Lecture 1. Catalan's Conjecture. The case $q = 2$

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We use $\equiv$ as the defining equality sign; $x \equiv y$ defines the new symbol x by the already known expression y. Sometimes x and y may exchange their roles. $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{Z}$ are the integers, $\mathbb{Q}$ are the fractions, $\mathbb{R}$ are the real numbers and $\mathbb{C}$ are the complex numbers. For $n \in \mathbb{N}$ we set $[n] \equiv \{1, 2, \dots, n\}$. For $m, n \in \mathbb{Z}$ we write $(m, n) = 1$ to express that m and n are coprime integers, their largest common divisor is 1. Letters p and q usually denote prime numbers.

From the perspective of history of mathematical terminology the Belgian mathematician *Eugène Catalan (1814–1894)* was very lucky. The *Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$, on which I lectured relatively extensively last semester, are named after him.

The *Catalan opening* in chess, 1. d4 Nf6 2. c4 e6 3. g3, is not named after him. It was invented by *Savielly (Xavier, Ksawery) Tartakower (1887–1956)* for the 1929 tournament in Barcelona.

But *Catalan's conjecture* in number theory, which today is a theorem, is named after him ([2]). Let us call a number $n \in \mathbb{N}$ a *pure power* if $n = k^l$ for some $k, l \in \mathbb{N}$ with $l \geq 2$. Let us mark pure powers in the sequence 1, 2, … of natural numbers by the bold font:

1, 2, 3, **4**, 5, 6, 7, **8**, **9**, 10, 11, 12, 13, 14, 15, **16**, 17, 18, 19, 20, 21, 22, 23, 24, **25**, 26, **27**, 28, 29, 30, 31, 32, 33, 34, 35, **36**, 37, 38, 39, 40, 41, …

Conjecture 1 (E. Catalan [2], 1844) *In this sequence the only two consecutive pure powers are* $8 = 2^3$ *and* $9 = 3^2$.

Said in a more algebraic notation, employed for example on the very first page of the book [10], the conjecture asserts that the only solution of the exponential Diophantine equation

$$x^u - y^v = 1$$

in natural numbers x, y, u and v with $u, v \geq 2$ is $3^2 - 2^3 = 1$. As for the above sequence, other close encounters of pure powers which did not fit in it are $2^7 - 5^3 = 128 - 125 = 3$ and $13^3 - 3^7 = 2197 - 2187 = 10$.

You can read much more on the history of Catalan's conjecture and partial results concerning it in the books [1, 10, 11] but in my course I do not have time for it and have to jump directly to the man who solved it. By [9], *Preda*

Mihăilescu (1955) was born in Bucharest and after leaving (bleak Ceauşescu's) Romania in 1973 he studied mathematics and computer science on ETH Zürich in Switzerland and received PhD degree there in 1997. For several years he did research at the University of Paderborn in Germany. Since 2005 he has held a professorship at the University of Göttingen. In 2002 he proved Catalan's conjecture and published his results in 2004 in the same journal where Catalan's note [2] appeared 160 years earlier. I give, following [11, Chapter 1], a rough top view outline of Mihăilescu's proof.

Clearly, if $x^u - y^v = 1$ for numbers $x, y \in \mathbb{Z}$ and $u, v \in \mathbb{N}$ with $u, v \geq 2$ then we may assume that the exponents are prime numbers $u = p$ and $v = q$. Solutions with $p = 2$ or $q = 2$ can be excluded relatively easily and it remains to handle the case of all remaining odd primes $p, q > 2$.

So suppose that $x^p - y^q = 1$ for $x, y \in \mathbb{Z} \setminus \{0\}$ and odd primes p and q.

Theorem 2 (PM [6]) *Then*

$$p^{q-1} \equiv 1 \pmod{q^2} \text{ and } q^{p-1} \equiv 1 \pmod{p^2}\,.$$

Theorem 3 (PM [7]) *Then*

$$p \equiv 1 \pmod q \text{ or } q \equiv 1 \pmod p\,.$$

Theorem 4 (PM [8]) *Then*

$$p < 4q^2 \text{ and } q < 4p^2\,.$$

From these three theorems we deduce, following [11, pp. 4–5], Catalan's conjecture.

Corollary 5 (Cc proved) *The only solutions of the equation $x^u - y^v = 1$ in integers $u, v \geq 2$ and nonzero integers x, y are given by $u = 2$, $v = 3$, $x = \pm 3$ and $y = 2$.*

Proof. Bla □

Theorem 6 (PM [8]) *If p and q are prime numbers such that $2 < p \leq 41$ or $2 < q \leq 41$ then the equation $x^p - y^q = 1$ has no solution in nonzero integers x, y.*

Theorem 7 (V. Lebesgue [5], 1850) *For no $u \in \mathbb{N}$ with $u \geq 2$ the equation $x^u - y^2 = 1$ has a solution in nonzero integers x, y.*

Proof. Bla □

Theorem 8 (Chao Ko [4], 1965) *For no prime $q \geq 5$ the equation $x^2 - y^q = 1$ has a solution in nonzero integers x, y.*

Theorem 9 (L. Euler [3], 1738) *The only solutions to the equation $x^2 - y^3 = 1$ in nonzero fractions $x, y \in \mathbb{Q}$ are $x = \pm 3$ and $y = 2$.*

References

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