences. Their proofs were inspired by some ideas and techniques of Tarjan [58] who pioneered applications of $\alpha(n)$ in computer science. This method gave bounds for both 5-DS sequences and compression schemes, but it was technically complicated. Soon it turned out that the translation is not really necessary and that one can work directly with DS sequences. This approach is adopted in all subsequent works. (For information on compression schemes and their relation to 5-DS sequences, see the book of Sharir and Agarwal [52].) Komjáth [37] proved the superlinear lower bound $\lambda_3(n) \gg n\alpha(n)$ by a construction purely in terms of sequences. Wiernik and Sharir [64] gave a simpler construction and, more importantly and remarkably, they proved that the 5-DS sequences produced by it can be realized as minimazing sequences of appropriate systems of plane segments. Thus there do exist systems of n plane segments whose lower envelopes have $\gg n\alpha(n)$ portions. We come to the natural but open

Problem 2. Can every 5-DS sequence be realized as the minimazing sequence of a system of plane segments? □

In [52], the authors express their opinion that the correct answer is negative. It is easy to realize every 5-DS sequence as the minimazing sequence of a system of *pseudosegments*. These are graphs of continuous functions defined on intervals, each two of them intersecting in at most one point.

Example 4 ([64, 52]). Following [52], we decribe the construction of [64] proving $\lambda_3(n) \gg n\alpha(n)$. One defines, by double induction, a two-dimensional array $S : \mathbf{N} \times \mathbf{N} \to A^*$ of sequences. Before giving the precise inductive definition, we have to say that the sequences $S(k,m)$ have no immediate repetition and are of the form

$$S(k,m) = u_1 v_1 u_2 v_2 \dots u_N v_N,$$

where every u_i is a sequence of length m containing m distinct symbols, and $v_1, \dots, v_N$ are possibly empty intermediate sequences. The sequences u_i are called *fans* or *m-fans* and v_i are called *separating sequences*. The key property of fans is this: every symbol of $S(k,m)$ appears in exactly one fan and this is its leftmost appearance in $S(k,m)$. The number $N = N(k,m)$ will be defined inductively in the construction. The sequences u_i and v_i depend on k and m as well, of course, but to avoid cumbersome notation we do not mark this dependence.

The first row $k = 1$ consists of the sequences $S(1,m) = u_1 = 12\dots m$, and $N(1,m) = 1$. If the row $k \ge 1$ is already defined, we define $S(k+1,1)$ to be identical with $S(k,2)$, except that every 2-fan in $S(k,2)$ is now regarded as two neighbouring 1-fans in $S(k+1,1)$. Thus $N(k+1,1) = 2N(k,2)$.

Let now the whole row $k \ge 1$ be already defined, as well as the sequences in the row $k+1$ up to the column $m \ge 1$. Let the same hold for the values of $N(x,y)$. We define $S(k+1,m+1)$ and $N(k+1,m+1)$. We denote $w_0 = S(k, N(k+1,m))$. We set $M = N(k, N(k+1,m))$ and create M copies $w_1, w_2, \dots, w_M$ of the sequence $S(k+1,m)$, renaming the symbols so that no two of the $M+1$ sequences $w_0, w_1, \dots, w_M$ share a symbol. We have as many copies of $S(k+1,m)$ as fans in w_0, and any fan in w_0 has as many elements as $S(k+1,m)$ has fans. By duplicating the last term in every fan in every $w_i, i=0,1,\ldots,M$, we create sequences w_i'. We set

$$S(k+1,m+1) = w_1'w_2'\ldots w_M' + w_0' = w_1^*z_1w_2^*z_2\ldots w_M^*z_M,$$

where the $+$ indicates the following interleaving of $w_1'w_2'\ldots w_M'$ and w_0', which preserves the order of terms in both sequences. The elements of the first $N(k+1,m)$-fan of w_0' are used to separate the twins on the ends of the $N(k+1,m)$ m-fans of w_1'; this yields w_1^*. The sequence z_1 consists of the last term of the first fan of w_0', followed by the first separating sequence of w_0'. In the same way we use the second fan of w_0' to separate the twins in w_2', which yields w_2^*, and so on. The resulting sequence $S(k+1,m+1)$ has no immediate repetition and its $(m+1)$-fans are the old m-fans in $w_1',\ldots,w_M'$, each enlarged by one term coming from the fans of w_0'. Thus

$$N(k+1,m+1) = N(k+1,m)\cdot N(k,N(k+1,m)).$$

One can easily check that the key property of fans is preserved during this step.

Note that $S(k,m)$ uses exactly $m\cdot N(k,m)$ symbols. Using the key property of fans, it is easy to show by double induction that every $S(k,m)$ is a 5-DS sequence. One can define, for details consult [64] or [52], a sequence of numbers $1\le m_1 < m_2 < \cdots$ such that, writing n_k for $\|S(k,m_k)\| = m_k\cdot N(k,m_k)$, the inequality $|S(k,m_k)|\ge n_k\alpha(n_k)-3n_k$ holds. (We owe the superlinear growth of $|S(k,m_k)|$ to the duplications.) Hence, for every $k\in\mathbf{N}$,

$$\lambda_3(n_k)\ge n_k\alpha(n_k)-3n_k. \tag{11}$$

A simple interpolation argument of [52] shows that

$$\lambda_3(n)\ge \tfrac{1}{2}n\alpha(n)-2n$$

holds for all $n\in\mathbf{N}$. □

Hart and Sharir [21] proved the lower bound in (10) with the constant $\frac{1}{4}+o(1)$. The constants achieved in the upper bound were $52+o(1)$ in [21] and $68+o(1)$ in [52]. (The objective of these works was not really to obtain the best constants.) Klazar [26] obtained the constant $4+o(1)$ and in [31] he proved that

$$\lambda_3(n) < 2n\alpha(n)+O(n\sqrt{\alpha(n)}). \tag{12}$$

Problem 3. Does the limit

$$\lim_{n\to\infty} \frac{\lambda_3(n)}{n\alpha(n)}$$

exist? $\square$

If it exists, (11) and (12) show that it lies in the interval $[1, 2]$.

We answer in positive the question from the MR review of Burkowski and Ecklund [12] that we quoted in the previous section. We know that $\lambda_3(n) > cn$ for large n for every constant $c > 0$. We deduce from it that for every $k \in \mathbf{N}$ there exists a 5-DS sequence v in which every symbol appears at least k times. Let $v \in \mathrm{DS}_5$ be such that $|v| \geq (k+1)\|v\|$ and $\|v\|$ is as small as possible. If some symbol $a \in S(v)$ occurs in v less than k times, we eliminate all a-occurrences by deleting at most $k - 1 + 2 = k + 1$ terms (as in the third proof in Example 2) and obtain a sequence $w \in \mathrm{DS}_5$ such that $|w| \geq (k+1)\|w\|$ and $\|w\| \leq \|v\| - 1$. But w contradicts the minimality of $\|v\|$. Therefore v has the stated property. Note that for $k = 5$ the sequence v must use at least 22 symbols, because Mills' table in Section 2 shows that $\lambda_3(n) < 5n$ for $n < 22$.

Bounds on $\lambda_{\mathbf{s}}(\mathbf{n})$ for $\mathbf{s} > 3$. The next obvious step was to extend the new techniques to $\lambda_s(n)$ for $s > 3$. Sharir in [50] proved the upper bound

$$\lambda_s(n) \ll n\alpha(n)^{c_s\alpha(n)^{s-3}}$$

and in [51] the lower bound

$$\lambda_{2s-1}(n) \gg_s n\alpha(n)^{s-1}.$$

Since $\lambda_{2s}(n) \geq \lambda_{2s-1}(n)$, this gives lower bounds for every $\lambda_s(n)$.

This line of research culminated in 1989 in the long and technical work of Agarwal, Sharir and Shor [4]. For $s = 4$ they proved the estimate

$$n2^{\alpha(n)} \ll \lambda_4(n) \ll n2^{\alpha(n)}. \tag{13}$$

For $s > 4$ they obtained strong bounds as well but they could not match completely the precision of (10) and (13). Their lower bound says that

$$\lambda_{2s}(n) \gg_s n2^{c_s\alpha(n)^{s-1}+Q_s(n)}, \tag{14}$$

where $c_s = 1/(s-1)!$ and $Q_s(n)$ is a polynomial in $\alpha(n)$ of degree at most $s-2$. As for the upper bound, they proved that

$$\lambda_{2s+1}(n) \le n2^{\alpha(n)^{s-1}\log(\alpha(n))+C_{2s+1}(n)} \quad \text{and} \quad \lambda_{2s}(n) \le n2^{\alpha(n)^{s-1}+C_{2s}(n)}, \qquad (15)$$

where $C_k(n)$ equals 6 and 11 for k equal to 3 and 4, respectively, $C_{2s+1}(n) = O(\alpha(n)^{s-1})$, and $C_{2s}(n) = O(\alpha(n)^{s-2}\log(\alpha(n)))$. We remark that in these bounds (and the whole [4]) $\log n$ denotes the *binary logarithm* with base 2, whereas in Example 2 and (5) we have the natural logarithm.

Let us summarize the current best bounds on $\lambda_s(n)$. Cases $s \le 1$ are trivial. The formula $\lambda_2(n) = 2n-1$ was proved by Davenport and Schinzel in [14], see Example 2. The functions $\lambda_3(n)$ and $\lambda_4(n)$ grow, up to multiplicative constants, as $n\alpha(n)$ and $n2^{\alpha(n)}$, respectively, as proved by Hart and Sharir [21] and Agarwal, Sharir and Shor [4]. The bounds (14) and (15) of [4] estimate $\lambda_s(n)$ for $s > 4$.

Problem 4. What are the exact speeds of growth of $\lambda_5(n)$ and $\lambda_6(n)$? And of the other $\lambda_s(n)$ for $s > 4$? □

By (14) and (15),

$$n2^{\alpha(n)} \ll \lambda_5(n) \ll n\alpha(n)^{(1+o(1))\alpha(n)}$$

and

$$n2^{(1+o(1))\alpha(n)^2/2} \ll \lambda_6(n) \ll n2^{(1+o(1))\alpha(n)^2}.$$

Bounds on $\lambda_s(n)$ found many applications in problems and algorithms of computational geometry. We suggest to the interested reader works [3] and [52] of Agarwal and Sharir for detailed information and many references. We remark that the "Web of Science" [65] listed in the middle of the year 2002 more than 110 citations of [21], which documents the big impact of this work.

4 A generalization of $\lambda_s(n)$ to any forbidden subsequence

A containment of sequences. The extremal function $\lambda_s(n)$ corresponds to the (forbidden) alternating sequence $ababa\ldots$ of length $s+2$. Now we associate with every sequence, not just with the alternating ones, an extremal function. For this we need to define a general containment of sequences.

Recall that our sequences are finite and are over A, where A is an infinite alphabet such that $A \supset \mathbf{N}$ and $a, b, c, d, \ldots$ lie in A. Recall that two sequences $u = a_1 a_2 \ldots a_l$ and $v = b_1 b_2 \ldots b_l$ of the same length are isomorphic, if for some injection $f : A \to A$ we have $a_i = f(b_i)$, $i = 1, 2, \ldots, l$. This is an equivalence relation and each class of isomorphic sequences contains exactly one normal sequence (see the definition before Example 1). We shall refer to elements of A by the leters $a, b, c, d, \ldots$ and to sequences over A by the letters $u, v, w, \ldots$.

Let u and v be two sequences. We say that u *contains* v and write $u \supset v$, if u has a subsequence isomorphic to v. For example, $u = a_1 a_2 \ldots a_l$ contains *abccba* if and only if there are six indices $1 \le i_1 < \cdots < i_6 \le l$ such that $a_{i_1} = a_{i_6}$, $a_{i_2} = a_{i_5}$, $a_{i_3} = a_{i_4}$, and these are the only equality relations among $a_{i_1}, \ldots, a_{i_6}$. The containment is a nonstrict partial order on classes of isomorphic sequences. If u does not contain v, we say that u is *v-free*.

The extremal function $\mathbf{Ex(v, n)}$. A sequence $u = a_1 a_2 \ldots a_l$ is called *k-sparse* if $a_i = a_j, i > j$, implies $i - j \ge k$. In other words, in every interval in u of length at most k all terms are distinct. For $k = 2$ we get the condition 1 from the definition of DS sequences. Recall that $|u|$ is the length of a sequence u and $\|u\|$ is the number of symbols used in u.

Let v be any sequence and $n \in \mathbf{N}$. We associate with v the extremal function

$$\mathrm{Ex}(v, n) = \max\{|u| : \ u \not\supset v \ \& \ u \text{ is } \|v\|\text{-sparse} \ \& \ \|u\| \le n\}. \tag{16}$$

It extends $\lambda_s(n)$: if al_s denotes the alternating sequence $abab\ldots$ of length s, then $\lambda_s(n) = \mathrm{Ex}(\mathrm{al}_{s+2}, n)$. The condition that u is $\|v\|$-sparse is necessary to ensure that $\mathrm{Ex}(v, n) < \infty$; note that $12\ldots k12\ldots k12\ldots$ is an infinite sequence that is k-sparse and contains no u with $\|u\| \ge k + 1$.

$\mathrm{Ex}(v, n)$ was introduced, albeit in a different notation, in 1992 by Adamec, Klazar and Valtr [1]. $\mathrm{Ex}(v, n)$ is always well defined because a modification of the argument proving (2) gives

$$\mathrm{Ex}(v, n) < \|v\| \cdot \left(\binom{n}{\|v\|}(|v| - 1) + 1\right) \ll_v n^{\|v\|}. \tag{17}$$

Before proceeding to further general properties of $\mathrm{Ex}(v, n)$, we derive two specific bounds to convey to the reader the flavour of arguments used to handle $\mathrm{Ex}(v, n)$, and we present a historical remark.

Example 5 ([26]). We determine $\mathrm{Ex}(abba, n)$ exactly and then prove a linear upper bound on $\mathrm{Ex}(a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k, n)$.

We prove that, for all $n \in \mathbf{N}$,

$$\mathrm{Ex}(abba, n) = 3n - 2$$

(cf. the next historical remark). The sequences

$$u = 1\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 4\ 5\ 4\ \ldots (n-2)\ (n-1)\ (n-2)\ (n-1)\ n\ (n-1)\ n$$

show that $\mathrm{Ex}(abba, n) \geq 3n-2$. We prove the opposite inequality. For $n = 1$ it is true. For $n > 1$ we use induction on n. Let $u = a_1 a_2 \ldots a_l$ be 2-sparse, $abba$-free, and $\|u\| \leq n$. Suppose first that some symbol $a \in S(u)$ appears in u at least four times. We select four indices $1 \leq i_1 < \cdots < i_4 \leq l$ such that $a_{i_1} = a_{i_2} = a_{i_3} = a_{i_4} = a$ and $a_j \neq a$ for $j \in [i_2 + 1, i_3 - 1]$. A moment of thought reveals that the symbol $b = a_{i_2+1}$ is distinct from a and appears in u only once. Deleting a_{i_2+1} and also a_{i_2} if $i_3 = i_2+2$, we decrease $\|u\|$ by one and obtain by induction the even stronger bound $|u| \leq 3(n-1)-2+2 = 3n-3$. Thus we may assume that every symbol appears in u at most three times, which gives $|u| \leq 3n$. If $a = a_1$ appears in u three times, we can still apply the deletion argument to $b = a_2$ and conclude that $|u| \leq 3n - 3$. The same if $a = a_l$ appears in u three times. If $a_1 = a_l = a$, only a may be repeated in u and $|u| \leq 2n - 1$. Thus we may assume in addition that $a_1 \neq a_l$ and both symbols a_1 and a_l appear in u at most twice. We conclude that $|u| \leq 3n-2$.

Let $v_k = a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k$ where $a_1, \ldots, a_k$ are k distinct symbols from A. We prove that

$$\mathrm{Ex}(v_k, n) \ll_k n.$$

Notice that $\mathrm{Ex}(v_1, n) = n$ and $\mathrm{Ex}(v_2, n) = \lambda_2(n) = 2n - 1$. Let $k, n \in \mathbf{N}$ and k be fixed. We set $K = (k-1)^4 + 1$ and $L = \mathrm{Ex}(v_k, K-1) + 1$. The number L exists by the rough general bound (17). Let u be a k-sparse sequence ($\|v_k\| = k$) with $\|u\| \leq n$. We assume that $|u| \geq (2n+1)L$ and show that it implies $u \supset v_k$. We split u into $2n+1$ disjoint intervals, each of length at least L. One of these intervals, let us call it I, contains neither the first nor the last appearance of any symbol $a \in S(u)$ because these are only at most $2n$ in number. If $\|I\| < K$, the definitions of L and $|I|$ imply $I \supset v_k$ and we are done. So $\|I\| = |S(I)| \geq K$. By the property of I, every $a \in S(I)$ appears before I, in I, and after I. Applying twice the classical Erdős–Szekeres lemma, which says that every sequence of $(k-1)^2+1$ numbers contains a monotone subsequence of length k, we see that there is a subset $Y \subset S(I)$ of k elements, $Y = \{y_1, y_2, \ldots, y_k\}$, such that $y_1, y_2, \ldots, y_k$ appear before I in this order, in I in this or in the opposite oder, and after I also in this or in the opposite order. But then two of the three orders agree, which gives $u \supset v_k$. Thus $|u| \geq (2n+1)L$ always forces the containment $u \supset v_k$. We conclude that $\mathrm{Ex}(v_k, n) < (2n+1)L$. $\square$

A historical remark. The author of this survey wondered from time to time why for so long (until the apperance of [1][1]) nobody tried to generalize $\lambda_s(n)$ and everybody followed so faithfully the original formulation of the problem that forbids only alternating subsequences. The extension (16) of (1) is relatively straightforward and Example 5 shows that with a modest effort one can obtain for $\mathrm{Ex}(v, n)$ results of some interest. On the other hand, to prove (6) or (10), say, is difficult. Of course, retrospect views are often dubious. We will not delve into psychological speculations but we want to present a little historical discovery.

Surprisingly, revisionists appeared already in the very beginning and it was nobody else but Davenport and Schinzel who in 1965, besides the famous [14], published also [15] hinting on a generalization of $\lambda_s(n)$. The latter forgotten note is missing in all bibliographies of DS sequences we know of ([3], [20, problem E20], [26], [52], [55], ...) and probably is not refered to anywhere. It is accesible in Davenport [9] where we found it. Davenport and Schinzel derive in [15] an inequality on lentghs of subsequences of a 2-sparse sequence. In the last paragraph they say:

> The inequality is of some interest in connection with sequences which, in addition to having no immediate repetition, satisfy some prescribed "hereditary" conditions, that is, some condition which if valid for a sequence is necessarily valid for every subsequence. Take as an illustration the condition that the sequence contains no subsequence
>
> $$\ldots,\ a, \ldots,\ b, \ldots,\ b, \ldots,\ a,\ \ldots\ (b \neq a)\ .$$
>
> Then the length of any such a sequence is at most $2n(n-1)$; for we can apply (1) [they refer to the inequality] with $m = 2$, in which case $M \leq 4$. (Actually in this particular case the maximum length is $3n - 2$.)

[1] I learned about DS sequences in the fall of 1988 in the Prague combinatorial seminar that was then led by J. Nešetřil and J. Matoušek. They suggested to us, a group of undergraduate students of Charles University, to investigate generalizations of $\lambda_s(n)$. This eventually resulted in [1] and some other works.

Nobody followed the hint then.

An almost linear bound on $\mathbf{Ex(v,n)}$. As for strengthening of (17), Klazar [23] showed by a simple combinatorial argument that $\mathrm{Ex}(v,n) \ll_v n^2$. The main result of [23] says that if v is a sequence with $\|v\| = k \ge 2$ and $|v| = l \ge 5$, then for every $n \in \mathbf{N}$

$$\mathrm{Ex}(v,n) \le n \cdot k2^{l-3} \cdot (10k)^{2\alpha(n)^{l-4}+8\alpha(n)^{l-5}}. \tag{18}$$

If $k = 1$ or $l \le 4$, it is easy to show that $\mathrm{Ex}(v,n) \ll_v n$. This general bound was derived by adapting the techniques from Sharir [50].

A slight generalization: $\mathbf{Ex(v,k,n)}$. One can investigate the more general extremal function $\mathrm{Ex}(v,k,n)$, which is defined as the maximum length of a v-free and k-sparse sequence u with $\|u\| \le n$. It is clear that $\mathrm{Ex}(v,\|v\|,n) = \mathrm{Ex}(v,n)$ and $\mathrm{Ex}(v,k,n) = \infty$ whenever $k < \|v\|$ and $n \ge k$ — the infinite sequence $12\ldots k12\ldots k12\ldots$ is k-sparse and does not contain v. Thus one has to have $k \ge \|v\|$. For the asymptotics it brings nothing new because, as proved in [1],

$$\mathrm{Ex}(v,n) \ll_{v,k} \mathrm{Ex}(v,k,n) \le \mathrm{Ex}(v,n) \tag{19}$$

for every sequence v and every $k \ge \|v\|$; the latter inequality is trivial. As for the precise values, in Klazar [28] it was proved that for $n \ge k \ge 2$,

$$\mathrm{Ex}(abab,k,n) = 2n - k + 1 \ \text{ and } \ \mathrm{Ex}(abba,k,n) = 2n + \left\lfloor \frac{n-1}{k-1} \right\rfloor - 1.$$

For $n \le k-1$ both functions equal to n.

As one expects, the asymptotic order of $\mathrm{Ex}(v,n)$ respects the containment order of sequences:

$$u \subset v \Rightarrow \mathrm{Ex}(u,n) \ll_{u,v} \mathrm{Ex}(v,n). \tag{20}$$

For $\|u\| = \|v\|$ this is triviality, since then even $\mathrm{Ex}(u,n) \le \mathrm{Ex}(v,n)$. For $\|u\| < \|v\|$ this follows immediately from the first bound in (19), and $\ll$ cannot in general be replaced with $\le$.

Blow-ups. If $a \in A$ and $i \in \mathbf{N}$, we write a^i for the sequence $aa\ldots a$ of i a's. We call u a *chain* if $\|u\| = |u|$, that is, u has no repetition. Obviously, $\mathrm{Ex}(a^i,n) = (i-1)n$ for any $i \in \mathbf{N}$ and $\mathrm{Ex}(v,n) = \min(|v|-1,n)$ for any chain v.

In the extremal theory of sequences we want to determine, for as many sequences as possible, the extremal functions or at least their orders of growth. At present our knowledge is still very fragmentary. The most succesful approach so far turned out to be the following one. We start with some sequences $v_1, \ldots, v_r$ and combine them, by a specific operation, into a new sequence w. If certain conditions are satisfied, we can bound $\mathrm{Ex}(w, n)$ in terms of the functions $\mathrm{Ex}(v_i, n)$. We have found three such operations; two are unary ($r = 1$) and one is binary ($r = 2$). They have two common features: the resulting w always contains all v_i and if $\mathrm{Ex}(v_i, n) \ll n$ for all $i = 1, \ldots, r$ then $\mathrm{Ex}(w, n) \ll n$ as well. We begin with one of the unary operations, the blow-up.

Every sequence u expresses uniquely in the form $u = a_1^{i_1} a_2^{i_2} \ldots a_r^{i_r}$, where $a_j \in A$, $a_j \neq a_{j+1}$, and $i_j \in \mathbf{N}$. The *blow-up* of u is any sequence isomorphic to $a_1^{k_1} a_2^{k_2} \ldots a_r^{k_r}$, where $k_1 \geq i_1$, $k_2 \geq i_2, \ldots, k_r \geq i_r$. For example, 1221111 and $a^3 b^3 a = aaabbba$ are blow-ups of $\mathrm{al}_3 = aba$.

If v is not a chain, then $\mathrm{Ex}(v, n) \geq |1\ 2\ \ldots\ n| = n$. On the other hand, we mentioned above that for every chain u the function $\mathrm{Ex}(u, n)$ is eventually constant. Thus the extremal function $\mathrm{Ex}(v, n)$ of every proper blow-up v of a chain u grows substantially faster than $\mathrm{Ex}(u, n)$. There are reasons to believe that, except of this trivial situation, blowing up a sequence cannot change the asymptotics of its extremal function:

Problem 5. Prove (or disprove) that if u is not a chain and v is a blow-up of u, then

$$\mathrm{Ex}(v, n) \ll_{u,v} \mathrm{Ex}(u, n).$$

□

The lower bound $\mathrm{Ex}(v, n) \geq \mathrm{Ex}(u, n)$ is trivial.

Adamec, Klazar and Valtr [1] proved that the bound in Problem 5 often holds. Namely, if a is a symbol and u, v sequences, then

$$\mathrm{Ex}(a^2 u, n) - \mathrm{Ex}(au, n) \ll_{au} n \ \text{ and } \ \mathrm{Ex}(ua^3 v, n) \ll_{ua^2 v} \mathrm{Ex}(ua^2 v, n). \quad (21)$$

By symmetry, also $\mathrm{Ex}(ua^2, n) - \mathrm{Ex}(ua, n) \ll_{ua} n$. Let $v = a_1^{k_1} a_2^{k_2} \ldots a_r^{k_r}$ be any blow-up of $u = a_1^{i_1} a_2^{i_2} \ldots a_r^{i_r}$ such that $k_j > i_j$ implies $i_j \geq 2$ or $j = 1$ or $j = r$. Applying the bounds (21), it is easy to see that then $\mathrm{Ex}(v, n) \ll \mathrm{Ex}(u, n)$. Both extremal functions are then of the same asymptotic order. For example, $a^4 bc^5 ab^2$ and $abc^2 ab$ have extremal functions with the same asymptotic order.

In Problem 5, it remains to settle the case when uav is not a chain and is blown-up to ua^2v (here u and v are nonempty sequences, u does not end with a, and v does not begin with a). This operation is not covered by the results (21) and whether it changes asymptotics is unknown. Surprisingly, this does happen for certain tree generalizations of $\mathrm{Ex}(v, n)$, as shown by Valtr in [61]. This is a contrary evidence showing that perhaps blow-ups may change asymptotics.

Example 6 ([1]). We prove the first bounds of (19) and (21). We begin with (19). Let v be a fixed sequence, $k \ge \|v\|$ be a number, and u be a $\|v\|$-sparse and v-free sequence. It suffices to show that u has a k-sparse subsequence w such that $|w| \gg_{v,k} |u|$. We set w to be the longest k-sparse subsequence of u. Let I be any of the intervals in u that are disjoint to w and are maximal to this property. Suppose, for a while, that $\|I\| \ge 2k-1$. Then there must be an $a \in S(I)$ that differs from the $k-1$ terms of w preceding I and also from the $k-1$ terms of w following after I. Such an a could be added to w, in contradiction with the maximality of w. Hence $\|I\| \le 2k-2$ for every I. Thus $|I| \le \mathrm{Ex}(v, 2k-2)$ for every I and $|w| \ge 1 + |u|/(1 + \mathrm{Ex}(v, 2k-2))$. This finishes the proof of the first bound in (19).

Now we prove that $\mathrm{Ex}(a^2u, n) = \mathrm{Ex}(au, n) + O(n)$, where the constant in O depends on the sequence au. Let $k = \|a^2u\|$ and v be a k-sparse and a^2u-free sequence with $\|v\| \le n$. First, we show that there is a constant $c > 0$ depending only on a^2u and with the following property. For every term x of v there is a k-sparse subsequence w of v avoiding x and of length $|w| \ge |v| - c$. In other words, x plus some other $O(1)$ terms of v can be deleted so that the k-sparseness is preserved. To prove it, we fix an arbitrary interval I in v containing x and of length $|I| = \mathrm{Ex}(a^2u, 3k-3) + 1$. So $\|I\| \ge 3k-2$ and there must be a subset $Y \subset S(I)$, $|Y| = k-1$, such that every $y \in Y$ is distinct from x, from the $k-1$ terms preceding I, and from the $k-1$ terms following after I. We fix in I one appearance for each $y \in Y$ and we delete from v the rest of I. The resulting sequence w is clearly k-sparse, x was deleted, and $|w| \ge |v| - (\mathrm{Ex}(a^2u, 3k-3) + 2 - k)$. We can set $c = \mathrm{Ex}(a^2u, 3k-3) + 2 - k$.

In this way we delete from v, one by one, the first appearances of all $x \in S(v)$. At most cn elements are deleted and the resulting subsequence w is k-sparse. Clearly, $w \not\supset au$ because otherwise $v \supset a^2u$ would be forced. Thus $|v| \le |w| + cn \le \mathrm{Ex}(au, n) + cn$ and $\mathrm{Ex}(a^2u, n) \le \mathrm{Ex}(au, n) + O(n)$. Trivially, $\mathrm{Ex}(a^2u, n) \ge \mathrm{Ex}(au, n)$. $\square$

The two-letter theorem. If u has at most two symbols, then either $u \supset ababa$ and $\mathrm{Ex}(u,n) \ge \lambda_3(n) \gg n\alpha(n)$, or u is contained in a blow-up of $abab$. The main result of [1] says that for the latter u's we have $\mathrm{Ex}(u,n) \ll_u n$. We obtain the characterization ([1, Theorem 5]) that is in [61] called the *two-letter theorem*: If $\|u\| \le 2$ then

$$\mathrm{Ex}(u,n) \ll_u n \iff u \not\supset ababa. \tag{22}$$

It follows from (20) and from the discussion after Problem 5 that the proof of (22) (given $\lambda_3(n) \gg n\alpha(n)$) reduces to proving that $\mathrm{Ex}(ab^2a^2b,n) \ll n$. By (21), this implies $\mathrm{Ex}(a^ib^ja^kb^l,n) \ll_m n$, where $m = \max(i,j,k,l)$, for every choice of the exponents $i,j,k,l \in \mathbf{N}_0$.

More results and errors on blow-ups. In [36, p. 467] the first author wrote: "However, it may be checked that the method [of Hart and Sharir] $(\ldots)$ works for $a^ib^ia^ib^ia^i$ as well and so $Ex(a^ib^ia^ib^ia^i,n) = \Theta(n\alpha(n))$.". However, after some time he realized that no matter how hard he tried he could not recollect the proof anymore and therefore we have the following problem.

Problem 6. Prove (or disprove) that $\mathrm{Ex}(v,n) \ll_v n\alpha(n)$ for every blow-up v of $ababa$. That is, prove (or disprove) that

$$\mathrm{Ex}(ab^2a^2b^2a,n) \ll n\alpha(n).$$

$\Box$

This is a special case of Problem 5. We are not done with the forbidden 5-term alternating subsequence yet! The applications of the conjectural bound $\mathrm{Ex}(a^ib^ia^ib^ia^i,n) \ll_i n\alpha(n)$ in [36, 3.2] must be considered as unproved.

A simpler proof of the two-letter theorem was given in Klazar [27]. In particular, he proved that

$$7n - 9 \le \mathrm{Ex}(ab^2a^2b,n) \le 8n - 7$$

and $\mathrm{Ex}(a^ib^ia^ib^i,n) < (1+o(1))32i^2 \cdot n$. The method of [27] was extended in Klazar [25] to the blow-ups of the sequences v_k of Example 5. In [25] he proved that, for $i \in \mathbf{N}$ and k distinct symbols $a_1,a_2,\ldots,a_k$,

$$\mathrm{Ex}(a_1^ia_2^i\ldots a_k^ia_1^ia_2^i\ldots a_k^i,n) \ll_{i,k} n. \tag{23}$$

Two general questions on the growth of $\mathrm{\mathbf{Ex}}(\mathbf{v},\mathbf{n})$. Is the equivalence (22) valid for sequences using more than two symbols? Klazar and Valtr realized that the answer was "no". A specific example is given in Klazar [24]: $u_1 = abcbadadbcd \not\succ ababa$ but $\mathrm{Ex}(u_1, n) \gg n\alpha(n)$. A more extremal example in this direction is given in [31]:

$$u_2 = abcbadadbecfcfedef \not\succ ababa \text{ but } \mathrm{Ex}(u_2, n) \gg n2^{\alpha(n)}.$$

Thus the containment of $\mathrm{al}_5 = ababa$ is not the sole cause of superlinearity. These examples are obtained by showing that the constructions of [64] and [4] produce sequences avoiding not only *ababa* and *ababab*, respectively, but even u_1 and u_2. Some new construction might help to solve the following problem.

Problem 7. We conjecture that for every constant $c > 0$ there exists a sequence u such that

$$u \not\succ ababa \text{ but } \mathrm{Ex}(u, n) \gg n2^{\alpha(n)^c}.$$

□

In other words, in view of (18), we conjecture that every extremal function $\mathrm{Ex}(v, n)$ is majorized (for $n > n_0$) by some $\mathrm{Ex}(u, n), u \not\succ ababa$. Below we will see that $u \not\succ abab$ implies $\mathrm{Ex}(u, n) \ll_u n$.

Another attractive but at present hopeless question is whether $n\alpha(n)$ is the smallest superlinear growth of extremal functions.

Problem 8. Is it true that for every sequence u either $\mathrm{Ex}(u, n) \ll n$ or $\mathrm{Ex}(u, n) \gg n\alpha(n)$? □

Is there an extremal function whose restrictions to two infinite subsequences of $\mathbf{N}$ have different orders of growth? Said more explicitly, we ask if there exist a sequence u, two infinite subsequences $1 \le n_1 < n_2 < \ldots$ and $1 \le m_1 < m_2 < \ldots$ of $\mathbf{N}$, and two increasing functions $f, g : \mathbf{N} \to \mathbf{R}^+$ with $f(n)/g(n) \to \infty$, such that $f(n_i) \ll \mathrm{Ex}(u, n_i) \ll f(n_i)$ and $g(m_i) \ll \mathrm{Ex}(u, m_i) \ll g(m_i)$ as $i \to \infty$. Valtr [61, Proposition 3] shows that for every sequence u,

$$\limsup_{n\to\infty} \frac{\mathrm{Ex}(u, n)}{n} = \infty \Longrightarrow \liminf_{n\to\infty} \frac{\mathrm{Ex}(u, n)}{n} = \infty.$$

So in any case one cannot select $g(n) = n$. If u is irreducible, which means that there is no nontrivial decomposition $u = u_1 u_2$ with $S(u_1) \cap S(u_2) = \emptyset$, then it is easy to prove the stronger result that $\lim_{n\to\infty} \mathrm{Ex}(u, n)/n$ exists (it may equal ∞).

Insertions and intertwinings. We proceed to the remaining two operations which compose w from $v_1, \ldots, v_r$ so that $\mathrm{Ex}(w, n)$ can be bounded in terms of $\mathrm{Ex}(v_1, n), \ldots, \mathrm{Ex}(v_r, n)$. They constitute the main result of Klazar and Valtr [36].

Let a and b be two distinct symbols and u_1, u_2, and v be three sequences. The *insertion* of v in $u = u_1 a^2 u_2$ gives the sequence $w = u(v) = u_1 a v a u_2$. The *intertwining* of $u = u_1 a^2 u_2 a$ and b gives the sequence $w = u[b] = u_1 a b^2 a u_2 a b$. We can bound $\mathrm{Ex}(u(v), n)$ only under the assumption that $S(u_1 a^2 u_2) \cap S(v) = \emptyset$. Similarly, we can bound $\mathrm{Ex}(u[b], n)$ only under the assumption that $b \notin S(u_1 a^2 u_2 a)$. The insertion is a binary operation in the sense that it is determined completely by two sequences u, v and an immediate repetition $\ldots aa \ldots$ in u. In the similar sense the intertwining is an unary operation.

Let $u = u_1 a^2 u_2$ and v be two sequences with $S(u) \cap S(v) = \emptyset$, and $u(v)$ arise by inserting v in u. If v is a chain, it is easy to see that $\mathrm{Ex}(u(v), n) \le \mathrm{Ex}(u, n)$. If v is not a chain then, as proved in [36],

$$\mathrm{Ex}(u(v), n) = \mathrm{Ex}(u_1 a v a u_2, n) \ll_{u,v} \mathrm{Ex}(v, \mathrm{Ex}(u, n)). \tag{24}$$

Let $u = u_1 a^2 u_2 a$ be a sequence, b be a symbol such that $b \notin S(u)$, and $u[b]$ arise by intertwining u with b. Then, as proved in [36],

$$\mathrm{Ex}(u[b], n) = \mathrm{Ex}(u_1 a b^2 a u_2 a b, n) \ll_u \mathrm{Ex}(u, n). \tag{25}$$

By symmetry, we have the analogous statement for $b a u_1 a b^2 a u_2$. The lower bounds

$$\mathrm{Ex}(u(v), n) \gg_{u,v} \max(\mathrm{Ex}(u, n), \mathrm{Ex}(v, n)) \quad \text{and} \quad \mathrm{Ex}(u[b], n) \gg_u \mathrm{Ex}(u, n)$$

are immediate from (20). Simpler proofs of (24) and (25) were given in Valtr [61]. Blowing-up $u[b]$, we obtain from (25) for $i, j \in \mathbf{N}$ the bounds

$$\mathrm{Ex}(u_1 a b^i a u_2 a b^j, n) \ll_{u,i,j} \mathrm{Ex}(u, n)$$

(remember the condition $b \notin S(u)$).

Linear sequences. It is natural (but difficult) to investigate the class of the *linear sequences*

$$\mathrm{Lin} = \{v : \ \mathrm{Ex}(v, n) \ll n\}.$$

We have already met several members of it: $abab$ (Example 2), $abba$ and v_k (Example 5), a^i and chains (trivial). On the other hand, $ababa \not\in \mathrm{Lin}$ by (10).

Problem 9. What are the elements of Lin? □

The bounds (21), (24), and (25) give a simple method to obtain many members of Lin: start with a^i and by repeated blow-ups, insertions, and intertwinings generate linear sequences. To formulate it more precisely, let L be the smallest class of sequences that is closed on the isomorphism and has the following closure properties: (i) for every $i \in \mathbf{N}$, $a^i \in L$; (ii) if $au \in L$ then $a^i u \in L$ for every $i \in \mathbf{N}$, and if $ua \in L$ then $ua^i \in L$ for every $i \in \mathbf{N}$; (iii) if $ua^2v \in L$ then $ua^iv \in L$ for every $i \in \mathbf{N}$; (iv) if $u = u_1a^2u_2, v \in L$ and $S(u) \cap S(v) = \emptyset$ then $u_1avau_2 \in L$; and (v) if $u = u_1a^2u_2a \in L$ and $b \not\in S(u)$ then $u_1ab^2au_2ab \in L$, and if $u = au_1a^2u_2 \in L$ and $b \not\in S(u)$ then $bau_1ab^2au_2 \in L$. Then ([36]),

$$L \subset \mathrm{Lin}.$$

Observe that, by the way of definition, L is closed to the containment and to all blow-ups.

Problem 10. Decide if there is a sequence $u \in \mathrm{Lin} \backslash L$. □

By (20), $u \subset v \in \mathrm{Lin}$ implies that $u \in \mathrm{Lin}$. Hence we can characterize Lin by means of the class of minimal nonlinear sequences

$$B = \{u : \ u \not\in \mathrm{Lin} \text{ but } v \in \mathrm{Lin} \text{ whenever } v \subset u \ \& \ |v| < |u|\}.$$

Namely, we have the equivalence $u \in \mathrm{Lin} \iff \forall v \in B \ [v \not\subset u]$. Knowing (effectively) B, we could hope to draw more information about Lin. Unfortunately, at present we know only two significant properties of B: (i) $ababa \in B$ and (ii) B has at least two elements. The first property follows from the facts that $\lambda_3(n) \gg n\alpha(n)$ (by (10)), $abab \in \mathrm{Lin}$ (Example 2), $ab^2a \in \mathrm{Lin}$ (Example 5), and $aba^2 \in \mathrm{Lin}$ (trivial). As for (ii), it follows from the examples given after the bound (23). Might B be infinite?

Problem 11. Is the set B of minimal nonlinear sequences finite or infinite? $\Box$

One might hope to prove the finiteness of the set B, which is an antichain to the containment, by proving that the quasiordering $(A^*, \subset)$ has no infinite antichains at all. However, one can easily construct infinite antichains in $(A^*, \subset)$ and thus B still might be infinite. An infinite antichain is presented by Klazar [24] who also determines certain well quasiordered subsets of A^*.

Interesting sequences in the class L. Let us return to the class L of the sequences whose linearity we can prove. It is rich enough to contain several interesting families of sequences. First we prove by induction on $|u|$ that every $abab$-free sequence u falls in L. If u is $abab$-free then u is isomorphic to $au_1au_2a\dots au_k$, where $u_1, \dots, u_k$ are possibly empty sequences, the sets $S(u_i)$ are mutually disjoint, and $a \notin S(u_i)$ for every $i = 1, \dots, k$. By induction, $u_i \in L$ for every $i = 1, \dots, k$. Inserting in a^{k+1} the sequences $u_1, \dots, u_k$, we conclude (applying k times the closure property (iv)) that $u \in L$. Thus $\mathrm{Ex}(u, n) \ll_u n$ for every $abab$-free sequence u. In particular,

$$\mathrm{DS}_4 \subset \mathrm{Lin}.$$

Recall that $u_1 = abcbadadbcd \notin \mathrm{Lin}$ and $u_1 \not\supset ababa$. Thus $\mathrm{DS}_5 \not\subset \mathrm{Lin}$.

Intertwinings and blow-ups yield an immediate proof of the two-letter theorem: $a^3 \in L$ by (i), $ab^2a^2b \in L$ by (v), and $a^ib^ia^ib^i \in L$ ($i \geq 2$) by (ii) and (iii). Hence every $a^{i_1}b^{i_2}a^{i_3}b^{i_4}$ ($i_j \geq 0$) is linear.

Intertwinings bring in L sequences more complicated than $abab$-free sequences. Repeated intertwinings and blow-ups show that the sequence

$$u' = 1\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 4\ 5\ 4\ \dots (k-2)\ (k-1)\ (k-2)\ (k-1)\ k\ (k-1)\ k$$

of Example 5 lies in L. Hence this longest $abba$-free 2-sparse sequence is linear for every k, as well as its every blow-up. On the other hand, $abcadbcd \not\supset abba$ but $abcadbcd \notin L$. At present we are not able to prove the linearity of all $abba$-free sequences.

Similarly, intertwinings and blow ups show that for every k and i the N-shaped sequence

$$u_N(k,i) = 1^i2^i \dots (k-1)^ik^i(k-1)^i \dots 2^i1^i2^i \dots (k-1)^ik^i$$

belongs to L and thus is linear. The bound

$$\mathrm{Ex}(u_N(k,i), n) \ll_{i,k} n \tag{26}$$

is an important result of [36] and a considerable strengthening of (23).

It follows by a case analysis that every *ababa*-free sequence u with $\|u\| \le 3$ is contained in a blow-up of one of the three sequences $v_1 = ababcbc$, $v_2 = abcbabc$, and $v_3 = abacacbc$. All blow-ups of v_1 and v_2 are in L and thus are linear; v_1 is the $k = 3$ instance of the above u' and $v_2 = u_N(3,1)$. But $v_3 = abacacbc \notin L$.

Problem 12. Is it true that $\mathrm{Ex}(abacacbc, n) \ll n$? And what about the blow-ups of *abacacbc*? □

In fact, the minimal subsequences of $v_3 = abacacbc$ lying outside L are *abacabc*, *abcacbc*, *abacacb*, and *bacacbc*. The first two are, up to the isomorphism, reversals of one another and hence have equal extremal functions. The same holds for the last two sequences. Writing $\overline{u}$ for the reversal of u, we have this partial characterization of the linear sequences over three symbols: for $\|u\| \le 3$,

$$u \text{ and } \overline{u} \text{ contain none of } \{ababa, abacabc, abacacb\} \Rightarrow \mathrm{Ex}(u,n) \ll_u n.$$

In the opposite direction we know only that $ababa \subset u$ implies the nonlinearity of u.

5 Geometric graphs, colored trees, 0-1 matrices, ordered bipartite graphs, permutations, and set partitions

Geometric graphs. Generalized DS sequences found interesting applications in the combinatorics of *geometric graphs*. These are particular planar realizations of graphs: the vertices of a graph are represented by some points in the plane lying in the general position and the edges are represented by possibly crossing straight segments. Two edges of a geometric graph *cross* if their relative interiors intersect, and they are *parallel* if they form two opposite sides of a convex quadrilateral.

Katchalski and Last [22] proved, using the bound (4), that any geometric graph with n vertices and no two parallel edges has at most $2n - 1$ edges. Valtr [60] lowered this bound to $2n - 2$, which proves the conjecture of Y. S. Kupitz from 1979; Figure 2 shows geometric graphs attaining this number of

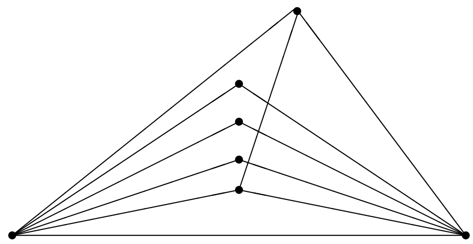


Figure 2: Geometric graphs with n vertices and $2n-2$ edges, no two of them "parallel".

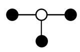
edges. (Another nice application of (4) in combinatorial geometry was given by Edelsbrunner and Sharir in the article [17] with a self-explaining title.) Applying the N-sequence bound (26), Valtr proved in [60] more generally that every geometric graph with n vertices and no k pairwise parallel edges has $\ll_k n$ edges. From this he derived that if k pairwise crossing edges are forbidden, then the number of edges is $\ll_k n \log n$. (For these results see also [61].) This improves the previous bound $\ll_k n \log^{2k-6} n$ (for $k > 2$) of Agarwal et al. [2] (based on the bound $\ll_k n \log^{2k-4} n$ of Pach, Shahrokhi and Szegedy [43]). Whether the bound $\ll_k n$ holds is open. It is known to hold only for $k = 2$ (the classical case of planar graphs) and $k = 3$ ([2]).

Precisely speaking, the bounds $\ll_k n \log^{2k-6} n$ of [2] and $\ll_k n \log n$ of [60] are in a sense incomparable. The former bound holds, as stated in [2], for the more general representation of edges by curves (details are in [2] given only for the case of segments). The latter stronger bound applies on the more special representation by segments. In [59] Valtr extended it to the situation when edges are represented by x-monotone curves, but the case of general curves seems out of reach of his method.

For further applications of DS sequences in computational and combinatorial geometry, see [3] and [52].

Colored trees. From the viewpoint of graph theory, sequences can be regarded as undirected colored paths, where colors are the symbols used. For example, *abcabc* is the path of six vertices $v_1 v_2 \dots v_6$ where v_1 and v_4 are colored a, v_2 and v_5 are colored b, and v_3 and v_6 are colored c. To work with more exciting objects, we regard colored paths just as special cases of colored trees. Can one extend in a reasonable way the definition (16) to colored trees? Can one prove for colored trees an analogue of, say, $\lambda_2(n) = \mathrm{Ex}(abab, n) = 2n - 1$? We begin with the latter problem.

Let $T(abab, n)$ be the maximum number of vertices in a tree $T = (V, E)$ that can be vertex-colored by at most n colors so that three conditions hold:

1. The coloring is proper, which means that no edge is monochromatic.
2. No subgraph of the colored tree T is a subdivision of the properly 2-colored 4-vertex path .
3. No subgraph of the colored tree T is a subdivision of the properly 2-colored 4-vertex star  .

The condition 1 is the analogy of 2-sparseness. The condition 2 forbids in T the color pattern $abab$ and it requires, for the coloring $f : V \to \mathbf{N}$, that there are no four distinct vertices $v_1, \ldots, v_4 \in V$ such that $f(v_1) = f(v_3) \neq f(v_2) = f(v_4)$ and the v_1-v_4 path contains, in this order, the vertices v_2 and v_3. The condition 3 requires that there are no three distinct vertices $v_1, \ldots, v_3 \in V$ such that $f(v_1) = f(v_2) = f(v_3) \neq f(v_4)$ and the three v_i-v_4 paths are disjoint except for v_4. For colored trees the conditions 1 and 2 alone do not suffice to bound the number of vertices, as shown by arbitrarily large properly colored stars. Therefore we add the condition 3. To reformulate it, define for a color c and a colored tree T the tree $T(c)$ as the smallest subtree of T containing all c-colored vertices. The condition 3 then says that, for every color c, in the tree $T(c)$ all vertices with degrees at least 3 must be colored c. Notice that if we restrict T to paths, then $T(abab, n)$ coincides with $\mathrm{Ex}(abab, n)$. This is due to the fact that the sequence $abab$ is isomorphic to its reversal.

Example 7 ([29]). We prove that for every $n \in \mathbf{N}$,

$$T(abab, n) = 2n - 1. \tag{27}$$

So if T ranges over the larger set of all trees, $T(abab, n)$ still equals $\mathrm{Ex}(abab, n) = \lambda_2(n) = 2n - 1$.

The lower bound $T(abab, n) \geq 2n-1$ is achieved already on colored paths. More strongly, we can color any tree U on $2n-1$ vertices with n colors so that the conditions 1, 2, and 3 are satisfied: we color with 1 two arbitrary leaves of U, then we color with 2 two arbitrary leaves of the uncolored subtree of U, and so on until the whole U is colored.

We prove the opposite inequality $T(abab, n) \leq 2n - 1$. Let $T = (V, E)$ be a tree and $f : V \to \{1, 2, \ldots, n\}$ be a coloring satisfying the conditions 1, 2, and 3. We show that $|V| \leq 2n - 1$. We may assume that $n > 1$ and that $b(T) = \#\{v \in V : \deg_T(v) \geq 3\} > 0$; else T is a path and $|V| \leq \mathrm{Ex}(abab, n) = 2n - 1$. Formally, we proceed by induction on the sum $n + b(T)$. We use the operation of *smoothing out* a vertex u of degree 1 or 2. For $\deg_T(u) = 1$ this is just the deletion of u. For $\deg_T(u) = 2$ we delete u and connect its two neighbours by an edge.

Observe that there is a vertex v in T with degree at least 3 and such that $T - v = P_1 \cup P_2 \cup \ldots \cup P_l \cup C$ where $l \geq 2$, every P_i is a path, and C is a tree which may not be a path. First we show that f may be assumed to be injective on the set $V(T) \backslash V(C) = \{v\} \cup V(P_1) \cup \ldots \cup V(P_l)$. If this is not the case, then $f(u_1) = f(u_2)$ for two distinct $u_1, u_2 \in V(T) \backslash V(C)$, and either (i) $\{u_1, u_2\} \subset \{v\} \cup V(P_i)$ for some i or (ii) $u_1 \in V(P_i)$ and $u_2 \in V(P_j)$ for some $i \neq j$. In the case (i), there must be a vertex u_3 between u_1 and u_2 whose color $c = f(u_3)$ does not appear elsewhere in T; else we would have in T the color pattern *abab*. Smoothing out u_3 and, if necessary, one of its neighbours (lest we create a monochromatic edge), we get rid of the color c and keep the three conditions satisfied. By induction, $|V| \leq 2(n - 1) - 1 + 2 = 2n - 1$. In the case (ii) we may assume that the color $c = f(u_1) = f(u_2)$ does not appear elsewhere in T; else we have the case (i) or the condition 3 is violated. If u_1 has two neighbours, they have distinct colors for else we would have the *abab* pattern. The same holds for u_2. We get rid of c by smoothing out both u_1 and u_2; this creates no monochromatic edge. The three conditions are satisfied and we conclude again by induction that $|V| \leq 2(n - 1) - 1 + 2 = 2n - 1$.

Thus we may assume that on the vertices in $V(T) \backslash V(C)$ no color is repeated. We transform the colored tree (T, f) into a new colored tree (T^*, f^*) by splitting the paths $P_1, \ldots, P_l$ into individual vertices, assembling from them a single colored path P, and joining P back to v. During the transformation every vertex keeps its color. The number of colors has not changed, $b(T^*) = b(T) - 1$ because $\deg_{T^*}(v) = 2$, and (T^*, f^*) clearly satisfies the conditions 1 and 3. It remains to find an appropriate order for the vertices in P so that (T^*, f^*) does not contain the color pattern *abab*. Then we apply the inductive assumption and conclude that $|V(T)| = |V(T^*)| \leq 2n - 1$.

To this end we define a binary relation R on the set of colors appearing in

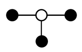
$V(P_1)\cup\ldots\cup V(P_l)$. We set aRb iff $a \neq b$ and there is a path $Q = (v_0, \ldots, v_k)$ in $C \cup \{v\}$, $v_0 = v$, such that, for some $i < j$, $f(v_i) = a$ and $f(v_j) = b$. We show that R is a strict partial ordering. First we prove that R is antisymmetric. Suppose, for the contradiction, that aRb and bRa, witnessed by paths Q_1 and Q_2, respectively. Let w be the merging vertex of Q_1 and Q_2. One case is that a and b appear on Q_1 in this order after w (if we go in the v-w direction), and the same holds for the appearances of b and a on Q_2. Since both colors appear also in the paths P_i, we have a contradiction with the condition 3: w should have both colors a and b. If this case does not occur (which includes the possibility $Q_1 = Q_2$), then Q_1 or Q_2 must contain the pattern aba or bab. But then (T, f) contains the pattern $abab$, which contradicts the condition 2. Thus R is antisymmetric. The transitivity of R can be proved by very similar arguments which we omit.

R is a strict partial ordering. Any occurrence of the pattern $abab$ in (T^*, f^*) would have to use two vertices of $C \cup \{v\}$ and two vertices of P. We order the vertices in P in a linear extension of R so that if aRb then the vertex in P colored a is closer to v than the vertex colored b. Then no $abab$ pattern can appear. $\square$

For a general sequence $u \in A^*$ with $\|u\| > 1$, we define $T(u, n)$ as the maximum number of vertices of a tree T that can be vertex-colored by at most n colors so that three conditions hold:

1. Two distinct vertices with the same color have distance at least $\|u\|$ edges.

2. No subgraph of the colored tree T is a subdivision of the path of $|u|$ vertices that is colored according to u.

3. No subgraph of the colored tree T is a subdivision of the properly 2-colored 4-vertex star  .

We keep the condition 3 and modify the first two conditions in the obvious way. For $\|u\| > 1$ this works fine, $T(u, n) < \infty$. For $u = a^i$ the first condition is void and for $i \geq 4$ we would still have $T(a^i, n) = \infty$ (consider monochromatic stars). Therefore in the special case of $u = a^i$ we require the coloring of T to be proper. We must not forget that after replacing sequences by paths we lose the unique left-right order of terms. Forbidding a sequence

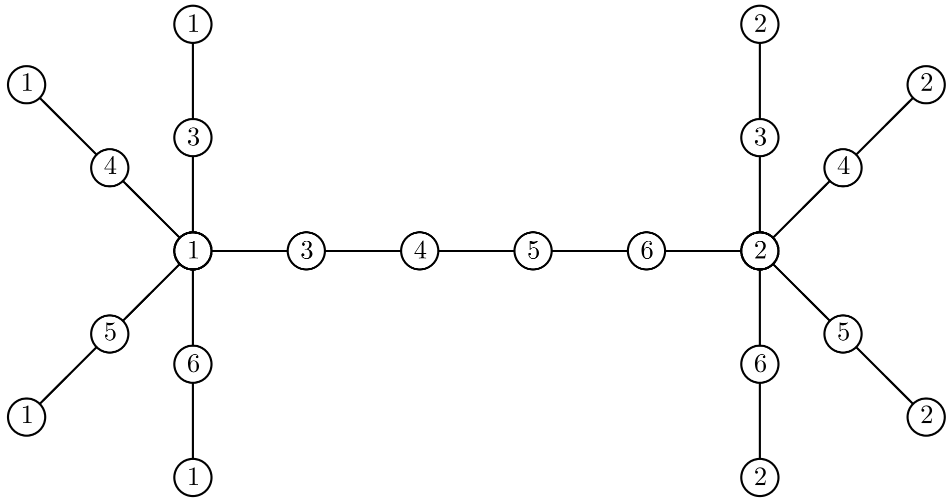


Figure 3: The bound $T(abba,n) \geq 5n-8$ for $n=6$.

u we forbid its reversal $\overline{u}$ as well. For u isomorphic to $\overline{u}$, if T are restricted to paths then $T(u,n) = \mathrm{Ex}(u,n)$. We say that then $T(u,n)$ *extends* $\mathrm{Ex}(u,n)$.

This generalization of $\mathrm{Ex}(u,n)$ to colored trees, $T(u,n)$, was considered first in [26]. From the results on $T(u,n)$ proved in Klazar [30] we mention the extension of (18) to colored trees and the exact values ($n > 1$)

$$T(a^i,n) = \begin{cases} (2i-3)n-2i+4 & \dots \quad i \geq 2 \text{ is even} \\ \\ (2i-4)n-2i+6 & \dots \quad i \geq 3 \text{ is odd} \end{cases}$$

(recall that for the monochromatic i-path a^i the coloring of T is required to be proper). The next example shows that, unlike $abab$, $T(abba,n) \neq \mathrm{Ex}(abba,n)$.

Example 8 ([62]). We show that $T(abba,n) \geq 5n-8$, in contrast with $\mathrm{Ex}(abba,n) = 3n-2$. (Since $abba$ is isomorphic to its reversal, $T(abba,n)$ extends $\mathrm{Ex}(abba,n)$.) We hope that the construction of (T,f) due to P. Valtr and independently discovered also by Ch. Vogt, is clear enough from its instance $n=6$ visualized in Figure 3.

The coloring is proper, contains no subdivision of $abba$, and satisfies the condition 3. In general the horizontal path has n vertices and the rays contribute $4(n-2)$ vertices. Together we have $5n-8$ vertices. In Valtr [62] a more general example is given showing that

$$T(a^i b^i a^i, n) \geq 9in - O(i + n).$$

□

In [26] and [29] we posed as a problem to show that $T(abba, n) \ll n$. This was accomplished in [62] by Valtr who proved more generally that

$$T(a^i b^i a^i, n) \leq 24in.$$

Thus $5n - 8 \leq T(abba, n) \leq 48n$.

Problem 13. Improve these bounds. Or, better, determine the function $T(abba, n)$ exactly. □

The theory of tree extremal functions was much advanced by Valtr in [61] and [63]. He introduced other generalizations of $\mathrm{Ex}(v, n)$ to colored trees, which are better to work with than $T(v, n)$, and he proved analogues of most of the results that we described in the previous section. In particular, he extended blow-ups, insertions, and intertwinings to colored trees. To discuss properly his results would mean to write another Section 4; the interested reader is refered for details to [61] and [63]. Here I only say that, contrary to my original expectations, the behaviour of tree extremal functions often turns out to be much different compared to $\mathrm{Ex}(v, n)$. For example, the two-letter theorem for colored trees ([61], [63]) says: for $\|u\| \leq 2$,

$$T(u, n) \ll_u n \iff u \not\succ ababa \ \&\ u \not\succ ab^2a^2b.$$

In fact ([61], [63]), $T(ab^2a^2b, n) \gg n\alpha(n)$. Comparing this with (27), we see that for colored trees blow-ups change asymptotics.

0-1 matrices and ordered bipartite graphs. Recall the notation $[n] = \{1, 2, \ldots, n\}$ and $[a, b] = \{a, a+1, \ldots, b\}$. Füredi and Hajnal [18] investigated the following class of extremal problems for 0-1 matrices. Let $N : [k] \times [l] \to \{0, 1\}$ and $M : [m] \times [n] \to \{0, 1\}$ be two 0-1 matrices of types $k \times l$ and $m \times n$, respectively. We say that M contains N if there are increasing injections $f : [k] \to [m]$ and $g : [l] \to [n]$ such that, for all $i \in [k]$ and $j \in [l]$, $M(f(i), g(j)) = 1$ whenever $N(i, j) = 1$. In other words, M has a (not necessarily contiguous) $k \times l$ submatrix that has 1 on every position where N has 1, and that has 0 or 1 on every position where N has 0. It is convenient to write in the matrices blanks instead of zeros. Let $f(m,n;N)$ be the maximum number of 1's in an $m \times n$ 0-1 matrix M not containing N, and let $f(n;N) = f(n,n;N)$. It is easy to see that for every N with at most three 1's one has $f(n;N) \ll n$. For four 1's the situation is much more complicated. Performing on N the obvious automorphisms preserving $f(n;N)$, one is left with 37 matrices N with four 1's and no zero row or column. Füredi and Hajnal investigated $f(n;N)$ for each of these N. One of their results related to DS sequences says that

$$n\alpha(n) \ll f(n; \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \end{pmatrix}) \ll n\alpha(n);$$

the upper bound is obtained from (10) by reduction to 5-DS sequences and for the lower bound they give a construction of their own. They prove the same lower and upper bounds for

$$N = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & 1 \end{pmatrix}$$

and the upper bound $f(n;N) \ll n\alpha(n)$ for

$$N = \begin{pmatrix} & 1 & & \\ & & & 1 \\ 1 & & 1 & \end{pmatrix} \text{ and } N = \begin{pmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{pmatrix}.$$

In the end of [18] the authors pose a question whether $f(n;P) \ll_P n$ holds for all permutation matrices P. In [34] we pointed out that this conjecture, if true, would imply the enumerative Stanley–Wilf conjecture (which we formulate in a moment).

One can naturally reformulate the matrix setting in terms of ordered bipartite graphs; these are bipartite graphs with linear orders on both parts. M is understood as the bipartite graph $\mathcal{G} = ([n], [n+1, n+m], H)$ where, for all $i \in [m]$ and $j \in [n+1, n+m]$, $\{i, j\} \in H$ iff $M(i, j-n) = 1$. The matrix containment translates into the usual subgraph relation, with the important additional condition that the linear orders of vertices are preserved. The extremal function $f(n;\mathcal{H})$ is defined as the maximum number $|H|$ of the edges of a bipartite graph $\mathcal{G} = ([n], [n+1, 2n], H)$ such that $\mathcal{G}$ does not contain $\mathcal{H}$.

For a permutation $p = a_1 a_2 \dots a_k$ of $[k]$, the permutation bipartite graph $\mathcal{G}_p$ is defined as

$$\mathcal{G}_p = ([k], [k+1, 2k], \{\{i, k + a_i\} : \ i = 1, 2, \dots, k\}).$$

The question of Füredi and Hajnal asks if, for any fixed permutation p, every $\mathcal{G} = ([n], [n+1, 2n], H)$ not containing $\mathcal{G}_p$ as an ordered subgraph must have $\ll n$ edges.

Permutations. Alon and Friedgut [5] applied generalized DS sequences to a problem in enumerative combinatorics. Let $p = a_1 a_2 \dots a_m$ and $q = b_1 b_2 \dots b_n$ be permutations of $[m]$ and $[n]$, respectively. We say that q contains p if q has a subsequence $b_{i_1} b_{i_2} \dots b_{i_m}$, $1 \le i_1 < i_2 < \dots < i_m \le n$, such that $b_{i_r} < b_{i_s} \iff a_r < a_s$ for every r and s. Else we say that q *avoids* p. Let $S_n(p)$ be the number of permutations of $[n]$ avoiding p. The *Stanley–Wilf conjecture* (stated, for example, in Bóna [10]) asserts that for any given permutation p,

$$S_n(p) < c^n \tag{28}$$

holds for every $n \in \mathbf{N}$ and a constant $c > 1$ depending only on p. Using the general bound (18), Alon and Friedgut proved for every p the upper bound

$$S_n(p) < \beta_p(n)^n, \tag{29}$$

where $\beta_p(n)$ is an extremely slowly growing function defined in terms of $\alpha(n)$. Using the N-sequence bound (26), they proved also that (28) holds for all unimodal p. (Recall that $p = a_1 a_2 \dots a_m$ is unimodal if it first decreases and then increases or vice versa.) Bóna [11] proved that (28) holds for all permutations p of the form $p = a_1 a_2 \dots a_m = s_1 s_2 \dots s_k$ where the s_i's are decreasing sequences and, for every i, all terms of s_i are smaller than those of s_{i+1}.

Example 9 ([34]). We give a simpler proof of the bound (29). We translate the problem from permutations to ordered bipartite graphs $\mathcal{G}$ and work with the above described extremal function $f(n; \mathcal{G})$ and the permutation graphs $\mathcal{G}_p$. Let, for a permutation p, $G_n(p)$ be the number of all ordered bipartite graphs $\mathcal{G} = ([n], [n+1, 2n], H)$ such that $\mathcal{G} \not\supset \mathcal{G}_p$. Clearly,

$$S_n(p) \le G_n(p)$$

because now we count many more p-free objects. Let us suppose that we have a bound $f(n; \mathcal{G}_p) < n\gamma(n)$, where $\gamma(n) = \gamma_p(n)$ is a nondecreasing function.

Let $n \in \mathbf{N}$ be fixed and $m = \lceil n/2 \rceil$. We claim that

$$G_n(p) < 15^{m\gamma(m)} \cdot G_m(p).$$

To prove this inductive inequality, we transform every $\mathcal{G} = ([n], [n+1, 2n], H)$ counted by $G_n(p)$ to a $\mathcal{G}' = ([m], [m+1, 2m], H')$ counted by $G_m(p)$. For $i \in [m]$ and $j \in [m+1, 2m]$, we let $\{i, j\} \in H'$ iff in $\mathcal{G}$ the sets $\{2i-1, 2i\}$ and $\{2j-1, 2j\}$ are connected by at least one edge. In other words, to get $\mathcal{G}'$, we identify in $\mathcal{G}$ the vertices in each of the pairs $(1, 2), (3, 4), \ldots$ and $(n+1, n+2), (n+3, n+4), \ldots$ and replace the arising multiple edges by simple edges. Every edge of $\mathcal{G}'$ can be obtained in at most $2^{2 \cdot 2} - 1 = 15$ ways. Thus if $\mathcal{G}'$ has e edges, there are at most 15^e graphs $\mathcal{G}$ that transform to $\mathcal{G}'$. Also, $\mathcal{G} \not\supset \mathcal{G}_p$ implies $\mathcal{G}' \not\supset \mathcal{G}_p$, and therefore $e \leq f(m; \mathcal{G}_p) < m\gamma(m)$. Hence we obtain the inductive inequality. Iterating it until $m = 1$ and denoting $m_0 = n, m_i = \lceil m_{i-1}/2 \rceil$, we obtain the bound

$$S_n(p) \leq G_n(p) < 2 \cdot 15^{\sum_{i \geq 1} m_i \gamma(m_i)} < 2 \cdot 15^{\gamma(n) \sum_{i \geq 1} m_i} < 15^{2n\gamma(n)} = \left(225^{\gamma(n)}\right)^n.$$

We have obtained (29) with $\beta_p(n) = 225^{\gamma_p(n)}$. If $\gamma_p(n)$ is almost constant, so is $\beta_p(n)$. To find such a $\gamma_p(n)$, we reduce bipartite graphs to sequences. We associate with every $\mathcal{G} = ([n], [n+1, 2n], H)$ the sequence $u = N_1 N_2 \ldots N_n \in [n+1, 2n]^*$, where N_i is the (arbitrarily ordered) list of the neighbours of $i \in [n]$ in $\mathcal{G}$. Let p be a fixed permutation of $[k]$. It follows that $\mathcal{G} \not\supset \mathcal{G}_p$ implies $u \not\succ w$ where $w = a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k \ldots a_1 a_2 \ldots a_k$ consists of $2k$ repetitions of the segment of k distinct symbols $a_1, \ldots, a_k$. The problem that u may not be k-sparse is easily fixed: deleting at most $k-1$ terms from the beginnings of $N_2, N_3, \ldots, N_n$, we obtain a subsequence v of u, $|v| \geq |u| - (k-1)(n-1)$, which is k-sparse. Invoking (18), we obtain the required bound:

$$\begin{aligned} |H| &= |u| \leq (k-1)(n-1) + |v| < kn + \mathrm{Ex}(w, n) \\ &< kn + n \cdot k 2^{l-3} \cdot (10k)^{2\alpha(n)^{l-4} + 8\alpha(n)^{l-5}} =: n\gamma_p(n), \end{aligned}$$

where $l = 2k^2$. □

Set partitions. From the viewpoint of algebraic combinatorics, sequences can be regarded as set partitions. For example, *abcabc* is the partition of [6] in the blocks $\{1, 4\}$, $\{2, 5\}$, and $\{3, 6\}$. Generally, we assign to a sequence $u = a_1 a_2 \ldots a_l$ of length l the partition P of $[l]$ such that i and j are in the same block of P iff $a_i = a_j$. Then the set of blocks of mutually isomorphic sequences of length l corresponds bijectively to the set of all partitions of $[l]$. Representing partitions by equivalence relations $\sim$, we can formulate the containment of sequences $\subset$ in the following way. A partition $u = ([k], \sim_u)$ is contained in another partition $v = ([l], \sim_v)$, if there is an *increasing* injection $f : [k] \to [l]$ such that the equivalence $x \sim_u y \iff f(x) \sim_v f(y)$ holds for every $x, y \in [k]$.

An important and interesting class of partitions is the *noncrossing partitions*. A partition P of $[l]$ is noncrossing, if there are no four numbers $1 \le x_1 < x_2 < x_3 < x_4 \le l$ and no two distinct blocks B_1 and B_2 of P such that $x_1, x_3 \in B_1$ and $x_2, x_4 \in B_2$. More briefly, P is noncrossing if it does not contain $abab = \{\{1, 3\}, \{2, 4\}\}$. Noncrossing partitions and *abab*-free sequences (4-DS sequences with 2-sparseness dropped) are two ways of looking at the same thing. Simion [53] wrote an interesting survey of results on noncrossing partitions and related topics. The seminal work introducing noncrossing partitions was that of Kreweras [38], followed shortly by Poupard [45]. In the same year 1972, Mullin and Stanton published independently their article [42] on enumeration of 4-DS sequences. Other enumerative works on 4-DS sequences are Roselle [47], Gardy and Gouyou-Beauchamps [19], and Klazar [28]. See also Klazar [32] for a more general approach to the enumeration of u-free set partitions.

Although one can read in the MR review of [31] (with the main result (12)) that "The author improves previous results to show that the number $N_5(n)$ $[\lambda_3(n)]$ of finite sequences ...", unfortunately, to my knowledge, no significant enumerative results on 5-DS sequences are known. Some should be discovered! In this connection it is interesting that Alon and Onn [6] applied in an enumerative problem (of bounding the numbers of separable partitions of points on the moment curve) the extremal bounds (2) and (7).

Problem 14. Let r_l be the number of *ababa*-free partitions of $[l]$. In other words,

$$r_l = \#\{u : \ u \not\supset ababa \ \& \ |u| = l \ \& \ u \text{ is normal}\}.$$

What can be said about the numbers r_l? What is their asymptotics? □

The numbers r_l grow superexponentially. To see it, note that no partition of $[l]$ into blocks of at most two elements contains $ababa = \{\{1, 3, 5\}, \{2, 4\}\}$.

Thus

$$r_l \geq \sum_{i \geq 0} \binom{l}{2i} \cdot (2i-1)!!$$

where $(2i-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2i-1)$ and $(-1)!! = 1$.

Having in mind the variety of enumerative formulas for noncrossing partitions, we state the following problem.

Problem 15. Let s_l be the number of *abcabc*-free partitions of $[l]$. In other words,

$$s_l = \#\{u : \ u \not\succ abcabc \ \& \ |u| = l \ \& \ u \text{ is normal}\}.$$

What can be said about the numbers s_l? What is their asymptotics? $\square$

The hypergraph bound (31) (proved in [33]) covers the partition case and implies that $s_l < c^l$ for a constant $c > 1$. It is easy to see that $s_{k+l} \geq s_k s_l$ for every $k, l \in \mathbf{N}$. Hence $s_l^{1/l}$ tends to a finite limit.

6 Hypergraphs

A hypergraph containment. The containment of sequences discussed through this survey is a special case of the hypergraph containment introduced in Klazar [33]. In Section 6 we survey some of the results of [33] and [35].

A *hypergraph* $\mathcal{H} = (E_i : \ i \in I)$ is a finite list of finite nonempty subsets E_i of $\mathbf{N} = \{1, 2, \ldots\}$, which are called *edges*. The edges may be repeated (we allow $E_i = E_j$ for $i \neq j$). *Simple* hypergraphs have no repeated edges. The elements of $\bigcup \mathcal{H} = \bigcup_{i \in I} E_i \subset \mathbf{N}$ are called *vertices*. Hypergraphs include partitions as a special case: partitions are the hypergraphs with mutually disjoint edges. Now, for partitions we have the important notion of the sequential containment. Could not it be "lifted" to hypergraphs? We propose the following definition ([33, 35]).

A hypergraph $\mathcal{H}' = (E'_i : \ i \in I')$ *is contained* in another hypergraph $\mathcal{H} = (E_i : \ i \in I)$, in symbols $\mathcal{H}' \prec \mathcal{H}$, if there is an *increasing* injection $F : \bigcup \mathcal{H}' \to \bigcup \mathcal{H}$ and an injection $f : I' \to I$ such that the implication

$$x \in E'_i \Longrightarrow F(x) \in E_{f(i)}$$

holds for every vertex $x \in \bigcup \mathcal{H}'$ and every index $i \in I'$. If $\mathcal{H}' \not\subset \mathcal{H}$, we say also that $\mathcal{H}$ is *$\mathcal{H}'$-free*. If $\mathcal{H}'$ and $\mathcal{H}$ are partitions, the hypergraph containment coincides with the sequential containment. To help the reader to get used to the former, we give two examples. Let $\mathcal{H}_1 = (E_1, E_2)$, $E_1 = E_2 = \{1\}$, be the hypergraph consisting of the singleton edge $\{1\}$ repeated twice. Then $\mathcal{H}$ is $\mathcal{H}_1$-free iff $\mathcal{H}$ is a partition. Let $\mathcal{H}_2 = (\{1,3\}, \{2,4\})$. Then $\mathcal{H} = (E_i :\ i \in I) \supset \mathcal{H}_2$ iff there are four vertices $x_1, \ldots, x_4 \in \bigcup \mathcal{H}$, $x_1 < x_2 < x_3 < x_4$, and two (not necessarily distinct) edges E_i, E_j in $\mathcal{H}$, $i \neq j$, such that $x_1, x_3 \in E_i$ and $x_2, x_4 \in E_j$. ($\mathcal{H}_2$-free hypergraphs generalize noncrossing partitions.)

Let $\mathcal{F}$ be a fixed hypergraph. One can ask two extremely natural questions. First, how many $\mathcal{F}$-free hypergraphs are there. Second, how large $\mathcal{F}$-free hypergraphs may be. We discuss first the enumerative aspect and then in more details the extremal aspect.

Exponential and almost exponential bounds. For a fixed hypergraph $\mathcal{F}$ and $n \in \mathbf{N}$, we are interested in the number

$$a(\mathcal{F}, n) = \#\{\mathcal{H} :\ \mathcal{H} \text{ is simple } \& \bigcup \mathcal{H} = [n] \ \&\ \mathcal{H} \not\supset \mathcal{F}\}.$$

The simplicity of $\mathcal{H}$ is needed to make $a(\mathcal{F}, n)$ finite. The forbidden hypergraph $\mathcal{F}$, however, may be arbitrary, not necessarily simple. (If $\mathcal{F} = \mathcal{H}_1 = (\{1\}, \{1\})$ from the above example, $a(\mathcal{F}, n)$ counts the partitions of $[n]$ and equals to the nth Bell number.) One of the basic problems here is to determine all hypergraphs $\mathcal{F}$ for which $a(\mathcal{F}, n) < c^n$ for a constant $c > 1$. The candidates for such $\mathcal{F}$ are the *permutation hypergraphs* $\mathcal{H}_p$ (with added singleton edges); these are slightly modified graphs $\mathcal{G}_p$. For a permutation $p = a_1 a_2 \ldots a_k$ of $[k]$ we define

$$\mathcal{H}_p = (\{i, k + a_i\} :\ i = 1, \ldots, k).$$

For example, $\mathcal{H}_{1,3,2} = (\{1,4\}, \{2,6\}, \{3,5\})$. If a hypergraph $\mathcal{F}$ either (i) has an edge with at least three elements or (ii) has two intersecting edges or (iii) has two two-element edges E_1 and E_2 such that $E_1 < E_2$, then every permutation hypergraph $\mathcal{H}_p$ is $\mathcal{F}$-free. Thus for such an $\mathcal{F}$ we have

$$a(\mathcal{F}, n) \geq (\lfloor n/2 \rfloor)! = \exp((\tfrac{1}{2} + o(1)) n \log n)$$

and the numbers $a(\mathcal{F}, n)$ grow superexponentially. It is clear that $\mathcal{F}$ satisfies neither of (i)–(iii) if and only if it is a disjoint union of several singleton edges and a hypergraph isomorphic to some $\mathcal{H}_p$. We say briefly that $\mathcal{F}$ is of the form $\mathcal{H}_p$+*singletons*. For example, we may have

$$\mathcal{F} = (\{1\}, \{3\}, \{7\}, \{2, 9\}, \{4, 6, \}, \{5, 8\}).$$

We conjecture that $a(\mathcal{F}, n) < c^n$ if and only if, for some permutation p, $\mathcal{F} = \mathcal{H}_p$+singletons. This strengthens the Stanley–Wilf conjecture.

To prove our conjecture, it is enough to prove $a(\mathcal{H}_q, n) < c^n$ for every permutation q; every $\mathcal{H}_p$+singletons is contained in an appropriate $\mathcal{H}_q$.

Problem 16. Prove (or disprove) that for any given permutation p,

$$a(\mathcal{H}_p, n) < c^n$$

holds for every $n \in \mathbf{N}$ and a constant $c > 1$. □

In [33] we proved, using (18), a slightly weaker bound: for every permutation p there is an almost constant function $\beta_p(n)$ defined in terms of the inverse Ackermann function $\alpha(n)$, such that

$$a(\mathcal{H}_p, n) < \beta_p(n)^n. \tag{30}$$

Note that this strengthens (29) (and our strengthening of (29) in Example 9) because now we count many more p-free objects. In [33] we also proved, using the N-sequence bound (26), that the exponential bound in Problem 16 holds for certain permutations: if $p = a_1a_2 \ldots a_k$ first dicreases and then increases, or if p^{-1} first increases and then dicreases, then

$$a(\mathcal{H}_p, n) < c^n \tag{31}$$

for all $n \in \mathbf{N}$ and a constant $c > 1$. Note that this gives an exponential upper bound on the numbers s_l of Problem 15 ($p = 1, 2, 3$). Can the reader give a reasonably simple direct proof of it?

Summarizing, we have the enumerative alternative

$$a(\mathcal{F}, n) \left\{ \begin{array}{lll} < \beta_p(n)^n & \ldots & \mathcal{F} \subset \mathcal{H}_p \text{ for some } p \\ & & \\ > n^{(1/2+o(1))n} & \ldots & \mathcal{F} \not\subset \mathcal{H}_p \text{ for every } p, \end{array} \right\} \tag{32}$$

we conjecture that $\beta_p(n)$ may be replaced with a constant, and can prove this for some particular permutations p.

Two hypergraph extremal functions. For a hypergraph $\mathcal{H} = (E_i : i \in I)$, we denote by $v(\mathcal{H}) = |\bigcup \mathcal{H}|$ the number of vertices, by $e(\mathcal{H}) = |I|$ the number of edges, and by $i(\mathcal{H}) = \sum_{i \in I} |E_i|$ the number of vertex-edge incidences. For a fixed hypergraph $\mathcal{F}$ and $n \in \mathbf{N}$, we define two extremal functions

$$\begin{aligned} H_e(\mathcal{F}, n) &= \max\{e(\mathcal{H}) : \mathcal{H} \not\supset \mathcal{F} \text{ \& } \mathcal{H} \text{ is simple \& } v(\mathcal{H}) \leq n\} \\ H_i(\mathcal{F}, n) &= \max\{i(\mathcal{H}) : \mathcal{H} \not\supset \mathcal{F} \text{ \& } \mathcal{H} \text{ is simple \& } v(\mathcal{H}) \leq n\}. \end{aligned}$$

The simplicity of $\mathcal{H}$ is again needed that $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ be well defined. The forbidden $\mathcal{F}$ may be arbitrary. Obviously, $H_e(\mathcal{F}, n) \leq H_i(\mathcal{F}, n)$ for every $\mathcal{F}$ and n. In most cases we have, up to a constant factor, also the opposite inequality:

Example 10 ([35]). Suppose that no two edges E_1 and E_2 of $\mathcal{F}$ satisfy $E_1 < E_2$. Let $p = v(\mathcal{F})$ and $q = e(\mathcal{F}) > 1$ (the case $q = 1$ is trivial). Then for every $n \in \mathbf{N}$,

$$H_i(\mathcal{F}, n) \leq (2p - 1)(q - 1) \cdot H_e(\mathcal{F}, n). \tag{33}$$

For the proof suppose that $\mathcal{H}$ is a simple and $\mathcal{F}$-free hypergraph with $v(\mathcal{H}) \leq n$. We transform $\mathcal{H}$ in a new hypergraph $\mathcal{H}'$. If $E = \{v_1, v_2, \ldots, v_s\}$ is an edge of $\mathcal{H}$, $v_1 < v_2 < \cdots < v_s$, we keep it if $s < p$. If $s \geq p$, we replace E with $t = \lfloor |E|/p \rfloor$ new edges $\{v_1, \ldots, v_p\}$, $\{v_{p+1}, \ldots, v_{2p}\}, \ldots,$ $\{v_{(t-1)p+1}, \ldots, v_{tp}\}$. The new edges have each p elements and are mutually separated in the way that is excluded in $\mathcal{F}$. The new hypergraph $\mathcal{H}'$ may not be simple. Therefore we define a simple hypergraph $\mathcal{H}''$ by keeping from every family of repeated edges of $\mathcal{H}'$ only one edge. We observe two things: (i) no edge of $\mathcal{H}'$ is repeated more than $q - 1$ times and (ii) $\mathcal{H}''$ is $\mathcal{F}$-free. If (i) were false, there would be q distinct edges $E_1, \ldots, E_q$ in $\mathcal{H}$ such that $|\bigcap_{i=1}^{q} E_i| \geq p$. But this implies the contradiction $\mathcal{F} \subset \mathcal{H}$. As for (ii), since the new p-element edges born from an edge E of $\mathcal{H}$ are separated, every copy of $\mathcal{F}$ in $\mathcal{H}''$ may use for every E only at most one of them. But then it would be a copy of $\mathcal{F}$ in $\mathcal{H}$ as well, which is again impossible. Both observations and the definitions of $\mathcal{H}'$ and $\mathcal{H}''$ imply

$$\begin{aligned} i(\mathcal{H}) &\leq \frac{(2p-1) \cdot i(\mathcal{H}')}{p} \leq \frac{(2p-1)(q-1) \cdot i(\mathcal{H}'')}{p} \\ &\leq (2p-1)(q-1) \cdot e(\mathcal{H}'') \\ &\leq (2p-1)(q-1) \cdot H_e(\mathcal{F}, n). \end{aligned}$$

□

The inequality (33) holds, for example, for all permutation hypergraphs $\mathcal{F} = \mathcal{H}_p$. On the other hand, there exists a class of somewhat singular hypergraphs $\mathcal{F}$ for which $H_i(\mathcal{F}, n) \not\ll H_e(\mathcal{F}, n)$. For $\mathcal{F} = (\{1\}, \{2\})$ one can quickly show that $H_e(\mathcal{F}, n) = 1$ but $H_i(\mathcal{F}, n) = n$. More generally, in [35] we have shown that if $\mathcal{F} = (\{1\}, \{2\}, \ldots, \{k\})$ then $H_e(\mathcal{F}, n) = 2^{k-1} - 1$ ($n \geq k-1$) and $H_i(\mathcal{F}, n) = (k-1)n - (k-2)$ (n is large enough).

Problem 17. Is it true that for every $\mathcal{F} \neq (\{1\}, \{2\}, \ldots, \{k\})$ one has the estimate $H_i(\mathcal{F}, n) \ll_{\mathcal{F}} H_e(\mathcal{F}, n)$? □

Linear and almost linear bounds on $\mathbf{H_i}(\mathcal{F}, \mathbf{n})$. In [33] we proved, by means of (18), an extremal analogue of (30): for every permutation p,

$$H_i(\mathcal{H}_p, n) < n \cdot \gamma_p(n) \tag{34}$$

where $\gamma_p(n)$ is defined in terms of $\alpha(n)$ and thus grows to infinity extremely slowly. (This proof now may be simplified by means of the inequality (33).) Similarly, in [33] we proved, by means of (26), that if $p = a_1 a_2 \ldots a_k$ is a permutation that first dicreases and then increases, or if p^{-1} first increases and then dicreases, then

$$H_i(\mathcal{H}_p, n) \ll_p n. \tag{35}$$

Problem 18. Prove (or disprove) that for every permutation p we have $H_i(\mathcal{H}_p, n) \ll_p n$. □

We have seen already in Example 9 that the extremal problem is, in a sense, more fundamental than the enumerative problem. This holds also on the hypergraph level: in [33] we prove first the bounds (34) and (35) and from them we derive, respectively, the bound (30) and (31) as corollaries. The derivation uses a variant of the inductive argument presented in Example 9. In the same vein, the bound in Problem 18 implies the bound in Problem 16.

What is the extremal analogue of (32)?

Problem 19. Identify the class of hypergraphs Ψ such that

$$H_e(\mathcal{F}, n), H_i(\mathcal{F}, n) \begin{cases} < n \cdot \gamma_{\mathcal{F}}(n) & \ldots \quad \mathcal{F} \in \Psi \\ \\ > n \cdot \delta_{\mathcal{F}}(n) & \ldots \quad \mathcal{F} \not\in \Psi \end{cases}$$

holds. Here the functions $\gamma_{\mathcal{F}}(n)$ and $\delta_{\mathcal{F}}(n)$ grow to infinity, any $\gamma_{\mathcal{F}}(n)$ is defined in terms of $\alpha(n)$ and thus is almost constant, and any $\delta_{\mathcal{F}}(n)$ is much faster than any $\gamma_{\mathcal{F}}(n)$. $\square$

Unlike in enumeration, now we cannot hope to replace every $\gamma_{\mathcal{F}}(n)$ with a constant. This, of course, comes as no surprise. By a reduction to 5-DS sequences, it is easily shown ([35]) that for $\mathcal{F} = (\{1,3\},\{1,5\},\{2,4\},\{2,6\})$ one has $H_e(\mathcal{F},n) \gg n\alpha(n)$. This $\mathcal{F}$ is an example of a *star forest*. These are simple hypergraphs $\mathcal{F}$ with only two-element edges, with no cycles, and with the components forming stars such that all centers of the stars precede all endvertices. In [35] we proved that for every star forest $\mathcal{F}$,

$$H_i(\mathcal{F},n) < n \cdot \gamma_{\mathcal{F}}(n) \tag{36}$$

where $\gamma_{\mathcal{F}}(n)$ is an almost constant function defined in terms of $\alpha(n)$. Note that by (33), it suffices to prove this bound for $H_e(\mathcal{F},n)$. So, in Problem 19, the class Ψ must contain all star forests. Does it consist only from star forests? Since star forests generalize permutation hypergraphs $\mathcal{H}_p$, the bound (36) generalizes the bound (34). However, (34) can be probably improved to a $\ll n$ bound.

One more almost linear bound and back to abab. In the definition (16), if expressed in terms of partitions, the number of vertices is maximized over all v-free partitions u with at most n edges (and u is moreover $\|v\|$-sparse). For partitions $\mathcal{H}$ we have $v(\mathcal{H}) = i(\mathcal{H})$ but the proper measure of size for hypergraphs is $i(\mathcal{H})$. We generalize the approach of (16) as follows. Suppose that $\mathcal{F}$ is a fixed partition with $q = e(\mathcal{F}) > 1$. Then for every $\mathcal{F}$-free hypergraph $\mathcal{H}$ (really every, even not simple) we have the inequality ([35])

$$i(\mathcal{H}) < (q-1)v(\mathcal{H}) + e(\mathcal{H}) \cdot \gamma_{\mathcal{F}}(e(\mathcal{H})) \tag{37}$$

where $\gamma_{\mathcal{F}}(n)$ is an almost constant function defined in terms of $\alpha(n)$. The bound (37) is an extension of (18) to hypergraphs. We can also apply (37) to bound $H_i(\mathcal{F},n)$ (almost linearly) in terms of $H_e(\mathcal{F},n)$ in situations when (33) does not apply.

We conclude our survey by returning to the pattern *abab* alias ●–○–●–○ alias $(\{1,3\},\{2,4\})$.

Example 11 ([33, 35]). We prove that, denoting $abab = (\{1,3\},\{2,4\})$, for every $n > 1$,

$$H_e(abab,n) = 4n-5 \quad \text{and} \quad H_i(abab,n) = 8n-12.$$

We begin with the case when $\mathcal{G}$ is a simple *abab*-free graph with the vertex set $[n]$ (so $\mathcal{G}$ has only two-element edges). We prove by induction on n that $e(\mathcal{G}) \leq 2n-3$. For $n=2$ this is true. Let $n>2$ and $\deg(1) \geq 2$; for $\deg(1)=1$ induction immediately applies. We split $[n]$ into two overlapping intervals $I_1=[k]$ and $I_2=[k,n]$, where k is the largest vertex in $[2,n-1]$ adjacent to 1. The restrictions of $\mathcal{G}$ to I_1 and I_2 are simple *abab*-free graphs and every edge of $\mathcal{G}$, except possibly of $\{1,n\}$, lies in I_1 or in I_2. By induction,

$$e(\mathcal{G}) \leq 1+2|I_1|-3+2|I_2|-3 = 2(|I_1|+|I_2|)-5 = 2n-3.$$

In the graph case, $e(\mathcal{G}) \leq 2v(\mathcal{G})-3$.

Let now $\mathcal{H}$ be a simple *abab*-free hypergraph with the vertex set $[n]$. We look at the *big* edges of $\mathcal{H}$ having 3 and more vertices. We claim that after deleting from each of them its first and last vertex, the resulting sets lie in $[2,n-1]$ and are mutually disjoint. The former claim is clear. If the resulting sets were not disjoint, we would have two distinct edges E_1, E_2 in $\mathcal{H}$ and five not necessarily distinct vertices $v_1,\ldots,v_5 \in \bigcup\mathcal{H}$ such that $v_2<v_3<v_4$, $v_1<v_3<v_5$, $\{v_1,v_3,v_5\} \subset E_1$, and $\{v_2,v_3,v_4\} \subset E_2$. Moreover, we may assume that $v_1 \neq v_2$ or $v_4 \neq v_5$ because $E_1 \neq E_2$ ($\mathcal{H}$ is simple). But then $\mathcal{H}$ contains *abab*, a contradiction. Thus the resulting sets must be disjoint and their number is at most $n-2$, which bounds the number of big edges in $\mathcal{H}$.

Not forgetting singleton edges, we conclude that

$$\begin{aligned} e(\mathcal{H}) &\leq n+(2n-3)+(n-2) = 4n-5 \\ i(\mathcal{H}) &\leq n+2(2n-3)+(n-2)+2(n-2) = 8n-12. \end{aligned}$$

The *abab*-free hypergraphs

$$(\{i\},\{j,j+1\},\{1,k\},\{1,j,j+1\}:\ i\in[n], j\in[2,n-1], k\in[2,n])$$

show that these bounds are tight. □

Interestingly, the previous proof needs only small adjustments to work also for the forbidden hypergraph $abba = (\{1,4\},\{2,3\})$. For $n>1$ we have $H_e(abba,n)=4n-5$ and $H_i(abba,n)=8n-12$ as well. This should be compared with the situation for sequences and colored trees when *abab* and *abba* have different extremal functions.

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