Combinatorial Counting 2025: the Ramsey theorem for pairs and the Catalan numbers

Martin Klazar

March 6, 2025

(lecture notes for the course taught in summer term 2025)

Contents

Introduction

These lecture notes

Notation. We use $\equiv$ as definitional equality; in $x \equiv y$ the new symbol x is being defined by the already known expression y. Sometimes x and y exchange their roles. Recall that $f : X \to Y$, where X and Y are sets, means that f is a map (function) from X to Y. So f is a set such that $f \subset X \times Y$ and for every $x \in X$ there is a unique $y \in Y$ for which $(x, y) \in f$, which is standardly written as $f(x) = y$. Let $f : X \to Y$ and Z be any set. The *restriction* of f to Z is the map $f \,|\, Z : X \cap Z \to Y$ with values $(f \,|\, Z)(x) \equiv f(x)$, $x \in X \cap Z$. Often we write instead of $f \,|\, Z$ just f. The *image* and the *preimage* of Z by f is the respective set

$$f[Z] \equiv \{f(x) :\ x \in X \cap Z\} \ \ (\subset Y) \ \text{ and } \ f^{-1}[Z] \equiv \{x \in X :\ f(x) \in Z\} \ \ (\subset X)\,.$$

If $Z = \{z\}$ is a singleton, we usually write (in analogy with $f(x)$) instead of $f^{-1}[\{z\}]$ just $f^{-1}[z]$.

We denote by $\mathbb{N} = \{1, 2, \dots\}$ the (infinite) set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are nonnegative integers. For $n \in \mathbb{N}$ we define $[n] \equiv \{1, 2, \dots, n\}$; we set $[0] \equiv \emptyset$. For any finite set X we denote by $|X|$ $(\in \mathbb{N}_0)$ the number of its elements. For any set X and $k \in \mathbb{N}_0$,

$$\tbinom{X}{k} \equiv \{Y :\ Y \subset X,\, Y \text{ is finite and } |Y| = k\}\,.$$

Thus $\binom{X}{0} = \{\emptyset\}$ and $\binom{X}{1} = \{\{x\} :\ x \in X\}$ $(\neq X)$. We have $\binom{X}{k} = \emptyset$ whenever X is finite and $|X| < k$. Also, for finite X the set $\binom{X}{k}$ is finite too and for $0 \le k \le |X|$ we have equalities

$$\left|\tbinom{X}{k}\right| = \tbinom{|X|}{k} = \tfrac{|X|(|X|-1)\dots(|X|-k+1)}{k!}\,.$$

Mostly we work with the sets $\binom{X}{2}$ and call their elements *edges*.

Chapter 1

The Ramsey theorem for pairs: simple bounds

Theorem 1.1 (Ramsey, 1930) *Let $r, p, k \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ such that for every map $\chi : \binom{[n]}{p} \to [r]$ there is a k-element set $Y \subset [n]$ for which the restriction $\chi \mid \binom{Y}{p}$ is constant.*

In Chapters 1 and 2 we deal with the function $R_r(k) : \mathbb{N}^2 \to \mathbb{N}$ corresponding to the pairs case $p = 2$ of the theorem. Its values are called *Ramsey numbers (for pairs)* and it is defined as follows.

Definition 1.2 *Let $r, k \in \mathbb{N}$. We define $R_r(k)$ to be the minimum $n \in \mathbb{N}$ such that for every union $\binom{[n]}{2} = \bigcup_{i=1}^{r} X_i$ there exists an index $i \in [r]$ and a k-element set $Y \subset [n]$ such that $\binom{Y}{2} \subset X_i$.*

In Proposition 1.5 we prove that $R_r(k)$ is defined for every r and k. Without loss of generality the sets X_i may be assumed to be pairwise disjoint and to form a partition (of $\binom{[n]}{2}$). This follows from the next proposition.

Proposition 1.3 *Let $r \in \mathbb{N}$, X be a set and $\{Y_i : i \in [r]\}$ be a set system such that $X = \bigcup_{i=1}^{r} Y_i$. Then there is a set system $\{Z_i : i \in [r]\}$ with the following properties.*

1. *$Z_i \subset Y_i$ for every $i \in [r]$.*
2. *$X = \bigcup_{i=1}^{r} Z_i$.*
3. *$Z_i \cap Z_j = \emptyset$ for every $i, j \in [r]$ with $i \neq j$.*

Proof. Let $Y_0 \equiv \emptyset$ and $Z_i \equiv Y_i \setminus \bigcup_{j=0}^{i-1} Y_j$, $i \in [r]$. It is not hard to see that this set system $\{Z_i : i \in [r]\}$ has the three stated properties. □

An exercise for the reader is to extend this proposition to infinite set systems.

So in other words, $R_r(k)$ is the minimum $n \in \mathbb{N}$ such that for every *coloring* (map) $\chi : \binom{[n]}{2} \to [r]$ there is a set $Y \subset [n]$ with $|Y| = k$ for which the restriction $\chi \,|\, \binom{Y}{2}$ is a constant map. We say that the set Y is *χ-homogeneous.* Yet another equivalent definition of $R_r(k)$ is that it is the minimum $n \in \mathbb{N}$ such that for every r-coloring of edges in the complete graph $K_n = (n, \binom{[n]}{2})$ there is a monochromatic k-clique.

The following *pigeonhole principles*, a finite and an infinite one, are Ramsey theorems for *singletons* (1-element sets). They were known, of course, long time before Theorem 1.1.

Proposition 1.4 (pigeonhole principles) *Let $k, r \in \mathbb{N}$. The following holds.*

1. *If $n \equiv r(k-1) + 1$, X is any finite set with $|X| = n$ and $\chi : X \to [r]$ is any map, then there is an $i \in [r]$ such that $|\chi^{-1}(i)| \ge k$.*
2. *If X is any infinite set and $\chi : X \to [r]$ is any map, then there is an $i \in [r]$ such that the set $\chi^{-1}(i)$ is infinite.*

Proof. 1. Since $X = \bigcup_{i \in [r]} \chi^{-1}(i)$ is a partition, if there were no such i then we would have the contradiction

$$n = |X| = \textstyle\sum_{i \in [r]} |\chi^{-1}(i)| \le \sum_{i \in [r]} (k-1) = r(k-1)\,.$$

2. If there were no such i, we would have the contradiction that the set $X = \bigcup_{i=1}^{r} \chi^{-1}(i)$ is finite as it is a finite (disjoint) union of finite sets. □

1.1 Finite case

In this section we obtain several elementary upper and lower bounds on the Ramsey numbers $R_r(k)$ which were introduced in Definition 1.2. First we show that $R_r(k)$ is defined for every $r, k \in \mathbb{N}$. If $\chi : \binom{[n]}{2} \to [r]$ and $Y \subset [n]$, we say that the set Y is *χ-min-homogeneous* if for every $e, f \in \binom{Y}{2}$ it holds that $\chi(e) = \chi(f)$ iff $\min e = \min f$. In the next Section 1.2 we consider a generalization of this kind of colorings.

Proposition 1.5 *Let $r, k \in \mathbb{N}$. Then $R_1(k) = k$, $R_r(1) = 1$ and for $r, k \ge 2$ we have the bound $R_r(k) \le r^{rk-2}$.*

Proof. The cases when $r = 1$ or $k = 1$ are clear. Let $r, k \ge 2$. We show, for the coloring form of Definition 1.2, that $n \equiv r^{rk-2}$ works. Let $\chi : \binom{[n]}{2} \to [r]$ be any map. We set $l \equiv r(k-1) + 1$ and define sets $A_0 \equiv [n]$, A_1, $\dots$, A_{l-1} such that $A_0 \supset A_1 \supset \dots \supset A_{l-1} \ne \emptyset$, $\min A_0 = 1 < \min A_1 < \dots < \min A_{l-1}$, that for every $i \in [l-1]$ the edges $\{\min A_{i-1}, x\}$, $x \in A_i$, have in χ the same color, and that $|A_i| = r^{rk-2-i}$, $i = 0, 1, \dots, l-1$. Suppose that $i \in [l-1]$ and that the sets A_0, A_1, $\dots$, A_{i-1} with the stated properties are already defined. By 1 of Proposition 1.4, at least $\lceil \frac{|A_{i-1}|-1}{r} \rceil = \lceil \frac{r^{rk-2-i+1}-1}{r} \rceil = r^{rk-2-i}$ edges $\{\min A_{i-1}, x\}$ with $x \in A_{i-1} \setminus \{\min A_{i-1}\}$ have in χ the same color; we define A_i to be some r^{rk-2-i} endpoints x of such edges. Thus we have sets A_0, A_1, ..., A_{l-1} with the stated properties; note that $A_{l-1} \neq \emptyset$ because $rk - 2 - (l-1) = r - 2 \geq 0$. We consider the l-element set

$$X \equiv \{\min A_{i-1} : \ i \in [l]\} .$$

It follows that X is χ-min-homogeneous and we can define the map $\psi : X \to [r]$ by setting $\psi(x) \equiv \chi(e)$ for any $e \in \binom{X}{2}$ with $\min e = x$; for $x = \max X$ when there is no such e we define $\psi(x)$ arbitrarily. By 1 of Proposition 1.4 there is a set $Y \subset X$ such that $|Y| = k$ and $\psi \,|\, Y$ is constant. It follows that Y is the sought for k-element χ-homogeneous set. □

In the next chapter we use this bound in the slightly weaker but simpler form $R_r(k) \leq r^{rk}$. Thus in the simplest nontrivial case $r = p = 2$ of Theorem 1.1 we have the following upper bound.

Corollary 1.6 *For every $k \in \mathbb{N}$,*

$$R_2(k) \leq 4^{k-1} .$$

1.2 The canonical Ramsey theorem

If $k \in \mathbb{N}$, $e = \{e_1, e_2, \ldots, e_k\}_<$ in $\binom{\mathbb{N}}{k}$ is a k-element set of natural numbers with the elements e_i listed increasingly and if $I \subset [k]$, we define

$$e : I \equiv \{e_i : \ i \in I\} .$$

If $X \subset \binom{\mathbb{N}}{k}$ and $\chi \colon X \to \mathbb{N}$ is any coloring of X (by infinitely many colors), then we call χ *canonical*, or more precisely *I-canonical*, if there is a set $I \subset [k]$ such that for every $e, f \in X$,

$$\chi(e) = \chi(f) \iff e : I = f : I .$$

This section is devoted to the function $\mathrm{ER}(k; l) \colon \mathbb{N}^2 \to \mathbb{N}$, especially for $k = 2$, defined as follows.

Definition 1.7 *Let $k, l \in \mathbb{N}$. Then $\mathrm{ER}(k; l)$ is the minimum $n \in \mathbb{N}$ such that for every coloring $\chi : \binom{[n]}{k} \to \mathbb{N}$ there exists an l-element set $Y \subset [n]$ such that the restriction $\chi \,|\, \binom{Y}{k}$ is canonical. We set $\mathrm{ER}(l) \equiv \mathrm{ER}(2; l)$.*

In 1950 P. Erdős and R. Rado proved in [3] that the numbers $\mathrm{ER}(k; l)$ exist for every $k, l \in \mathbb{N}$. For $k = 1$ these numbers are easily determined exactly.

Proposition 1.8 $\mathrm{ER}(1; l) = (l-1)^2 + 1$ *for every $l \in \mathbb{N}$.*

Proof. Let $l \in \mathbb{N}$, $n \equiv (l-1)^2+1$ and $\chi\colon [n] \to \mathbb{N}$. Since $n = \sum_{i\in\mathbb{N}} |\chi^{-1}(i)|$, we see that there is a set $X \subset [n]$ such that $|X| = l$ and $\chi \,|\, X$ is constant or 1-1 (injective). Thus

$$\mathrm{ER}(1;\, l) \le n = (l-1)^2 + 1\,.$$

On the other hand, if we set $n \equiv (l-1)^2$ and, for $i = 1, 2, \ldots, l-1$ and $(i-1)(l-1) < j \le i(l-1)$, define the coloring $\chi\colon [n] \to \mathbb{N}$ by $\chi(j) \equiv i$, we get the bound $\mathrm{ER}(1; l) > n = (l-1)^2$; for this χ there is no l-element canonical set (for $k = 1$). Thus we get the stated equality. □

In 1996 S. Shelah proved in [10] for any $k \ge 2$ a strong general upper bound on $\mathrm{ER}(k; l)$ in the form of an iterated $(k-1)$-fold exponential. For $k, l \in \mathbb{N}$ we set $\mathrm{tow}(1; l) \equiv 2^l$ and $\mathrm{tow}(k; l) \equiv 2^{\mathrm{tow}(k-1;l)}$ for $k \ge 2$.

Theorem 1.9 (Shelah, 1996) *There is a constant $c > 0$ such that for every $k, l \in \mathbb{N}$ with $k \ge 2$,*

$$\mathrm{ER}(k;\, l) \le \mathrm{tow}\big(k-1;\, cl^{8(2k-1)}\big)\,.$$

In the rest of this section we prove two theorems on $\mathrm{ER}(l) = \mathrm{ER}(2; l)$. We begin with a theorem due to H. Lefmann and V. Rödl. They obtained the easy lower bound in [6], and the harder to prove upper bound in [7].

Theorem 1.10 (Lefmann and Rödl, 1993 and 1995) *For some constants $c_1, c_2 > 0$ and every $l \in \mathbb{N}$ with $l \ge 2$,*

$$2^{c_1 l^2} \le \mathrm{ER}(l) \le 2^{c_2 l^2 \log l}\,.$$

We begin with the lower bound.

The lower bound $2^{c_1^2 l} \le \mathrm{ER}(l)$

We prove the lower bound of Lefmann and Rödl and begin with a lemma.

Lemma 1.11 *Let $k, l \in \mathbb{N}$ with $k \le l$ and $I \subset [k]$ with $I \ne \emptyset$. Then there exist $l-k+1$ sets $X_i \in \binom{[l]}{k}$, $i \in [l-k+1]$, such that the $l-k+1$ sets*

$$X_i : I,\ \ i \in [l-k+1]\,,$$

are mutually distinct

Proof. For $i = 0, 1, \ldots, l-k$ set $X_i \equiv \{i+1, i+2, \ldots, i+k\}$. □

Recall that for $t, k, l \in \mathbb{N}$ the (classical) Ramsey number $\mathrm{R}_t(k; l)$ is the minimum $n \in \mathbb{N}$ such that for every $\chi\colon \binom{[n]}{k} \to [t]$ there is an l-element set $X \subset [n]$ such that the restriction $\chi \,|\, \binom{X}{k}$ is constant.

Proposition 1.12 *For every $k, l \in \mathbb{N}$ with $k < l$,*

$$\mathrm{ER}(k;\, l) \ge \mathrm{R}_{l-k}(k;\, l)\,.$$

Proof. Let k and l be as stated, $n \equiv \mathrm{R}_{l-k}(k;l) - 1$ and let $\chi\colon \binom{[n]}{k} \to [l-k]$ be such that there is no l-element χ-monochromatic set. But then there is also no $I \subset [k]$ and no l-element set $X \subset [n]$ such that $\chi \,|\, \binom{X}{k}$ is I-canonical. For $I = \emptyset$ it follows from the non-existence of monochromatic set, and for $I \neq \emptyset$ it follows from Lemma 1.11 which shows that then at least $l - k + 1$ distinct colors would be needed. Hence $\mathrm{ER}(k;l) > n$ and we get the stated inequality. □

This concludes the proof of the lower bound.

The upper bound $2^{c_2 l^2 \log l} \geq \mathrm{ER}(l)$

We prove the upper bound of Lefmann and Rödl.

1.3 Infinite case

Theorem 1.13 (infinite Ramsey for pairs) *Let $r \in \mathbb{N}$. Then for every map $\chi\colon \binom{\mathbb{N}}{2} \to [r]$ there is an infinite χ-homogeneous set, an infinite set $Y \subset \mathbb{N}$ such that $\chi \,|\, \binom{Y}{2}$ is constant.*

Proof. Let r and χ be as stated. We define a sequence of infinite sets $A_0 \equiv \mathbb{N}$, $A_1, \ldots$ such that $A_0 \supset A_1 \supset \ldots$, $\min A_0 = 1 < \min A_1 < \ldots$ and that for every n the pairs $\{\min A_{n-1}, x\}$, $x \in A_n$, have in χ the same color. Suppose that $n \in \mathbb{N}$ and that the sets $A_0, A_1, \ldots, A_{n-1}$ with the stated properties are already defined. By 2 of Proposition 1.4, for infinitely many $x \in A_{n-1} \setminus \{\min A_{n-1}\}$ the edges $\{\min A_{n-1}, x\}$ have in χ the same color; we define A_n as the set of these numbers x. Thus we get a sequence of sets $(A_n)_{n \geq 0}$ with the stated properties. We define the infinite set

$$X \equiv \{\min A_{n-1} :\ n \in \mathbb{N}\}\,.$$

It follows that X is χ-min-homogeneous. As before we can define the map $\psi : X \to [r]$ by setting $\psi(x) \equiv \chi(e)$ for any $e \in \binom{X}{2}$ with $\min e = x$. By 2 of Proposition 1.4 there is an infinite set $Y \subset X$ such that $\psi \,|\, Y$ is constant. It follows that Y is an infinite χ-homogeneous set. □

Theorem 1.14 (compactness) *Let $r \in \mathbb{N}$. For every sequence (χ_n) of colorings $\chi_n : \binom{[n]}{2} \to [r]$ there exists a coloring $\chi : \binom{\mathbb{N}}{2} \to [r]$ with the following property. For every k there is an n, $n \geq k$, with*

$$\chi_n \,|\, \tbinom{[k]}{2} = \chi \,|\, \tbinom{[k]}{2}\,.$$

Proof. Let r and χ_n, $n \in \mathbb{N}$, be as stated. Let $F : \mathbb{N} \to \binom{\mathbb{N}}{2}$ be a bijection, thus the sequence $F(1)$, $F(2)$, $\ldots$ enumerates the edges of the countable complete graph $K_{\mathbb{N}}$. We define by induction on $j = 0, 1, \ldots$ infinite sets A_j such that $A_0 = \mathbb{N}$, $A_0 \supset A_1 \supset \ldots$ and that if $A_j = \{a_{1,j} < a_{2,j} < \ldots\}$, then for every $i \in [j]$ the values $\chi_{a_{1,j}}(F(i))$, $\chi_{a_{2,j}}(F(i))$, $\dots$ are all defined (that is, $a_{1,j} \geq \max F(i)$) and are all equal. In other words, for every $i \in [j]$ we have $|\{\chi_a(F(i)) : \ a \in A_j\}| = 1$. Suppose that $j \in \mathbb{N}$ and that the sets A_0, A_1, $\dots$, A_{j-1} (with the stated properties) are already defined. We define A_j as any infinite subset of A_{j-1} for which every value $\chi_a(F(j))$, $a \in A_j$, is defined and $|\{\chi_a(F(j)) : \ a \in A_j\}| = 1$. Such a subset exists by 2 of Proposition 1.4. Thus we get the sequence of sets $(A_j)_{j\geq 0}$ with the stated properties. We define the map $\chi : \binom{\mathbb{N}}{2} \to [r]$ for $e \in \binom{\mathbb{N}}{2}$ by setting, with $j \equiv F^{-1}(e)$,

$$\chi(e) \equiv \chi_a(e) \text{ for any } a \in A_j \; .$$

It is clear that this definition is correct — the color $\chi_a(e)$ does not depend on the element $a \in A_j$ — and we show that χ has the stated property. So let $k \in \mathbb{N}$. We take a $j \in \mathbb{N}$ such that $F[\,[j]\,] \supset \binom{[k]}{2}$ and take any $n \in A_j$. Let $e \in \binom{[k]}{2}$. Then by the definition of A_j and χ we have with $i \equiv F^{-1}(e)$ that $n \geq k$, $i \in [j]$, $n \in A_i$ and thus $\chi(e) = \chi_n(e)$, as required. □

We say that a finite set $X \subset \mathbb{N}$ is *big* if $|X| \geq \min X$.

Theorem 1.15 (big Ramsey for pairs) *Let $r \in \mathbb{N}$. Then for every k there is an n such that for every coloring $\chi : \binom{[n]}{2} \to [r]$ there exists a big and at least k-element χ-homogeneous set $Y \subset [n]$.*

Proof. Let $r, k \in \mathbb{N}$. Suppose for the contrary that for every n there is a coloring $\chi_n : \binom{[n]}{2} \to [r]$ that has no big and at least k-element χ_n-homogeneous set. It follows that the same holds for the coloring $\chi : \binom{\mathbb{N}}{2} \to [r]$ obtained from the sequence (χ_n) in Theorem 1.14. But this is a contradiction because it is easy to deduce from Theorem 1.13 that every r-coloring of $\binom{\mathbb{N}}{2}$ has a big and at least k-element homogeneous set. Indeed, if $\psi : \binom{\mathbb{N}}{2} \to [r]$ is any coloring and $\{a_1 < a_2 < \dots\} \subset \mathbb{N}$ is the infinite ψ-homogeneous set provided by Theorem 1.13, then

$$\{a_1 < a_2 < \dots < a_{k+a_1}\}$$

is a big and at least k-element ψ-homogeneous set. □

Theorem 1.16 (Erdős–Dushnik–Miller) *Suppose that κ is an infinite cardinal. Then for every partition $\binom{\kappa}{2} = A \cup B$ there exists a set $C \subset \kappa$ such that $|C| = \omega$ and $\binom{C}{2} \subset A$, or $|C| = \kappa$ and $\binom{C}{2} \subset B$.*

Proof.

□

Chapter 2

The Ramsey theorem for pairs: the BBCGHMST bound

Theorem 2.1 (BBCGHMST, 2024) *Let $r \geq 2$ and $\delta \equiv \frac{1}{2^{160} r^{12}}$. Then for every $k \geq 2^{160} r^{16}$,*

$$R_r(k) \leq \frac{r^{rk}}{\mathrm{e}^{\delta k}} .$$

Chapter 3

The Catalan numbers

Bibliography

[1] P. Balister, B. Bollobás, M. Campos, S. Griffiths, E. Hurley, R. Morris, J. Sahasrabudhe and M. Tiba, Upper bounds for multicolour Ramsey numbers, arXiv:2410.17197v1, 2024, 17 pp.

[2] M. Campos, S. Griffiths, R. Morris and J. Sahasrabudhe, An exponential improvement for diagonal Ramsey, arXiv:2303.09521v1, 2023, 57 pp.

[3] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249–255

[4] T. Jech, *Set Theory. The Third Millennium Edition, revised and expanded*, Springer-Verlag, Berlin 2003

[5] M. Klazar, On a proof of Ramsey theorem and of Erdös-Rado theorem for pairs, preprint KAM Series, 93–256, 1993, 5 pp.

[6] H. Lefmann and V. Rödl, On canonical Ramsey numbers for complete graphs versus paths, *J. Combin. Theory* Ser. B **58** (1993), 1–13

[7] H. Lefmann and V. Rödl, On Erdős-Rado numbers, *Combinatorica* **15** (1995), 85–104

[8] Ch. Misak, *Frank Ramsey. A Sheer Excess of Powers*, Oxford University Press, Oxford 2020

[9] L. Paulson, Formalising new mathematics in Isabelle: diagonal Ramsey, arXiv:2501.10852v1, 2025, 22 pp.

[10] S. Shelah, Finite canonization, *Comment. Math. Univ. Carolin.* **37** (1996), 445–456.