Lecture 5. The function Γ. Dirichlet series

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In the fifth lecture we arrive in the second part *Complex Analysis Methods* of G. Tenenbaum's book [8]. We cover Chapter II.0. *The Euler Gamma function* and Chapter II.1. *Generating functions: Dirichlet series*, up to page 217.

Chapter II.0. The Euler Gamma function

The following are Theorems 0.1 (Euler) and 0.2 (Functional equation) and Corollary 0.3 in [8]. Two equivalent definitions of the function Γ are due to L. Euler: for complex numbers $s = \sigma + i\tau$,

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\frac{(1+1/n)^s}{1+s/n}\ \ (s \notin \mathbb{Z}\setminus\mathbb{N})\ \text{ and }\ \Gamma(s) = \int_0^{+\infty} t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t\ \ (\sigma > 0)\,.$$

Both by [8, p. 169] and [9, p. 101], the function Γ appeared first in a letter of L. Euler to Ch. Goldbach in 1729.

Theorem 1 *For $n \in \mathbb{N}$ let*

$$\Gamma_n(s) = \int_0^n \big(1 - t/n\big)^n t^{s-1}\mathrm{d}t \quad (\sigma > 0)\,.$$

Then we have

$$\Gamma_n(s) = \frac{n^s n!}{s(s+1)\dots(s+n)}\,,$$

and

$$\lim_{n\to\infty}\Gamma_n(s) = \Gamma(s) = \int_0^{+\infty} t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad (\sigma > 0)\,.$$

Theorem 2 *We have*

$$\Gamma(s+1) = s\Gamma(s) \quad (\sigma > 0)\,.$$

Corollary 3 *For all $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$.*

The following is Theorem 0.4 in [8]. Recall that a function $f\colon I \to (0, +\infty)$, where $I \subset \mathbb{R}$ is an interval, is *logarithmically convex (on I)* if the composition $\log(f)\colon I \to \mathbb{R}$ is a convex function.

Theorem 4 *The function* Γ *is logarithmically convex on* $(0, +\infty)$.

The following is Theorem 0.5 (Artin) in [8].

Theorem 5 *Suppose that* $\Phi\colon (0, +\infty) \to (0, +\infty)$ *is differentiable, logarithmically convex and that* $x\Phi(x) = \Phi(x+1)$ *for any* $x > 0$. *Then for any* $x > 0$,

$$\Phi(x) = \Phi(1)\Gamma(x)\,.$$

Theorem 5 is due to *Emil Artin (1898–1962)* who grew up in Reichenberg in Böhmen, today Liberec v Čechách (Czechia). In [7, Appendix C] the references given for Artin's theorem are [1, 2].

The following are Theorem 0.6 (Weierstrass) and Corollary 0.7 in [8]. As usual, $s = \sigma + i\tau$ and γ is the Euler constant.

Theorem 6 *For any* $\sigma > 0$,

$$\frac{1}{\Gamma(s)} = s\mathrm{e}^{\gamma s} \cdot \prod_{j=1}^{\infty} \big(1 + s/j\big)\mathrm{e}^{-s/j}\,.$$

The right side defines an entire continuation of $1/\Gamma(s)$.

This theorem is due to *Karl Weierstraß (1815–1897)*. [7, Appendix C] gives the reference [10].

Corollary 7 *We have* $\gamma = -\Gamma'(1)$.

The following are Theorem 0.8 and Corollaries 0.9, 0.10 (Real Stirling's formula) and 0.11 (Legendre duplication formula) in [8].

Theorem 8 *For any real* $x, y > 0$ *the beta function*

$$B(x,\,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.$$

Proof. This is the first of two proofs in [8] and goes by change of variables and the Fubini theorem. The second proof uses Artin's theorem. From

$$\Gamma(x)\Gamma(y) = \int_0^{+\infty}\int_0^{+\infty} t^{x-1}\mathrm{e}^{-t}u^{y-1}\mathrm{e}^{-u}\,\mathrm{d}t\,\mathrm{d}u$$

we get by introducing the variable v via $u = tv$ and by using the Fubini theorem that $\Gamma(x)\Gamma(y) = \int_0^{+\infty}\int_0^{+\infty} t^{x-1}\mathrm{e}^{-t}t^y v^{y-1}\mathrm{e}^{-vt}\,\mathrm{d}t\,\mathrm{d}v$ indeed equals

$$\begin{aligned}
&\int_0^{+\infty} v^{y-1}\int_0^{+\infty}\frac{(t(v+1))^{x+y-1}\mathrm{e}^{-(v+1)t}}{(v+1)^{x+y-1}}\,\mathrm{d}t\,\mathrm{d}v\\
=\;&\int_0^{+\infty}\frac{v^{y-1}\Gamma(x+y)}{(v+1)^{x+y}}\,\mathrm{d}v = \Gamma(x+y)\int_0^{+\infty}\big(\tfrac{v}{v+1}\big)^{y-1}\big(\tfrac{1}{v+1}\big)^{x-1}\frac{\mathrm{d}v}{(v+1)^2}\\
=\;&\Gamma(x+y)\int_0^1 w^{y-1}(1-w)^{x-1}\,\mathrm{d}w = \Gamma(x+y)B(x,\,y)\,.
\end{aligned}$$

□

Corollary 9 *For any real $x, y > 0$,*

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} (\sin\vartheta)^{2x-1}(\cos\vartheta)^{2y-1}\,\mathrm{d}\vartheta$$

In particular, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Corollary 10 *We have ($x \in \mathbb{R}$)*

$$\Gamma(x+1) \sim x^x \mathrm{e}^{-x}\sqrt{2\pi x} \qquad (x \to +\infty)\,.$$

Corollary 11 *For any $x > 0$,*

$$\Gamma(x/2)\cdot\Gamma((x+1)/2) = \sqrt{\pi}\cdot 2^{1-x}\Gamma(x)\,.$$

[7, Appendix C] gives for this duplication formula of *Adrien-Marie Legendre (1752–1833)* (for the troubles with his portrait see [6]) the reference [5].

The following are Theorem 0.12 (Complex Stirling's formula) and Corollaries 0.13 (Behavior in vertical strips), 0.14 (Mellin inversion formula), 0.15 (Reflection formula) and 0.16 (Euler) in [8].

Theorem 12 *For any $s \in \mathbb{C} \setminus (-\infty, 0]$,*

$$\mathrm{Log}(\Gamma(s)) = (s - \tfrac{1}{2})\mathrm{Log}\, s - s + \tfrac{1}{2}\log(2\pi) - \int_0^{+\infty} B_1(t)\frac{\mathrm{d}t}{s+t}\,.$$

Here $\mathrm{Log}\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is the so called *principal branch* of the complex logarithm: $\mathrm{Log}\big(r\exp(i\varphi)\big) = \log r + i\varphi$ for any real $r > 0$ and $\varphi \in (-\pi, \pi)$. On $(0, +\infty)$ it coincides with the real logarithm log. $B_1(t)$ is the 1-periodic extension of the first Bernoulli polynomial $b_1(t) = t - \frac{1}{2}\colon [0, 1) \to \mathbb{R}$, see the Euler–Maclaurin summation in Lecture 1.

Corollary 13 *Let $\sigma_2 > \sigma_1$ be real and $h_\sigma(\tau) = \tau\log|\tau| - \tau + \frac{1}{2}\pi(\sigma - \frac{1}{2})\mathrm{sgn}\tau$. Then it holds uniformly for $\sigma \in [\sigma_1, \sigma_2]$ and $|\tau| \geq 1$ that*

$$\Gamma(s) = \big(1 + O(1/\tau)\big)\sqrt{2\pi}\cdot|\tau|^{\sigma-1/2}\mathrm{e}^{-\pi|\tau|/2}\mathrm{e}^{ih_\sigma(\tau)}\,.$$

Here as usual $s = \sigma + i\tau$.

Corollary 14 *For any $x, \sigma > 0$,*

$$\mathrm{e}^{-x} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)x^{-s}\,\mathrm{d}s\,.$$

Corollary 15 *For any $s \in \mathbb{C} \setminus \mathbb{Z}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}\,.$$

Corollary 16 *For any $z \in C$,*

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big)\,.$$

The following are Theorem 0.17 (Hankel's formula) and Corollary 0.18 in [8]. *Hankel's contour* $H = H(R)$ for $R \in (0, 1)$ is the submap of

$$(0,\, +\infty) \times [-\pi,\, \pi] \ni (r,\, \varphi) \mapsto r\exp(i\varphi) = s \in \mathbb{C}$$

such that first r runs from $+\infty$ to R and $\varphi = -\pi$, then $r = R$ and φ runs from $-\pi$ to π, and finally r runs from R to $+\infty$ and $\varphi = \pi$. See [8, p. 179] for a picture.

Theorem 17 *Let H be a Hankel contour. Then for any $z \in \mathbb{C}$,*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_H s^{-z}\mathrm{e}^s\,\mathrm{d}s\,.$$

Corollary 18 *For $X > 1$, let $H(X)$ be the restriction of H obtained by replacing $+\infty$ with X. Then it holds uniformly in $z \in \mathbb{C}$ that*

$$\frac{1}{2\pi i}\int_{H(X)} s^{-z}\mathrm{e}^s\,\mathrm{d}s = \frac{1}{\Gamma(z)} + O\big(47^{|z|}\Gamma(1+|z|)\mathrm{e}^{-X/2}\big)\,.$$

Chapter II.1. Generating functions: Dirichlet series

Next in [8] come Definition 1.1 and Theorem 1.2. Recall that $*$ is the Dirichlet convolution.

Definition 19 *Let $f\colon \mathbb{N} \to \mathbb{C}$. The Dirichlet series of f is the function*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

defined for any $s \in \mathbb{C}$ where this series converges.

Here $n^s = \exp(s\log n)$ where $\exp z = \sum_{n\ge 0} z^n/n!$ for any $z \in \mathbb{C}$. The real $\log x\colon (0, +\infty) \to \mathbb{R}$ is the inverse of the real $\exp x\colon \mathbb{R} \to (0, +\infty)$.

Theorem 20 *Let $s \in \mathbb{C}$ and $f, g, h\colon \mathbb{N} \to \mathbb{C}$, with respective Dirichlet series F, G, H. If $h = f * g$ and both $F(s)$ and $G(s)$ absolutely converge, the so does $H(s)$ and $H(s) = F(s)G(s)$.*

Next in [8] come Theorem 1.3 and Proposition 1.4 (Euler's formula). In the latter proposition, $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

Theorem 21 *Let $s \in \mathbb{C}$ and $f\colon \mathbb{N} \to \mathbb{C}$ be multiplicative, with the Dirichlet series F, and let $\sum_p \sum_{\nu=1}^{\infty} |f(p^\nu)/p^{\nu s}| < +\infty$. Then*

$$F(s) = \sum_{n=1}^{\infty} f(n)/n^s \text{ absolutely converges and } F(s) = \prod_p \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^{\nu s}}\,.$$

An important special case is the following famous formula. By [9, p. 211], "In a paper [1] presented in 1737 and published in 1744, Euler reported on his stunning discovery (...)". The reference "[1]" is [3] here.

Proposition 22 *For $\sigma > 1$,*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

In the next theorem, which is not in [8], we handle absolutely convergent infinite products more generally.

Theorem 23 *Suppose that $(z_n) = (z_1, z_2, \dots) \subset \mathbb{C}$ is a sequence such that $|z_1| + |z_2| + \cdots < +\infty$. Then there is a number $z \in \mathbb{C}$ such that for every bijection $\rho\colon \mathbb{N} \to \mathbb{N}$,*

$$\lim_{n\to\infty} \prod_{j=1}^{n} \big(1 + z_{\rho(j)}\big) = z\,.$$

Proof. We set $p_n = \prod_{j=1}^{n}(1 + z_j)$ and show that the sequence (p_n) is bounded. We may assume that all $p_n \neq 0$ because if $p_n = 0$ then $1 + z_m = 0$ for some $m \le n$ and the theorem trivially holds with $z = 0$. Since $\log(1 + x) \le x$ for every real $x \ge 0$, for every n it holds that

$$|p_n| \le \prod_{j=1}^{n}(1 + |z_j|) \text{ and } \log(|p_n|) \le \sum_{j=1}^{n} \log(1 + |z_j|) \le \sum_{j=1}^{\infty} |z_j|\,.$$

Thus $|p_n| \le c$ for every n and a constant $c > 0$.

We show that the sequence (p_n) is Cauchy. Let an $\varepsilon \in (0, \frac{1}{2})$ be given. We take an n_0 such that $\sum_{n>n_0} \log(1+|z_n|) \le \varepsilon$. If $n > m \ge 1$ then by the triangle inequality

$$\left|\frac{p_n}{p_m} - 1\right| = \left|\prod_{j=m+1}^{n} (1 + z_j) - 1\right| \le \prod_{j=m+1}^{n} (1 + |z_j|) - 1 =: p(m, n) - 1\,.$$

If $n > m \ge n_0$ then

$$1 \le p(m, n) = \exp(\log(p(m, n))) \le \exp(\varepsilon) \le 1 + 2\varepsilon\,.$$

For $n > m \ge n_0$ then $|p_m - p_n| = |p_m| \cdot |1 - p_n/p_m| \le 2c\varepsilon$. So (p_n) is Cauchy and the limit $\lim p_n = z$ exists.

Let ρ be any permutation of $\mathbb{N}$. We set $q_n = \prod_{j=1}^n (1 + z_{\rho(j)})$, assume that the above selected n_0 is so large that also $|z - p_{n_0}| \le \varepsilon$, and take an n_1 such that $[n_0] \subset \{\rho(j) \mid j \in [n_1]\}$. If $n \ge n_1$ then by the triangle inequality

$$\left|\frac{q_n}{p_{n_0}} - 1\right| = \left|\prod_{j\in X}(1 + z_j) - 1\right| \le \prod_{j\in X}(1 + |z_j|) - 1 =: p(X) - 1\,,$$

for a finite set $X = X(n) \subset \mathbb{N}$ such that $X \cap [n_0] = \emptyset$. Like before $1 \le p(X) \le 1 + 2\varepsilon$. Then for any $n \ge n_1$ we have

$$|z - q_n| \le |z - p_{n_0}| + |p_{n_0} - q_n| \le \varepsilon + |p_{n_0}| \cdot |1 - q_n/p_{n_0}| \le (2c + 1)\varepsilon$$

and $\lim q_n = z$ as well. □

For the following Theorems 1.5 and 1.6 in [8] we need some notation. For $(a_n) \subset \mathbb{C}$ we set $A(t) = \sum_{n \le \exp t} a_n$. The Dirichlet series of (a_n) then can be expressed as

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \int_{0-}^{+\infty} \mathrm{e}^{-ts}\,\mathrm{d}A(t)$$

— this is the *Laplace–Stieltjes transform of* $A(t)$. By $\mathcal{V}$ we denote the class of functions $f\colon \mathbb{R} \to \mathbb{R}$ with bounded variation on any bounded interval. See Lecture 1 for the required definitions.

Theorem 24 *Let $A \in \mathcal{V}$ and let*

$$F(s) = \int_{0-}^{+\infty} \mathrm{e}^{-st}\,\mathrm{d}A(t)$$

be the Laplace–Stieltjes transform of A. The following hold.

1. *If the integral converges for $s = s_0 = \sigma_0 + i\tau_0$, then it converges on $\sigma > \sigma_0$ and the convergence is uniform in any sector $S(\vartheta) = \{s \in \mathbb{C} \mid |\arg(s - s_0)| \le \vartheta\}$, $\vartheta \in [0, \pi/2)$.*

2. *If the integral converges absolutely for $s = s_0$, then it converges absolutely and uniformly on $\sigma \ge \sigma_0$.*

3. *$F(s)$ is holomorphic on the domain of convergence of the integral and for any $k \in \mathbb{N}_0$,*
$$F^{(k)}(s) = \int_{0-}^{+\infty} (-t)^k \mathrm{e}^{-st}\,\mathrm{d}A(t)\,.$$

Here, as usual, $s = \sigma + i\tau$. Let σ_c (resp. σ_a) be the infimum of $\sigma_0 \in \mathbb{R}$ such that the given Dirichlet series $F(s)$ (resp. absolutely) converges on the half-plane $\sigma > \sigma_0$. We call σ_c (resp. σ_a) the *abscissa of (resp. absolute) convergence* of $F(s)$.

Theorem 25 *If the integral $F(s)$ in the previous theorem has a holomorphic extension $\widetilde{F}(s)$ from $\sigma > \sigma_c$ to some points on $\sigma = \sigma_c$, then the equality*

$$\widetilde{F}(s) = \int_{0^-}^{+\infty} \mathrm{e}^{-st} \, \mathrm{d}A(t)$$

holds at any point where the integral converges.

The following are Theorems 1.7, 1.8 and 1.9 (Phragmén–Landau), Corollary 1.10 and Theorems 1.11 and 1.12 in [8].

Theorem 26 *For any Dirichlet series it holds that $\sigma_c \le \sigma_a \le \sigma_c + 1$.*

Theorem 27 *If $F(s) = \sum_{n\ge1} a_n/n^s$ vanishes for every large σ then $a_n = 0$ for every n.*

Unlike power series which always have a singularity on the boundary of the disc of convergence, Dirichlet series need not have singularity on $\sigma = \sigma_c$ (since this line is not compact). In the following situations there is a singularity.

Theorem 28 *Let $A \in \mathcal{V}$ and let $F(s)$ be the integral in Theorem 24. If A is non-decreasing then $s = \sigma_c$ is a singularity of $F(s)$.*

Corollary 29 *If all a_n are real and all $a_n \ge 0$, than $s = \sigma_c$ is a singularity of $F(s) = \sum_{n\ge1} a_n/n^s$.*

This corollary is usually called the *Landau theorem*, after the German mathematician *Edmund Landau (1877–1938)*, and is the analogue of the Vivanti–Pringsheim theorem that for any power series with nonnegative coefficients the radius of convergence is a singularity of the associated function. The reference given in [7] for the Landau theorem is [4]. *Lars E. Phragmén (1863–1937)* was a Swedish mathematician. If $U \subset \mathbb{C}$ is an open set, $z_0 \in \partial U$ and $f \colon U \to \mathbb{C}$ is holomorphic, then we say that z_0 is a singularity of f if f cannot be holomorphically extended to any neighborhood of z_0.

From the Phragmén–Landau theorem one deduces the following two oscillation theorems.

Theorem 30 *If $A \colon (1, +\infty) \to \mathbb{R}$ is locally bounded and measurable, the integral*

$$H(s) = \int_1^{+\infty} \frac{A(t)}{t^{s+1}} \, \mathrm{d}t$$

has a finite abscissa of convergence σ_c and $H(s)$ has a holomorphic extension to $s = \sigma_c$, then for each $\varepsilon > 0$ we have

$$A(x) = \Omega_\pm\left(x^{\sigma_c - \varepsilon}\right).$$

Theorem 31 *Let $F(s) = \sum_{n\geq 1} a_n/n^s$ with real a_n have finite abscissa of convergence. Let a real $\sigma_0 > 0$ be such that $F(s)$ has holomorphic extension that includes the half-line $[\sigma_0, +\infty)$ and has a pole on $\sigma = \sigma_0$. Then we have*

$$\sum_{n\leq x} a_n = \Omega_{\pm}\big(x^{\sigma_0}\big)\,.$$

Here if $f, g\colon M \to \mathbb{R}$, where $M \subset \mathbb{R}$ has the limit point $+\infty$ and $g > 0$ on M, we write that $f = \Omega_+(g)$, resp. $f = \Omega_-(g)$, if $\limsup_{x\to+\infty} f(x)/g(x) > 0$, resp. $\liminf_{x\to+\infty} f(x)/g(x) < 0$.

The following are Theorems 1.13 and 1.14 in [8]. Let $A \in \mathcal{V}$, $A(0\pm) = 0$ and

$$F(s) = \int_{0^-}^{+\infty} \mathrm{e}^{-st}\,\mathrm{d}A(t)$$

Theorem 32 *Let σ_c be the abscissa of convergence of the above integral. The following hold.*

1. *For $\delta \in \mathbb{R}$, $A(x) \ll \mathrm{e}^{\delta x} \Rightarrow \sigma_c \leq \delta$.*
2. *If the above integral converges for $s = s_0$ with $\sigma_0 > 0 \Rightarrow A(x) = o\big(\mathrm{e}^{\sigma_0 x}\big)$ $(x \to +\infty)$.*
3. *$(\dots)$ with $\sigma_0 < 0 \Rightarrow A(x) = \alpha + o\big(\mathrm{e}^{\sigma_0 x}\big)$ $(x \to +\infty)$ for some $\alpha \in \mathbb{R}$.*

Theorem 33 *Let $\kappa = \limsup_{x\to+\infty} x^{-1}\log(|A(x)|)$. The following hold.*

1. *$\kappa \neq 0 \Rightarrow \sigma_c = \kappa$.*
2. *If $\kappa = 0$ then either finite $\lim_{x\to+\infty} A(x)$ does not exist and $\sigma_c = 0$, or this limit is $\alpha \in \mathbb{R}$ and $\sigma_c = \limsup_{x\to+\infty} x^{-1}\log(|A(x) - \alpha|) \leq 0$.*

The following are Theorems 1.15 and 1.16 in [8].

Theorem 34 *For $n \in \mathbb{N}$ we denote by $k(n)$ the product of prime divisors of n. Then for every $\varepsilon > 0$, the bound*

$$N(x,\, y) := |\{n \leq x \mid k(n) \leq y\}| \ll_{\varepsilon} yx^{\varepsilon}$$

holds uniformly for $x \geq y \geq 1$.

Theorem 35 *The bound*

$$N(x,\, y) \ll y(\log y)\exp\big(\sqrt{8\log(x/y)}\big)$$

holds uniformly for $x \geq y \geq 2$.

The following are Theorem 1.17, Lemma 1.18 (Dirichlet), Theorems 1.19, 1.20 and 1.21, and Corollary 1.22 in [8]. Her $\zeta(s) = \sum_{n\geq 1} 1/n^s$ is the zeta-function.

Theorem 36 *For every real $T > 0$ there is a real $\tau > T$ such that*

$$\sup\left(\{|\zeta(\sigma + i\tau)| \mid \sigma > 1\}\right) \geq \log\log(3 + \tau)/10 .$$

This is proven by means of the next lemma.

Lemma 37 *Let $N, D \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_N$ be in $\mathbb{R}$. Then for every $Q \in \mathbb{N} \setminus \{1\}$ there exist $q \in \mathbb{N}$ with $D \leq q \leq DQ^N$ such that*

$$\max_{1 \leq j \leq N} \|q\alpha_j\| \leq 1/Q .$$

Theorem 38 *Let $F(s) = \sum_{n \geq 1} a_n/n^s$ have finite σ_c, and let $\sigma_0 > \sigma_c$ and $\varepsilon > 0$. Then for $|\tau| \geq 1$ the bound ($s = \sigma + i\tau$)*

$$F(s) \ll |\tau|^{1-(\sigma-\sigma_c)+\varepsilon}$$

holds uniformly for $\sigma \in [\sigma_0, \sigma_c + 1]$.

If $D \subset \mathbb{C}$ is a domain (an open connected set) and $F\colon D \to \mathbb{C}$ is holomorphic, we say that F is of *finite order on D* if $F(s) \ll |\tau|^A$ ($|\tau| \geq 1$) on D for some $A > 0$ ($s = \sigma + i\tau$). By $\mu(\sigma) = \mu_F(\sigma)$ we then denote the infimum

$$\inf\left(\{\xi \in \mathbb{R} \mid F(s) \ll |\tau|^\xi \text{ for } s = \sigma + i\tau \in D \text{ and } |\tau| \geq 1\}\right) .$$

Theorem 39 *Let $F(s)$ be a function of finite order in the vertical strip $\sigma_1 \leq \sigma \leq \sigma_2$. Then the function $\mu(s)$ is convex in this interval. In particular, it is continuous on (σ_1, σ_2).*

Theorem 40 *For any Dirichlet series $F(s)$, $\mu(\sigma) = 0$ for $\sigma > \sigma_a$. Moreover, $\mu(\sigma)$ is non-increasing.*

Corollary 41 *Let $F(s)$ be a Dirichlet series and let $\sigma_0 < \sigma_a$. If $F(s)$ has finite order for every $\sigma > \sigma_0$ then $\mu(\sigma_a) = 0$.*

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