Lecture 6. Summation formulae. The Riemann zeta function

M. Klazar

April 5, 2024

In the sixth lecture we cover Chapter II.2. *Summation formulae* and Chapter II.3. *The Riemann zeta function*, up to page 261, in [5].

Chapter II.2. Summation formulae

The following are Theorem 2.1 (Perron's formula), Lemma 2.2, Theorem 2.3 (First effective Perron formula), Corollary 2.4 (Second effective Perron formula), Theorem 2.5, Lemma 2.6 and Theorem 2.7 in [5].

Recall that σ_c and σ_a are abscissas of convergence and of absolute convergence of a Dirichlet series $F(s)=\sum_{n\ge 1}a_n/n^s$. For real x we define $A^*(x)=\sum_{n<x}a_n+a_x/2$ where $a_x=0$ if $x\not\in\mathbb{N}$.

Theorem 1 *For $\kappa>\max(\{0,\sigma_c\})$ and $x>0$,*

$$A^*(x)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}F(s)x^s\,\frac{\mathrm{d}s}{s}$$

where the $\int$ conditionally converges if $x\not\in\mathbb{N}_0$ and converges "in the sense of Cauchy's principal value" if $x\in\mathbb{N}_0$.

Lemma 2 *Let $h(x)=0$ for $x\in(0,1)$, $\frac{1}{2}$ for $x=1$ and 1 for $x>1$. Then for any $\kappa,T,T'>0$,*

$$\left|h(x)-\int_{\kappa-iT'}^{\kappa+iT}x^s\,\frac{\mathrm{d}s}{s}\right|\le\frac{x^\kappa}{2\pi|\log x|}\cdot(1/T+1/T')\ \ldots\ x\ne 1$$

and

$$\left|h(1)-\int_{\kappa-iT}^{\kappa+iT}\frac{\mathrm{d}s}{s}\right|\le\frac{\kappa}{T+\kappa}\,.$$

Theorem 3 *For any $\kappa>\max(\{0,\sigma_a\})$ and $T,x\ge 1$,*

$$A(x)=\sum_{n\le x}a_n=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}F(s)x^s\,\frac{\mathrm{d}s}{s}+O\bigg(x^\kappa\sum_{n\ge 1}\frac{|a_n|}{n^\kappa(1+T|\log(x/n)|)}\bigg)\,.$$

Corollary 4 *Let $F(s) = \sum_{n\geq 1} a_n/n^s$ have finite σ_a. Suppose that for a real $\alpha \geq 0$ we have that $\sum_{n\geq 1} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha}$ for $\sigma \in (\sigma_s, \sigma_a + 1]$ and that $|a_n| \leq B(n)$ for $n \in \mathbb{N}$ and a non-decreasing function B. Then for $x, T \geq 2$, $\sigma \leq \sigma_a$ and $\kappa = \sigma_a - \sigma + 1/\log x$,*

$$\begin{aligned}\sum_{n\leq x} \frac{a_n}{n^s} &= \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(s+w)x^w \frac{\mathrm{d}w}{w} \\ &+ O\big(x^{\sigma_a-\sigma}(\log^\alpha x)/T + (B(2x)/x^\sigma)(1 + x(\log T)/T)\big)\,.\end{aligned}$$

Theorem 5 *For any $\kappa > \max(\{0, \sigma_c\})$ and $x \geq 1$,*

$$\sum_{n\leq x} a_n \log(x/n) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} F(s)x^s \frac{\mathrm{d}s}{s^2}$$

and

$$\int_0^x A(t)\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} F(s)x^{s+1} \frac{\mathrm{d}s}{s(s+1)}\,.$$

In the next lemma the function h is as in Lemma 2

Lemma 6 *For any $\delta > 0$ there exists $a = a(\delta), b = b(\delta) \in \mathbb{C}$ such that with ($s \in \mathbb{C}$ and $y > 0$) $w_\delta(s) = 1/s + a/(s+1) + b/(s+2)$ and $g_\delta(y) = h(y)(1 + a/y + b/y^2)$ we have uniformly for $y, \kappa > 0$ that*

$$\frac{1}{2\pi i}\int_{\kappa-i\delta}^{\kappa+i\delta} w_\delta(s)y^s\,\mathrm{d}s = g_\delta(y) + O\big(y^\kappa/(1+\log^2 y) + \kappa y^\kappa\big)\,.$$

Theorem 7 *Let $F(s) = \sum_{n\geq 1} a_n/n^s$ converge for $\sigma > 1$ and be such that $\sum_{n\leq x} |a_n| \ll x \cdot b(x)$ for $x \geq 2$ and a non-decreasing function $b(x)$. Then for $\kappa = \kappa_x = 1/\log x$,*

$$\sum_{n\leq x} \frac{a_n}{n} = \frac{1}{2\pi i}\int_{\kappa_x-i\delta}^{\kappa_x+i\delta} F(s+1)w_\delta(s)x^s\,\mathrm{d}s + O(b^*(x))$$

where $b^*(x) = \kappa_x \int_1^\infty b(t)/t^{1+\kappa_x}\,\mathrm{d}t$.

The following are Theorems 2.8 and 2.9 in [5]. For the function μ see the chapter on Dirichlet series.

Theorem 8 *Let $F(s) = \sum_{n\geq 1} a_n/n^s$ have finite abscissa σ_c. If $\sigma_0 \in \mathbb{R}$ is such that $F(s)$ has an analytic extension satisfying $\mu(\sigma) = 0$ for $\sigma > \sigma_0$ then $\sigma_c \leq \sigma_0$.*

This theorem is due to E. Landau [2] in 1909.

Theorem 9 *Let $F(s) = \sum_{n\geq 1} a_n/n^s$ converge for $\sigma > 1$, have an analytic extension to a neighborhood of $s = 1$ and $\sum_{n\leq x} a_n = o(x)$ ($x \to +\infty$), then $F(1)$ converges.*

This theorem is due to M. Riesz [4] in 1909.

Finally, the following are Theorem 2.10 and Corollaries 2.11 and 2.12 in [5].

Theorem 10 *Let* $F(s) = \sum_{n\geq 1} a_n/n^s$ *have* $\sigma_a =: \sigma_1$ *and* $G(s) = \sum_{n\geq 1} b_n/n^s$ *have* $\sigma_a =: \sigma_2$*. Then for any* $\alpha > \sigma_1$ *and* $\beta > \sigma_2$*,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(\alpha + i\tau)G(\beta - i\tau)\,\mathrm{d}\tau = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\alpha+\beta}}\,.$$

Corollary 11 *For any* $\sigma > \sigma_a$*,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(s)|^2\,\mathrm{d}\tau = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}\,.$$

Corollary 12 *For any* $\sigma > \sigma_a$*,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(s)n^s\,\mathrm{d}\tau = a_n\,.$$

Chapter II.3. The Riemann zeta function

This chapter is devoted to the function (Dirichlet series)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1\,.$$

Next in [5] come Theorems 3.1 and 3.2.

Theorem 13 $\zeta(s)$ *extends to* $\zeta(s) = \frac{1}{s-1} + f(s)$ *with entire* $f\colon \mathbb{C} \to \mathbb{C}$*.*

Theorem 14 *For* $n \in \mathbb{N}_0$*,*

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where B_n *are the Bernoulli numbers defined in Lecture 1. Hence* $\zeta(-2n) = 0$ *for* $n \in \mathbb{N}$*.*

Next in [5] come Theorems 3.3 and 3.4.

Theorem 15 *For any* $s \in \mathbb{C} \setminus \{1\}$*,*

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \cdot \Gamma(1-s) \cdot \zeta(1-s)\,.$$

Several proofs of the functional equation for $\zeta(s)$ can be found in [6].

Theorem 16 *For any $n \in \mathbb{N}$,*

$$\zeta(2n) = (-1)^{n-1} 2^{2n-1} \frac{B_{2n}}{(2n)!} \pi^{2n} .$$

The following are Theorem 3.5, Lemma 3.6, Corollary 3.7, Theorems 3.8 and 3.9 and Corollary 3.10 in [5]; $s = \sigma + i\tau$.

Theorem 17 *Let $\sigma_0 > 0$ and $\delta \in (0,1)$. Then it holds uniformly for $\sigma \geq \sigma_0$, $x \geq 1$ and $0 < |\tau| \leq (1-\delta)2\pi x$ that*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(1/x^{\sigma}) .$$

This theorem is due to G. H. Hardy and J. E. Littlewood [1] in 1921. The proof uses the next lemma.

Lemma 18 *For any $\sigma > 0$ and $N \in \mathbb{N}$,*

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s \int_N^{+\infty} \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t .$$

Corollary 19 *For $\tau \geq 1$ it holds that $\zeta(\frac{1}{2} + i\tau) \ll \tau^{1/6} \log \tau$.*

Theorem 20 *For $\zeta(s)$ we have the bound $\mu(\sigma) \leq (1-\sigma)/3$ for $\sigma \in [\frac{1}{2}, 1]$ and $\mu(\sigma) \leq (3 - 4\sigma)/6$ for $\sigma \in [0, \frac{1}{2}]$.*

Theorem 21 *For any $\alpha \in (0,1)$, $\sigma \geq \alpha$ and $|\tau| \geq 1$,*

$$|\zeta(s)| \leq \frac{3|\tau|^{1-\alpha}}{2\alpha(1-\alpha)} .$$

In particular, for any $c > 0$, $|\tau| \geq 2$ and $\sigma \geq 1 - c/\log(|\tau|)$,

$$\zeta(s) \ll \log(|\tau|) .$$

Corollary 22 *For any $c > 0$, $k \in \mathbb{N}_0$, $|\tau| \geq 2$ and $\sigma \geq 1 - c/\log(|\tau|)$,*

$$\zeta^{(k)}(s) \ll \big(\log(|\tau|)\big)^{k+1} .$$

The following are Theorem 3.11 (Mertens), Corollary 3.12, Theorem 3.13 and Corollary 3.14 in [5].

Theorem 23 *Let $F(s) = \sum_{n \geq 1} a_n/n^s$ have $a_n \geq 0$ and $\sigma_c \in \mathbb{R}$. Then for any $\sigma > \sigma_c$,*

$$3F(\sigma) + 4\mathrm{re}(F(\sigma + i\tau)) + \mathrm{re}(F(\sigma + 2i\tau)) \geq 0.$$

Proof. The left side is $\sum_{n \geq 1} a_n V(\tau \log n)/n^{\sigma}$ where $V(\vartheta) = 3 + 4\cos\vartheta + \cos(2\vartheta) = 2(1 + \cos\vartheta)^2 \geq 0$. □

Corollary 24 *For any $\sigma > 1$,*

$$\zeta(\sigma)^3 \cdot |\zeta(\sigma + i\tau)|^4 \cdot |\zeta(\sigma + 2i\tau)| \geq 1 \,.$$

Proof. Apply the theorem to the Dirichlet series ($\sigma > 1$)

$$F(s) = \log(\zeta(s)) = -\sum_p (1 - p^{-s}) = \sum_{n\geq 2} \frac{\Lambda(n)}{n^s \log n} \,.$$

□

Theorem 25 *$\zeta(s) \neq 0$ for $\sigma \geq 1$.*

Proof. For contrary, let $\zeta(1 + i\tau_0) = 0$. Then $\tau_0 \neq 0$ and $\zeta(s)$ is holomorphic in a neighborhood of $1 + i\tau_0$. Thus for $\sigma > 1$, $\zeta(\sigma + i\tau_0) \ll \sigma - 1$. On the other hand, for $\sigma > 1$ we have that $\zeta(\sigma) \ll 1/(\sigma - 1)$ and $\zeta(\sigma + 2i\tau_0) \ll 1$ ($s = 1$ is a simple pole). Hence for $\sigma > 1$,

$$\zeta(\sigma)^3 \cdot |\zeta(\sigma + i\tau_0)|^4 \cdot |\zeta(\sigma + 2i\tau_0)| \ll \sigma - 1$$

which contradicts the corollary. □

Corollary 26 *In the halfplane $\sigma \leq 0$ the only zeros of $\zeta(s)$ are in $s = -2n$ for $n \in \mathbb{N}$.*

The following are Theorems 3.15 (Jensen's formula) and 3.16 (Real part lemma, Borel–Carathéodory) and Corollary 3.17 in [5].

Theorem 27 *Let $R > 0$ and $F(s)$ be holomorphic for $|s| \leq R$ with $F(0) = 1$. Then*

$$\int_0^R \frac{|\{s \in \mathbb{C} \mid |s| \leq r \wedge F(s) = 0\}|}{r} \,\mathrm{d}r = \frac{1}{2\pi} \int_0^{2\pi} \log\left(|F(R\mathrm{e}^{i\vartheta})|\right) \mathrm{d}\vartheta \,.$$

Theorem 28 *Let $R > 0$ and $F(s)$ be holomorphic for $|s| \leq R$ with $F(0) = 0$. If $\max_{|s|=R} \mathrm{re}(F(s)) \leq A$ then for any $k \in \mathbb{N}_0$ and s with $|s| < R$,*

$$\left|F^{(k)}(s)\right| \leq \frac{2AR \cdot k!}{(R - |s|)^{k+1}} \,.$$

Corollary 29 *Let $R > 0$, $F(s)$ be holomorphic for $|s| \leq R$ with $F(0) = 1$ and let $|F(s)| \leq M$ for $|s| = R$. Let Z be the finite sequence of zeros of $F(s)$ in $|s| \leq R/2$, each counted with its multiplicity. Then for any $s \in \mathbb{C}$ with $|s| < R/2$,*

$$\left|\frac{F'(s)}{F(s)} - \sum_{\rho\in Z} \frac{1}{s - \rho}\right| \leq \frac{4R \log M}{(R - 2|s|)^2} \,.$$

The following are Theorems 3.18 and 3.19 in [5]. Let

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)\,.$$

The non-trivial zeros of $\zeta(s)$ are denoted as $\rho = \beta + i\gamma$. For $T > 0$ we denote by $N(T)$ the number of ρ (counted with multiplicity) with $0 \le \gamma \le T$.

Theorem 30 *For $T \ge 2$,*

$$N(T+1) - N(T) \ll \log T\,.$$

Theorem 31 *For $T \ge 2$,*

$$N(T) = (T/2\pi)\log(T/2\pi) - T/2\pi + O(\log T)\,.$$

This asymptotics had been conjectured by B. Riemann and was proven by H. von Mangoldt [3] in 1895.

Finally, the following are Theorems 3.20 (Hadamard product formula), 3.21 and 3.22 in [5].

Theorem 32 *For $a = \frac{1}{2}\log(4\pi) - \frac{1}{2}\gamma - 1$, $b = \log(2\pi) - \frac{1}{2}\gamma - 1$ and every $s \in \mathbb{C}$,*

$$\xi(s) = \mathrm{e}^{as}\prod_\rho (1 - s/\rho)\mathrm{e}^{s/\rho} \;\; \textit{and} \;\; \zeta(s) = \frac{\mathrm{e}^{bs}}{2(s-1)} \cdot \frac{1}{\Gamma(s/2+1)}\prod_\rho (1 - s/\rho)\mathrm{e}^{s/\rho}$$

$(s \ne 1)$.

Theorem 33 *There is a $c > 0$ such that $\sigma \ge 1 - c/\log(2 + |\tau|) \Rightarrow \zeta(s) \ne 0$.*

Theorem 34 *There is a $c > 0$ such that if $|\tau| \ge 3$ and $\sigma \ge 1 - c/\log(|\tau|)$ then*

$$\frac{\zeta'(s)}{\zeta(s)},\, \frac{1}{\zeta(s)} \ll \log(|\tau|) \;\; \textit{and} \;\; |\log\zeta(s)| \le \log\log(|\tau|) + O(1)\,.$$

References

[1] G. H. Hardy and J. E. Littlewood, The zeros of Riemann's zeta function on the critical line, *Math. Z.* **10** (1921), 283–317

[2] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Teubner, Leipzig 1909

[3] H. von Mangoldt, Zu Riemanns Abhandlung 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse', *Journal für die reine und angewandte Mathematik* **114** (1895), 255–305

[4] M. Riesz, Sur les séries de Dirichlet et les séries entières, *C. R. Acad. Sci. Paris* **149** (1909), 909–912

[5] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)

[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford 1986 (Second Edition, revised by D. R. Heath-Brown)