Lecture 1

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October 7, 2025

Catalan's conjecture
=
Mihăilescu's theorem

The purpose of this series of 11 lectures on Catalan's conjecture is to get me started so that I eventually, some time, write down a *complete* proof of Mihăilescu's theorem.

Catalan's conjecture

Let $\mathbb{N} = \{1, 2, \dots\}$.

Conjecture 1 (E. Catalan, 1842). *If $x, y, m, n \in \mathbb{N} \setminus \{1\}$ are such that*

$$x^m - y^n = 1\,,$$

then $x = n = 3$ and $y = m = 2$.

Mihăilescu's theorem

Catalan's conjecture was affirmatively resolved by P. Mihăilescu in [2] twenty years ago.

Theorem 2 (P. Mihăilescu, 2006). *Catalan's conjecture is true.*

Mihăilescu's proof is a pen-and-paper proof, it is not computer-assisted, but it relies on large swaths of algebraic number theory.

The equation $x^2 - y^3 = 1$

The simplest and already non-trivial special case of Catalan's conjecture is to prove the following theorem. (By "solution" we mean in these lectures always solution in the integers, in $\mathbb{Z}$.)

Theorem 3 (L. Euler, 1738). *The only solutions of*

$$x^2 - y^3 = 1$$

are $\langle \pm 1, 0 \rangle$, $\langle \pm 3, 2 \rangle$ *and* $\langle 0, -1 \rangle$.

Euler proved in fact more strongly that $\langle \pm 3, 2 \rangle$ is the only solution of the equation in nonzero fractions, in $\mathbb{Q}^*$.

The equation $x^4 - 3y^2 = 1$

References

[1] M. Klazar, O řešení diofantické rovnice $x^2 - y^3 = \pm 1$, *Matematické obzory* **32** (1989), 47–53 (On solving Diophantine equation $x^2 - y^3 = \pm 1$)

[2] P. Mihăilescu,

[3] E. Notari,