Combinatorial Counting 2026

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(lecture notes for the course in summer term 2026)

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1.1 The symbolic method. Proper bracketings

The symbolic method

The fundamental *symbolic method in enumerative combinatorics*, briefly the *symbolic method*, addresses the problem to enumerate — both by exact counting formulas and by approximate asymptotic formulas — a given countable (infinite) set A of combinatorial objects. A is equipped with a *size function* $s\colon A \to \mathbb{N}_0$ $(= \{0, 1, \dots\})$ and it is assumed that for every $n \in \mathbb{N}_0$ the set of size n objects in A,

$$A_n = \{a \in A\colon\ s(a) = n\},$$

is finite. We assign to A and s the generating function, briefly gf,

$$A(x) = \sum_{a \in A} x^{s(a)} = \sum_{n \geq 0} |A_n| x^n \ \ (\in \mathbb{C}[[x]]),$$

where $|X|$ denotes the number of elements of a finite set X. $A(x)$ is a *formal power series*, briefly fps. It is a map

$$A(x)\colon \mathbb{N}_0 \to \mathbb{C},\ A(x)(n) = |A_n|.$$

We review basic algebraic properties of fps in the next section.

Besides the function $s(a)$ we often have a *weight function* $h\colon A \to R$, where R is a ring. The weighted version of $A(x)$ is then

$$A_h(x) = \sum_{n \geq 0} \big(\sum_{a \in A_n} h(a) \big) x^n \ \ (\in R[[x]]).$$

The symbolic method works in two phases. In the *first formal phase* relations between objects in A are translated into algebraic, differential or functional

equations. By formal (algebraic) manipulations with involved fps we obtain in this phase counting formulas for the coefficients $|A_n|$ or $\sum_{a \in A_n} h(a)$. In the *second analytic phase*, using analytic methods we obtain from the equations derived in the first phase asymptotic formulas for the mentioned coefficients.

Proper bracketings

We exemplify the symbolic method by enumerating the set A of proper bracketings. A *proper bracketing* is a word $b = b_1 b_2 \dots b_{2n}$ over the alphabet $\{(,)\}$ with length $2n$, $n \in \mathbb{N}_0$, and with the following structure.

1. b has n (s and n)s.

2. There is a partition P_b of the index set $[2n] = \{1, 2, \dots, 2n\}$ in n two-element blocks such that for every block $B = \{i < j\} \in P_b$ we have $b_i = ($ and $b_j =)$, and for every two distinct blocks $B, C \in P_b$ we have neither $\min B < \min C < \max B < \max C$ nor $\min C < \min B < \max C < \max B$.

The following is a fundamental result in mathematics.

Proposition 1.1 *The partition P_b is uniquely determined by the word b.*

Proof.

$\square$

The syntax of both propositional and predicate logic is based on Proposition 1.1. Note that if $b, c \in A$, then the concatenation bc and the insertion (b) are again in A.

For $b = b_1 b_2 \dots b_{2n} \in A$ with $n \in \mathbb{N}$ we set $s(b) = n$, and define $s(\emptyset) = 0$. The size function $s(b)$ is the number of pairs of brackets. For example, $A_0 = \{\emptyset\}$ and
$$A_3 = \{()()(), ()(()), (())(), (()()), ((()))\} \,.$$
We denote $C_n = |A_n|$ and $C(x) = A(x)$, and derive an equation for the gf
$$C(x) = \sum_{n \geq 0} C_n x^n \,.$$
We consider the map
$$f \colon A \setminus A_0 \to A \times A, \;\; f(b) = \langle b_1, \, b_2 \rangle \,,$$
defined by the decomposition $b = (b_1)b_2$. Here $(\dots)\dots$ is the first block (pair of brackets) in b. Note that $s(b) = 1 + s(b_1) + s(b_2)$. It is easy to see that for every $n \in \mathbb{N}_0$,
$$f \mid A_{n+1} \colon A_{n+1} \to \bigcup_{j=0}^{n} A_j \times A_{n-j}$$
is a bijection. Thus $C_0 = 1$ and
$$C_{n+1} = \sum_{j=0}^{n} C_j C_{n-j}$$

for $n \in \mathbb{N}_0$. This yields the equation

$$C(x) = 1 + xC(x)^2.$$

For a fps $B(x) = \sum_{n \geq 0} b_n x^n$ and $m \in \mathbb{N}_0$ we recall the notation $[x^m]B(x) = b_m$.

Proposition 1.2 *For every $n \in \mathbb{N}_0$,*

$$|A_n| = C_n = [x^n]C(x) = [x^n]A(x) = \frac{1}{n+1}\binom{2n}{n}.$$

Proof.

$\square$

1.2 Basics on $\mathbb{C}[[x]]$

1.3 Why $\sqrt{1-4x} = \sum_{n \geq 0} \binom{1/2}{n}(-4x)^n$?

1.4 Pólya's theorem via the symbolic method

Chapter 2

Numbers of SAWs in the hexagonal grid

Chapter 3

Combinatorics and Complexity of Partition Functions

Bibliography

[1] A. Barvinok, *Combinatorics and Complexity of Partition Functions*, Springer, Cham, Switzerland 2016

[2] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, Cambridge, UK 2015