Lecture 8. Tauberian Theorems

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In the eight lecture we cover Chapter II.7. *Tauberian Theorems* in G. Tenenbaum's book [1], up to page 359.

Chapter II.7. Tauberian Theorems

The following are Theorems 7.1 (Abel) and 7.2 in [1]. For $\vartheta \in [0, \frac{\pi}{2})$ let $S(\vartheta) = \{z \in \mathbb{C} \mid |z| < 1 \wedge |\arg(1-z)| \le \vartheta\}$.

Theorem 1 *Suppose that $f(z) = \sum_{n\ge 0} a_n z^n \in \mathbb{C}[[z]]$ has radius of convergence 1 and that $f(1) = \sum_{n\ge 0} a_n$ converges. Then for any $\vartheta \in [0, \frac{\pi}{2})$,*

$$\lim_{z\to 1} f(z) \mid S(\vartheta) = f(1)\,.$$

Theorem 2 *Suppose that $F(s) = \sum_{n\ge 1} a_n/n^s$ converges for $\sigma > a \ge 0$ and that there exist $c \in \mathbb{C}$ and $\omega > -1$ such that $\sum_{n\le x} a_n \sim \frac{c}{\Gamma(\omega+1)} \cdot x^a (\log x)^\omega$ $(x \to +\infty)$. Then for $\sigma \to a^+$,*

$$F(\sigma) \sim \frac{ca}{(\sigma - a)^{\omega+1}}\,.$$

The following are Theorems 7.3 (Tauber) and 7.4 in [1].

Theorem 3 *Suppose that $f(z) = \sum_{n\ge 0} a_n z^n \in \mathbb{C}[[z]]$ has radius of convergence 1, $\sum_{n\le x} n a_n = o(x)$ $(x \to +\infty)$ and that $\lim_{z\to 1} f(z) \mid [0,1) = \ell$. Then*

$$\sum_{n=0}^{\infty} a_n = \ell\,.$$

Proof. For $x > 0$ we set

$$A(x) = \sum_{n\le x} a_n \ \text{ and } \ \alpha(x) = \int_0^x t\,\mathrm{d}A(t)\,.$$

For $u > 0$ we set

$$G(u) = \frac{\mathrm{e}^{-u} - 1}{u},\ g(u) = -G'(u) = \frac{(1+u)\mathrm{e}^{-u} - 1}{u^2},\ H(u) = \frac{\mathrm{e}^{-u}}{u}$$

and $h(u) = -H'(u) = \frac{(1+u)\mathrm{e}^{-u}}{u^2}$. We have

$$
\begin{aligned}
f(\mathrm{e}^{-1/x}) - A(x) &= \frac{1}{x}\int_0^x G(t/x)t\,\mathrm{d}A(t) + \frac{1}{x}\int_x^{+\infty} H(t/x)t\,\mathrm{d}A(t) \\
&= \frac{1}{x}\int_0^x \int_{t/x}^{+\infty} g(u)\,\mathrm{d}u\,\mathrm{d}\alpha(t) + \frac{1}{x}\int_x^{+\infty}\int_{t/x}^{+\infty} h(u)\,\mathrm{d}u\,\mathrm{d}\alpha(t) \\
&= \frac{1}{x}\int_0^x g(u)\,\mathrm{d}u \int_0^{\min(1,u)x} \mathrm{d}\alpha(t) + \frac{1}{x}\int_1^{+\infty} h(u)\,\mathrm{d}u \int_x^{ux} \mathrm{d}\alpha(t) \\
&= \int_0^1 g(u)\frac{\alpha(ux)}{x}\,\mathrm{d}u - \frac{\alpha(x)}{x} + \int_1^{\infty} h(u)\frac{\alpha(ux)}{x}\,\mathrm{d}u\,.
\end{aligned}
$$

We have $\alpha(ux)/x \ll u$ uniformly in x since $\alpha(x)/x$ is bounded for $x > 0$. Since $\lim_{x\to+\infty} \alpha(ux)/x = 0$ for any fixed u and since $ug(u)$ and $uh(u)$ are integrable on $[0,1]$ and $[1,+\infty)$, respectively, the dominated convergence gives that

$$\lim_{x\to+\infty} \big(f\big(\mathrm{e}^{-1/x}\big) - A(x)\big) = 0\,.$$

This is what we wanted to prove. □

Theorem 4 *Suppose that $\omega > 0$, $A\colon (0,+\infty) \to \mathbb{R}$ has bounded variation on any bounded interval and that*

$$F(\sigma) = \int_0^{+\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t)$$

converges for $\sigma > 0$ and satisfies $F(\sigma) = o(1/\sigma^\omega)$ $(\sigma \to 0^+)$. Then

$$A(x) - A(0) = o(x^\omega) \iff \int_0^x t\,\mathrm{d}A(t) = o(x^{\omega+1}) \qquad (x \to +\infty)\,.$$

The following are Theorems 7.5 (Karamata), 7.6 (Landau), 7.7 (Generalized Hardy–Littlewood–Karamata), Corollaries 7.8 (Hardy–Littlewood) and 7.9 (Hardy–Littlewood–Karamata).

Theorem 5 *Let $A(t)$ be a non-decreasing function such that the integral $F(\sigma) = \int_0^{+\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t)$ converges for $\sigma > 0$. If there exist $c \geq 0$, $\omega > 0$ with $F(\sigma) \sim c/\sigma^\omega$ $(x \to 0^+)$, then*

$$A(x) \sim \frac{cx^\omega}{\Gamma(\omega+1)} \qquad (x \to +\infty)\,.$$

Theorem 6 *Let f'' exist on $(0,+\infty)$, $\alpha \in \mathbb{R}$, $M > 0$, $f(x) = o(x^\alpha)$ (both for $x \to 0^+$ and $x \to +\infty$) and $f''(x) \leq Mx^{\alpha-2}$ $(x > 0)$. Then $f'(x) = o(x^{\alpha-1})$ (both for $x \to 0^+$ and $x \to +\infty$).*

We define $\mathcal{V}^*(\mathbb{R}^+)$ to be the functions $A\colon \mathbb{R}^+ \to \mathbb{R}$ with bounded variation on any bounded interval and with the integral $\int_0^{+\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t)$ converging for $\sigma > 0$. We define $\mathcal{K}(\omega)$ to be the subclass of those functions A that for some $c \in \mathbb{R}$, $\sigma \to 0^+$ and $x \to +\infty$ satisfy that

$$F(\sigma) := \int_0^{+\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t) \sim c/\sigma^\omega \quad \text{and} \quad A(x) - A(0) \sim cx^\omega/\Gamma(\omega+1)\,.$$

Theorem 7 *Let $\omega \geq 0$ and $A \in \mathcal{V}^*(\mathbb{R}^+)$. If for some $c \in \mathbb{R}$ we have that*

$$F(\sigma) := \int_0^{\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t) \sim \frac{c}{\sigma^\omega} \qquad (x \to 0^+)$$

and $B \in \mathcal{K}(\omega+1)$ is such that the measure $t\,\mathrm{d}A(t) + \mathrm{d}B(t)$ is positive, then $A \in \mathcal{K}(\omega)$.

Corollary 8 *Suppose that $K > 0$ and that $f(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{R}[[z]]$ has radius of convergence 1 and $na_n \geq -K$. Then*

$$\lim_{z\to 1^-} f(z) = \ell \Rightarrow \sum_{n\geq 0} a_n = \ell\,.$$

Corollary 9 *Suppose that $F(s) = \sum_{n\geq 1} a_n/n^s$ converges for $\sigma > 1$, there exist real numbers $c, K > 0, \omega \geq 0$ such that $a_n \geq -K(\log n)^{\omega-1}$ $(n \geq 2)$ and that $F(\sigma) \sim c/(\sigma-1)^\omega$ $(\sigma \to 1^+)$. Then for $x \to +\infty$,*

$$\sum_{n\leq x} \frac{a_n}{n} \sim \frac{c}{\Gamma(\omega+1)}(\log x)^\omega\,.$$

The following are Theorems 7.10 (Karamata–Freud), 7.11 and Lemma 7.12 in [1].

Theorem 10 *Suppose that $A(t)$ is non-decreasing and such that the integral*

$$F(\sigma) := \int_0^{\infty} \mathrm{e}^{-\sigma t}\,\mathrm{d}A(t)$$

converges for $\sigma > 0$. Suppose that $c \geq 0$, $\omega > 0$ and $\psi(t)$ is non-decreasing with $\lim_{t\to+\infty} \psi(t) = +\infty$, $\psi(t)/t^\omega$ is non-decreasing for large t and $F(\sigma) = \big(c + O(1/\psi(1/\sigma))\big)\sigma^{-\omega}$ $(\sigma \to 0^+)$. Then

$$A(x) = \big(c + O(1/\log(\psi(x)))\big)\frac{x^\omega}{\Gamma(\omega+1)}\,. \qquad (x \to +\infty)$$

In the following $\ell(p)$ denotes the sum of absolute values of the coefficients of a polynomial p.

Theorem 11 *Suppose that $f(t)$ has bounded variation on $[0,1]$. Then there exist constants A_1 and A_2 (depending only on f) such that for any $n \in \mathbb{N}$ there exist polynomials $p, q \in \mathbb{R}[t]$ with degrees at most n such that*

$$p \leq f \leq q \quad (\textit{on } [0,1]) \wedge \int_0^1 (q-p) \leq \frac{A_1}{n} \wedge \ell(p) + \ell(q) \leq A_2^n \,.$$

Lemma 12 *For $n \in \mathbb{N}$ let the n-th Chebyshev polynomial T_n be defined by $T_n(\cos x) = \cos(nx)$. Then*

$$T_n(x) = \tfrac{1}{2} \sum_{0 \leq m \leq n/2} (-1)^m \frac{n}{n-m} \binom{n-m}{m} (2x)^{n-2m}$$

and $\ell(T_n) \leq n(1+\sqrt{2})^n$.

The following are Theorems 7.13 ("Effective" Ikehara–Ingham–Delange theorem), 7.14 (Ganelius) and 7.15.

Theorem 13 *Let $A(t)$ be a non-decreaing function such that the integral*

$$F(s) = \int_0^{+\infty} \mathrm{e}^{-st} \, \mathrm{d}A(t)$$

converges for $\sigma > a > 0$. Suppose that there exist $c \geq 0$ and $\omega > -1$ such that the function

$$G(s) := \frac{F(s+a)}{s+a} - \frac{c}{s^{\omega+1}} \qquad (\sigma > 0)$$

satisfies

$$\eta(\sigma, T) := \sigma^\omega \int_{-T}^{T} |G(2\sigma + i\tau) - G(\sigma + i\tau)| \, \mathrm{d}\tau = o(1) \qquad (\sigma \to 0^+)$$

for each fixed $T > 0$. Then

$$A(x) = \big(c/\Gamma(\omega+1) + O(\rho(x))\big) \mathrm{e}^{ax} x^\omega \qquad (x \geq 1)$$

with

$$\rho(x) := \inf_{T \geq 32(a+1)} \big(1/T + \eta(1/x, T) + 1/(Tx)^{\omega+1}\big) \,.$$

The big O constant for $A(x)$ depends only on a, c and ω. An admissible choice for this constant is

$$52 + 1652c(a+1)(\omega+2) + 69c(1 + \mathrm{e}^{1-\omega}(\omega+1)^{\omega+2})/\Gamma(\omega+1) \,.$$

Theorem 14 *Suppose that $g \colon \mathbb{R} \to \mathbb{R}$ is integrable and bounded, that for some $T > 0$ one has that*

$$\sup_{x \leq y \leq x+1/T} (g(y) - g(x)) \leq K < \infty$$

and that for $|\tau| \le T$*,*

$$\widehat{g}(\tau) := \int_{-\infty}^{\infty} \mathrm{e}^{-i\tau x} g(x)\,\mathrm{d}x = 0\,.$$

Then

$$\|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)| \le 16K\,.$$

Theorem 15 *Suppose that* $g\colon \mathbb{R} \to \mathbb{R}$ *is integrable and bounded and that for some* $T > 0$ *one has that*

$$\sup_{x\le y\le x+1/T} (g(y) - g(x)) \le K < \infty\,.$$

Then

$$\|g\|_\infty \le 16K + 6\int_{-T}^{T} |\widehat{g}(\tau)|\,\mathrm{d}\tau\,.$$

The next is Theorem 7.16 (Berry–Essen) in [1]. A function $F\colon \mathbb{R} \to [0, +\infty)$ is a *distribution function* if it is non-decreasing and

$$\lim_{x\to-\infty} F(x) = 0 \wedge \lim_{x\to+\infty} F(x) = 1\,.$$

The *characteristic function* of F is then

$$f(\tau) := \int_{-\infty}^{+\infty} \mathrm{e}^{i\tau x}\,\mathrm{d}F(x)\,.$$

Theorem 16 *Let* F *and* G *be two distribution functions with respective characteristic functions* f *and* g*. Suppose that on* $\mathbb{R}$ *the function* G *is differentiable with* G' *bounded. Then for any* $T > 0$*,*

$$\|F - G\|_\infty \le 16\frac{\|G'\|_\infty}{T} + 6\int_{-T}^{T} \left|\frac{f(\tau) - g(\tau)}{\tau}\right| \mathrm{d}\tau\,.$$

The following are Theorems 7.17, 7.18 (Fatou–Korevaar), 7.19 and 7.20.

Theorem 17 *Let* g *be an integrable and bounded function on* $\mathbb{R}$*, let* $C > 0$ *and let* $\psi\colon \mathbb{R} \to [1, +\infty)$ *be a non-decreasing function such that* $\psi(x) = 1$ *for* $x \le 0$ *and* $\psi(2x) \le C\psi(x)$ *for* $x \ge 0$*. Suppose that for some* $T > 0$ *we have that* (i) $\sup_{0\le y\le 1/T}(g(x + y) - g(x)) \le K/\psi(x)$ *(*$x \in \mathbb{R}$*) and* (ii) $\widehat{g} \in \mathcal{C}^\infty([-T, T])$*. Then there is an* $N = N(C, K) \in \mathbb{N}$ *and an* $A = A(C, K, T, M) > 0$*, where* $M = \sup_{n\le N} \|\widehat{g}^{(n)}\|_{[-T,T]}$*, such that for any* $x \in \mathbb{R}$*,*

$$|g(x)| \le A/\psi(x)\,.$$

Theorem 18 *Suppose that* $f(z) = \sum_{n\ge0} a_n z^n \in \mathbb{C}[[z]]$ *converges for* $|z| < 1$ *and has a holomorphic extension to* $z = 1$*. If there is a function* ψ *satisfying assumptions of Theorem 17 and such that* $a_n \ge -1/\psi(n)$ *then* $\sum_{n\ge0} a_n = f(1)$ *and for* $x \to +\infty$*,*

$$\sum_{n\le x} a_n = f(1) + O\big(1/\psi(x)\big)\,.$$

Theorem 19 *Suppose that* $F(s) = \sum_{n\ge1} a_n/n^s$ *converges for* $\sigma > 1$ *and has a holomorphic extension to* $s = 1$*. If there is a function* ψ *satisfying assumptions of Theorem 17 and such that* $a_n \ge -1/\psi(\log n)$ $(n \ge 2)$*, then for* $x \to +\infty$*,*

$$\sum_{n\le x} \frac{a_n}{n} = F(1) + O\big(1/\psi(\log x)\big)\,.$$

Theorem 20 *In Theorem 19 the conclusion persists under the assumption that for* $n \ge 2$*,*

$$\sum_{m\le n} a_m \ll n/\psi(\log n)\,.$$

Finally, the following are Theorems 7.21 (Ingham, 1945) and 7.22 (Landau, 1910), Corollary 7.23 (Ingham, 1945), Theorem 7.24 (Skałba, 1998), Corollary 7.25 (Skałba, 1998), Theorem 7.26 (Drmota, 1998) and Lemma 7.27 (Regularity of Lambert's summation method).

Let $f, g\colon \mathbb{N} \to \mathbb{C}$ be such that $f(n) = \sum_{d\,|\,n} g(d)$ and let $F(x) = \sum_{n\le x} f(n)$ and $G(x) = \sum_{n\le x} g(n)$.

Theorem 21 *If* $F(x) \sim ax$ $(x \to +\infty)$ *then for* $x \to +\infty$*,*

$$\sum_{n\le x} \frac{g(n)}{n} = \frac{G(x)}{x} + a + o(1)\,.$$

Theorem 22 *If* g *is real and for any* $n \in \mathbb{N}$ *one has that* $g(n) \ge -K$ *(for a constant* $K > 0$*) then the claim of Theorem 21 implies that*

$$\sum_{n\ge1} \frac{g(n)}{n} = a\,.$$

Corollary 23 *If* $F(x) \sim ax$ $(x \to +\infty)$ *and* $g(n)$ *is bounded from below then*

$$\sum_{n\ge1} \frac{g(n)}{n} = a\,.$$

Theorem 24 *If* $F(x) \sim ax$ $(x \to +\infty)$ *and* $f(n) \ll 1$ *then* $G(x) = o(x)$ $(x \to +\infty)$ *and*

$$\sum_{n\ge1} \frac{g(n)}{n} = a\,.$$

Corollary 25 *If* $f\colon \mathbb{R} \to \mathbb{C}$ *is* 2π*-periodic and Riemann integrable, then*

$$\frac{1}{2\pi}\int_0^{2\pi} f(u)\,\mathrm{d}u = \sum_{n\ge1} \frac{g(n)}{n}\,.$$

Theorem 26 *If f is real and bounded from below and*

$$\sum_{n\geq 1} \frac{g(n)}{n} = a$$

then $F(x) \sim ax$ $(x \to +\infty)$.

Lemma 27 *Let $\varphi(y) = y/(\mathrm{e}^y - 1)$. If $\sum_n a_n = a$ then for $y \to 0^+$,*

$$\sum_n a_n \varphi(ny) = a + o(1)\,.$$

References

[1] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)