Lecture 9. Primes in arithmetic progressions

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In the ninth lecture we cover Chapter II.8. *Primes in arithmetic progressions* in G. Tenenbaum's book [1], up to page 413.

Chapter II.8. Primes in arithmetic progressions

The following are Theorem 8.1, Definition 8.2, Theorem 8.3 (Orthogonality relations), Definition 8.4, Proposition 8.5, Definition 8.6, Theorems 8.7 and 8.8, Proposition 8.9 and Theorems 8.10 (Pólya–Vinogradov) and 8.11 in [1].

For an Abelian group $G = (G, 1_G, \cdot)$, a *character* χ is any group homomorphism $\chi\colon G \to \mathbb{C}^*$ where $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, 1, \cdot)$. It is *principal*, denoted χ_0, if $\chi(g) = 1$ for any $g \in G$. The set, in fact an Abelian group, of characters of G is denoted by $\widehat{G}$.

Theorem 1 *For any G, $|G| = n$, it holds that $\sum_{\chi\in\widehat{G}} \chi(\gamma)$ is n if $\gamma = 1_G$ and is 0 else, and that $\sum_{\gamma\in G} \chi(\gamma)$ is n if $\chi = \chi_0$ and is 0 else.*

Definition 2 *A Dirichlet character mod $q \in \mathbb{N}$ is a map $f\colon \mathbb{Z} \to \mathbb{C}$ such that for some $\chi \in \widehat{(\mathbb{Z}/q\mathbb{Z})^*}$ we have for any n that $f(n) = \chi(n \bmod q)$ if $(n, q) = 1$, and $f(n) = 0$ else.*

Theorem 3 *Let $q \in \mathbb{N}$. Then $\varphi(q)^{-1} \sum_{\chi \bmod q} \chi(n)\overline{\chi(m)}$ is 1 if n and m are the same element in $(\mathbb{Z}/q\mathbb{Z})^*$ and is 0 else, and $\varphi(q)^{-1} \sum_{n=1}^{q} \chi(n)\overline{\chi'(n)}$ is 1 if $\chi = \chi'$ and is 0 else.*

Definition 4 *A Dirichlet character χ mod q is primitive if there is no Dirichlet character χ_1 mod $q_1 < q$ such that $\chi(n) = \chi_1(n)$ if $(n, q) = 1$.*

Proposition 5 *Each imprimitive Dirichlet character χ mod q is induced by a unique primitive Dirichlet character χ_1 mod a proper divisor q_1 of q. If $(n, q_1) = 1$ then $\chi_1(n) := \chi(n + tq_1)$ where $t \in \mathbb{Z}$ is any integer such that $(n + tq_1, q) = 1$.*

Definition 6 *Let χ be a Dirichlet character mod q. The Gaussian sum associated with χ is ($n \in \mathbb{N}$, $\mathrm{e}(t) = \mathrm{e}^{2\pi i t}$)*

$$G(n, \chi) := \sum_{m=1}^{q} \chi(m)\mathrm{e}(mn/q)\,.$$

Theorem 7 *If χ is a primitive Dirichlet character mod q then for any $n \in \mathbb{N}$ it holds that $G(n, \chi) = \overline{\chi(n)}G(1, \chi)$.*

Theorem 8 *If χ is a primitive Dirichlet character mod q then $|G(1, \chi)| = \sqrt{q}$.*

Proposition 9 *Let χ be an imprimitive Dirichlet character mod q that is induced by the primitive Dirichlet character χ_1 mod q_1. Then with $r := q/q_1$ we have that $G(1, \chi)$ is $\mu(r)\chi_1(r)G(1, \chi_1)$ if $(q_1, r) = 1$ and is 0 else.*

Theorem 10 *If χ is a non-principal Dirichlet character mod $q \geq 2$ then*

$$H(\chi) := \max_{x \geq 1} \left| \sum_{n \leq x} \chi(n) \right| \leq 2\sqrt{q} \log q\,.$$

Theorem 11 *If χ is a primitive Dirichlet character mod q then*

$$H(\chi) \geq \tfrac{1}{4}\sqrt{q}\,.$$

The following are Theorems 8.12, 8.13 (Mertens) and 8.14, Corollary 8.15 and Theorems 8.16 and 8.17 (Siegel–Walfisz) in [1].

For any Dirichlet character χ we define $L(s, \chi) = \sum_{n \geq 1} \chi(n)/n^s$ $(\sigma > 1)$.

Theorem 12 *If $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ then for $\sigma > 1$,*

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \frac{\Lambda(n)}{n^s} = \frac{-1}{\varphi(q)} \sum_{\chi} \overline{\chi(a)} L'(s,\, \chi)/L(s,\, \chi)$$

where we sum over Dirichlet characters mod q.

For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ coprime to q let

$$c(a,\, q) := \tfrac{1}{\varphi(q)} \left(\gamma - \sum_p (\log(1/(1 - 1/p)) - 1/p) + \sum_{\chi \neq \chi_0} \overline{\chi(a)} \sum_p \chi(p)/p \right).$$

Theorem 13 *If $L(1, \chi) \neq 0$ for every non-principal Dirichlet character mod q then for any $a \in \mathbb{Z}$ with $(a, q) = 1$ and any $x \geq 2$,*

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{\log \log x}{\varphi(q)} + c(a,\, q) + O(1/\log x)\,.$$

Let $G(\chi) = \sum_{m=1}^{q} \chi(m)\mathrm{e}(m/q)$.

Theorem 14 *If χ is a real primitive Dirichlet character mod $q > 1$ then $L(1, \chi)$ is*

$$\frac{-i\pi}{qG(\chi)} \sum_{m=1}^{q} m\chi(m)$$

if $G(\chi) \in i\mathbb{R}$, and is

$$\frac{-1}{G(\chi)} \sum_{m=1}^{q} \chi(m) \log(2 \sin(\pi m/q))$$

if $G(\chi) \in \mathbb{R}$. Furthermore $\sum_{m=1}^{q} m\chi(m)$ is $-\frac{1-\chi(-1)}{2\pi i} qG(\chi)L(1, \chi)$, and this equals $\frac{iqG(\chi)}{\pi} L(1, \chi)$ in the former case, and is 0 in the latter case.

Let

$$M_2(\chi)^2 := \tfrac{1}{q}\int_0^q |\sum_{n\le x}\chi(n)|^2\,\mathrm{d}x\,.$$

Corollary 15 *If χ is a real primitive Dirichlet character mod $q>1$ then*

$$M_2(\chi)^2 \le q\big(L(1,\,\chi)^2/\pi^2 + 1/12\big)\,.$$

For $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ let

$$\psi(x,\,a,\,q) = \sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n)\,.$$

Theorem 16 *For any $A>0$ it holds uniformly in $x\ge 3$, $a\in\mathbb{Z}$ and $1\le q\le (\log x)^A$ with $(a,q)=1$ that*

$$\psi(x,\,a,\,q) = \frac{x}{\varphi(q)} + O\big(x(\log\log x)^{19/2}/\log x\big)\,.$$

In particular, the asymptotics $\psi(x,a,q)\sim\frac{x}{\varphi(q)}$ ($x\to+\infty$) holds for every $\varepsilon>0$ uniformly for $x\ge 3$ and $1\le q\le(\log x)^{1-\varepsilon}$.

Theorem 17 *Under the assumptions of the previous theorem,*

$$\psi(x,\,a,\,q) = \frac{x}{\varphi(q)} + O\big(x\mathrm{e}^{-c\sqrt{\log x}}\big)$$

where $c=c(A)>0$ is a constant.

The following are Theorems 8.18–8.22 in [1]. For $q\ge 1$ and $\tau\in\mathbb{R}$ we define $\mathcal{L}=\mathcal{L}(q,\tau):=\log(|\tau|+q+1)$.

Theorem 18 *For any $k\in\mathbb{N}_0$ and $\sigma\ge 1$, if $\chi\ne\chi_0$ then*

$$L^{(k)}(s,\,\chi) \ll_k \mathcal{L}^{k+1}\,.$$

Theorem 19 *For any $\sigma>1$ and $\tau\in\mathbb{R}$, if χ is a Dirichlet character then*

$$L(\sigma,\,\chi_0)^3\cdot|L(\sigma+i\tau,\,\chi)|^4\cdot|L(\sigma+2i\tau,\,\chi^2)|\ge 1\,.$$

Theorem 20 *If $\chi^2\ne\chi_0$ then for $\sigma\ge 1$ one has that $1/L(\sigma+i\tau,\chi)\ll\mathcal{L}^7$.*

Theorem 21 *If $\chi^2=\chi_0$ but $\chi\ne\chi_0$ then for $\sigma\ge 1$ one has that $L(\sigma,\chi)\ge 1/9\sqrt{q}$.*

Theorem 22 *There is an absolute constant $c_0>0$ such that if $\chi^2=\chi_0$ but $\chi\ne\chi_0$ then for $\sigma\ge 1$ one has for $|\tau|>c_0q^{-1/2}(\log(2q))^{-2}$ that*

$$\frac{1}{L(s,\,\chi)} \ll \mathcal{L}^6(\mathcal{L}+1/|\tau|)\,,$$

and else that $1/L(s,\chi)\ll\sqrt{q}$.

Next is Theorem 8.23 in [1].

Theorem 23 *Let* χ *be a primitive Dirichlet character mod* q *and* $\alpha(\chi) := \frac{1}{2}(1-\chi(-1))$. *Then* $L(s,\chi)$ *is entire if* $\chi \neq \chi_0$ *and* $L(s,\chi_0)\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$ *is meromorphic with a simple pole with residue* 1 *in* $s=1$. *If we set*

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-(s+\alpha)/2} \Gamma((s+\alpha)/2)L(s,\chi) \text{ and } E(\chi) := \frac{G(1,\chi)}{i^\alpha\sqrt{q}}$$

then we have for any $s \in \mathbb{C}$ *the functional equation*

$$\xi(s,\,\chi) = E(\chi)\xi(1-s,\,\overline{\chi})\,.$$

The following are Theorems 8.24 and 8.25 (Landau–Page) and Lemmas 8.26 and 8.27 (Landau) in [1].

Theorem 24 *If* χ *is a non-principal primitive Dirichlet character mod* q *then* $L(s,\chi)$ *has infinitely many zeros* ρ *in the critical strip and for any* $s \in \mathbb{C}$,

$$\xi(s,\,\chi) = \mathrm{e}^{A+Bs}\prod_\rho (1-s/\rho)\mathrm{e}^{s/\rho}$$

where the zeros are taken according to their multiplicities, the product absolutely converges,

$$A = A(\chi) := \mathrm{Log}\big(\Gamma((1+\alpha)/2)(q/\pi)^{(1+\alpha)/2}E(\chi)L(1,\,\overline{\chi})\big)$$

and $B = B(\chi) := (\xi'/\xi)(0,\chi) = -(\xi'/\xi)(1,\overline{\chi})$.

Theorem 25 $\exists\, c > 0$ *such that the region* $\{s \in \mathbb{C} \mid \sigma > 1 - \frac{c}{\log(q(|\tau|+2))}\}$ *contains at most one zero of* $\prod_{\chi \bmod q} L(s,\chi)$ *and at most one zero of*

$$\prod_{q_1 \le q} \prod_{\substack{\chi \bmod q_1 \\ \chi \text{ is primitive}}} L(s,\,\chi)\,.$$

In both cases the possible exceptional zero is real and simple, and corresponds to a real character.

We denote a generic zero of $L(s,\chi)$ by $\rho = \beta + i\gamma$.

Lemma 26 *There is an absolute constant* c_0 *such that if* $\chi \neq \chi_0$ *is a Dirichlet character mod* q *then for* $\sigma > 1$,

$$-\mathrm{Re}\big((L'/L)(s,\,\chi)\big) \le c_0\log(q(|\tau|+2)) - \sum_\rho \mathrm{Re}(1/(s-\rho))$$

where the series has non-negative terms.

Lemma 27 *Let χ_1 and χ_2 be two distinct primitive Dirichlet characters whose L-functions have real zeros β_1 and β_2. Then*

$$1 - \min(\beta_1, \beta_2) \geq c_{10}/\log(q_1 q_2)\,.$$

The following is Theorem 8.28 in [1]. Let $\psi(x,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n)$ and $\psi^*(x,\chi) := (\psi(x^-,\chi) + \psi(x,\chi))/2$. Let $\vartheta(\chi) = 1$ if χ is the possible real Dirichlet character mod q with the exceptional zero β_1, and let $\vartheta(\chi) = 0$ else. The quantities in which χ_1 and β_1 occur are 0 if these do not exist. Let $b(\chi) := \operatorname{Res}\big(L'(s,\chi)/sL(s,\chi), 0\big)$ and α be as in Theorem 23.

Theorem 28 *If χ is a non-principal Dirichlet character mod q then for any $x \geq 2$,*

$$\psi^*(x,\,\chi) = -\lim_{T\to+\infty} \sum_{\rho,\,|\mathrm{Im}(\rho)|\leq T} \frac{x^\rho}{\rho} - (1-\alpha)\log x - b(\chi) + \sum_{m\geq 1} \frac{x^{-2m+\alpha}}{2m-\alpha}\,.$$

Moreover, for any $q \geq 2$ and $2 \leq T \leq x$ we have that

$$\psi(x,\,\chi) = -\vartheta(\chi)\frac{x^{\beta_1}}{\beta_1} - \sum_{|\gamma|\leq T}^{*} \frac{x^\rho}{\rho} + R_q(x,\,T)$$

where $R_q(x,T) \ll x\log^2(qx)/T + x^{1/4}$ and where the asterisk indicates that if $\vartheta(\chi) = 1$, the zeros β_1 and $1-\beta_1$ are omitted from the sum.

Finally, the following are Theorem 8.29, Proposition 8.30, Corollary 8.31 and Theorems 8.32 and 8.33 (Siegel) in [1].

Theorem 29 *For any $\kappa > 0$ there exists a $c = c(\kappa) > 0$ such that for any $x \geq 2$, $1 \leq q \leq \mathrm{e}^{\kappa\sqrt{\log x}}$ and $a \in \mathbb{Z}$ with $(a,q) = 1$ it uniformly holds that*

$$\psi(x,\,a,\,q) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\beta_1} + O\big(x\mathrm{e}^{-c\sqrt{\log x}}\big)\,.$$

Proposition 30 *If $q \geq 3$ then*

$$1 - \beta_1 \gg L(1,\,\chi_1)/(\log q)^2 \gg 1/(\sqrt{q}(\log q)^2)\,.$$

Corollary 31 *There exists a $c > 0$ such that for any $h\colon \mathbb{R} \to \mathbb{R}$ for which $\lim_{x\to+\infty} h(x) = +\infty$ and for any $x \geq 3$, $1 \leq q \leq (\log x)^2/(h(x)^2(\log\log x)^6)$ and $a \in \mathbb{Z}$ with $(a,q) = 1$ it uniformly holds that*

$$\psi(x,a,q) = \frac{x}{\varphi(q)}\big(1 + O\big(1/(\log x)^{c\cdot h(x)}\big)\big)\,.$$

Theorem 32 *Suppose that $F(s) := \sum_{n\geq 1} a_n/n^s$ converges for $\sigma > 1$, that $a_1 \geq 1$ and $a_n \geq 0$ and that there is an $r \in (0,1)$ and an $M > 0$ such that*
(i) *$f(s) := F(s)(s-1)$ is holomorphic on $D := \{s \in \mathbb{C} \mid |s-1| \leq r\}$,*

(ii) $\sup_{s\in D}|f(s)| \le M$ *and*
(iii) $f(\beta) \ge 0$ *for some* $\beta \in [1 - r/2, 1)$.
Then

$$f(1) \ge \frac{c_1(1-\beta)}{M^{c_2(1-\beta)}}$$

where $c_i > 0$ *are constants depending only on* r.

Theorem 33 *Let* $q \ge 2$. *If* $\varepsilon > 0$ *then there is a* $c = c(\varepsilon) > 0$ *such that* $L(1,\chi) > c/q^{\varepsilon}$ *for all non-principal real Dirichlet characters* χ *mod* q. *I n particular*

$$1 - \beta_1 \gg_{\varepsilon} 1/q^{\varepsilon}\,.$$

References

[1] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)