Lecture 3. Selberg's sieve. Extremal orders

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I continue my survey of G. Tenenbaum's book [13]. In the third lecture we cover the last three sections of Chapter I.4. *Sieve methods*, and then Chapter I.5. *Extremal orders*, up to page 123. I follow notation explained in the first two lectures. As a rule I always (re)state each theorem so that the notation used is clear.

Chapter I.4. Sieve methods — continued

The following Theorems 4.15 and 4.16 (Brun–Titchmarsh) in [13] are obtained by the large sieve.

Theorem 1 *Let*

$$C = 2\prod_{p>2}\big(1-(p-1)^{-2}\big).$$

Then for every constant $c > C$ there is an $x_0 > 0$ such that for every $x > x_0$

$$|\{p \le x \mid p+2 \text{ is prime}\}| < \frac{8cx}{(\log x)^2}.$$

For real x and $a, q \in \mathbb{N}$ we denote by

$$\pi(x;\, a,\, q) = |\{p \le x \mid p \equiv a \pmod q\}|$$

the number of primes not exceeding x and lying in the residue class a modulo q.

Theorem 2 *For any constant $c > 2$ there is a constant $y_0 > 0$ such that for any $a, q \in \mathbb{N}$ and any $x, y > 0$ with $y/q > y_0$ we have*

$$\pi(x+y;\, a,\, q) - \pi(x;\, a,\, q) \le \frac{cy}{\varphi(q)\log(y/q)}.$$

(I somewhat altered the statement in [13].) Here $\varphi(\cdot)$ is of course Euler's totient function. In [8, Theorem 3.9] we find this formulation:

> Let a and q be integers with $(a, q) = 1$, and let x and y be real numbers with $x \ge 0$ and $y \ge 2q$. Then
>
> $$\pi(x+y;\, a,\, q) - \pi(x;\, a,\, q) \le \frac{2y}{\varphi(q)\log(y/q)}\big(1 + O(1/\log(y/q))\big).$$

By [8], E. C. Titchmarsh obtained this result, with a constant larger than 2, by Brun's method in [15] in 1930.

We proceed to *Selberg's sieve* which was invented by the Norwegian mathematician Atle Selberg (1917–2007) in [11] in 1947. The following are Definitions 4.17 and 4.18 in [13]. The first one is an interesting extension of multiplicativity of arithmetic functions to several variables.

Definition 3 *Let $r \in \mathbb{N}$. We say that $f\colon \mathbb{N}^r \to \mathbb{C}$ is multiplicative (in Selberg's sense) if the formal Dirichlet series*

$$F(s_1, \ldots, s_r) = \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{f(n_1, \ldots, n_r)}{n_1^{s_1} \ldots n_r^{s_r}}$$

expresses as $F(s_1, \ldots, s_r) = \prod_p F_p(p^{-s_1}, \ldots, p^{-s_r})$ where (the formal power series)

$$F_p(X_1, \ldots, X_r) = \sum_{\nu_1, \ldots, \nu_r = 0}^{\infty} f_p(\nu_1, \ldots, \nu_r) X_1^{\nu_1} \ldots X_r^{\nu_r} \in \mathbb{C}[[X_1, \ldots, X_r]]$$

are such that $F_p(0, \ldots, 0) = 1$ except for finitely many primes p.

We say that f is singular if $f(1, \ldots, 1) = 0$, regular if $f(1, \ldots, 1) \neq 0$ and normal if $f(1, \ldots, 1) = 1$.

In fact only the case $r \leq 2$ will be needed.

Definition 4 *We call $f\colon \mathbb{N}^2 \to \mathbb{C}$ symmetric if $f(m, n) = f(n, m)$ for every $m, n \in \mathbb{N}$. If $f(m, n) = 0$ whenever $n > m$, we call f lower triangular. If in addition $f(m, m) = 1$ for every m, we call f normal lower triangular.*

This terminology is used in the following three Propositions 4.19–4.21 in [13]. Let $t\colon \mathbb{N}^2 \to \mathbb{C}$ be normal lower triangular. For $1 \leq n \leq m$ we define the inverse $t^*\colon \mathbb{N}^2 \to \mathbb{C}$ of t as the solution of the infinite linear system

$$\sum_{k=n}^{m} t^*(m, k) t(k, n) = \delta_{m, n}$$

($\delta_{m,n} = 1$ for $m = n$ and is 0 else).

Proposition 5 *Let $t\colon \mathbb{N}^2 \to \mathbb{C}$ be normal lower triangular. The following hold.*

1. *If $f, g\colon \mathbb{N} \to \mathbb{C}$ satisfy $f(m) = \sum_{n=1}^{m} t(m, n) g(n)$ for every $m \in \mathbb{N}$, then*

   $$g(m) = \sum_{n=1}^{m} t^*(m, n) f(n).$$

2. *If $f, g\colon \mathbb{N} \to \mathbb{C}$ satisfy $f(n) = \sum_{m \geq n} t(m, n) g(m)$ where for every $n \in \mathbb{N}$ the series absolutely converges, then*

   $$g(n) = \sum_{m \geq n} t^*(m, n) f(m)$$

   provided that for every $n \in \mathbb{N}$ the series absolutely converges.

Proposition 6 *Let $t\colon \mathbb{N}^2 \to \mathbb{C}$ be normal lower triangular and multiplicative. Then so is its inverse t^*.*

The following is a useful special case of Proposition 5.

Proposition 7 *Let $t\colon \mathbb{N}^2 \to \mathbb{C}$ be normal lower triangular and multiplicative. The following hold.*

1. *If $f, g\colon \mathbb{N} \to \mathbb{C}$ satisfy $f(m) = \sum_{d\,|\,m} t(m,d)g(d)$ for every $m \in \mathbb{N}$, then*

$$g(m) = \textstyle\sum_{d\,|\,m} t^*(m,d)f(d).$$

2. *If $f, g\colon \mathbb{N} \to \mathbb{C}$ satisfy $f(n) = \sum_{n\,|\,m} t(m,n)g(m)$ where for every $n \in \mathbb{N}$ the series absolutely converges, then*

$$g(n) = \textstyle\sum_{n\,|\,m} t^*(m,n)f(m)$$

provided that for every $n \in \mathbb{N}$ the series absolutely converges.

Selberg's sieve(s) use quadratic forms. The next is Definition 4.22 in [13].

Definition 8 *We call $f\colon \mathbb{N}^2 \to \mathbb{R}$ positive definite if it is symmetric and if*

$$Q = Q(x_1,\, x_2,\, \dots) = \sum_{m,n=1}^{\infty} f(m,\, n)x_m x_n > 0$$

for any nonzero sequence $(x_n) \subset \mathbb{R}$ satisfying $x_n = 0$ for $n > n_0$.

The next is Proposition 4.23 in [13].

Proposition 9 *Let $f\colon \mathbb{N}^2 \to \mathbb{R}$ be positive definite and multiplicative. Then there exist multiplicative functions $g\colon \mathbb{N} \to (0, +\infty)$ and $t\colon \mathbb{N}^2 \to \mathbb{R}$ such that $t(m,n)$ is normal lower triangular and for any $m, n \in \mathbb{N}$,*

$$f(m,\, n) = \textstyle\sum_{d\,|\,(m,\,n)} g(d) \cdot t(m,\, d) \cdot t(n,\, d).$$

The functions g and t are uniquely determined by this condition.

The next Theorem 4.24 in [13] describes the solution of an optimization problem for quadratic forms that is needed in Selberg's sieve.

Theorem 10 *Let $N \in \mathbb{N}$ and let f, g and t be as in the previous proposition. Then for the constraint $x_1 = 1$ and $n > N \Rightarrow x_n = 0$ the quadratic form Q in Definition 8 has the minimum value*

$$Q^* = \frac{1}{\sum_{m=1}^{N} t^*(m,\, 1)^2/g(m)}\ ,$$

when $x_n = Q^* \sum_{m=1}^{N} t^*(m,n)t^*(m,1)/g(m)$, $n = 1, 2, \dots, N$.

The next Theorem 4.25 in [13] (reformulated by us) concerns the so called *Johnsen–Selberg prime power sieve* which originated in [4, 12]. We need for it some notation. Let $\mathcal{A} \subset \mathbb{Z}$ be a finite tuple of integers (which may be repeated), $z \geq 2$ be real, $P \subset \mathbb{P}$ and $P_z = \{p \in P \mid p \leq z\}$. For any prime power p^ν ($\nu \in \mathbb{N}$) we are given a set $W(p^\nu) \subset \mathbb{Z}$ that is a union of infinite arithmetic progressions of the form $m + p^\nu\mathbb{Z}$, $m \in \mathbb{Z}$; in other words it is a union of several residue classes modulo p^ν. We assume that $W(p^\mu) \cap W(p^\nu) = \emptyset$ if $\mu \neq \nu$. For $d \in \mathbb{N}$ we set $W(d) = \bigcap_{p^\nu \,\|\, d} W(p^\nu)$, with $W(1) = \mathbb{Z}$. We define

$$S(\mathcal{A},\, P,\, z) = |\{a \in \mathcal{A} \mid \forall\, p \in P_z\, \forall\, \nu \in \mathbb{N}:\ a \notin W(p^\nu)\}|.$$

For $n \in \mathbb{N}$ we denote by $P^+(n)$ the largest prime factor of n, with $P^+(1) = 1$.

Theorem 11 *With this notation we further suppose that $X \geq 0$ is real and that $w\colon \mathbb{N} \to [0, +\infty)$ is a multiplicative function such that* (i) *for any $p \notin P_z$ and $\nu \in \mathbb{N}$, $w(p^\nu) = 0$, and* (ii) *for any $p \in P_z$, $\sum_{\nu=1}^{\infty} w(p^\nu)/p^\nu < 1$. For any $d \in \mathbb{N}$ we set $r_d = |\mathcal{A} \cap W(d)| - w(d)X/d$. For any p and $\nu \in \mathbb{N}_0$ we define $\vartheta(p^\nu) = 1 - \sum_{\mu=1}^{\nu} w(p^\mu)/p^\mu$ (> 0) and consider the normal (i.e., $f(1) = 1$) multiplicative function $f\colon \mathbb{N} \to [0, +\infty)$ determined by the values ($\nu \in \mathbb{N}$)*

$$f(p^\nu) = \frac{w(p^\nu)}{\vartheta(p^\nu)\vartheta(p^{\nu-1})}.$$

Finally, for real $x, y \geq 1$ let $\psi_f(x, y) = \sum_{n \leq x, P^+(n) \leq y} f(n)/n$. Then for any $D \in \mathbb{N}$, $D > 1$, one has the upper bound

$$S(\mathcal{A},\, P,\, z) \leq \frac{X}{\psi_f(D,\, z)} + \sum_{\substack{m \leq D^2 \\ P^+(m) \leq z}} 3^{\omega(m)} |r_m|.$$

This theorem is due to G. Tenenbaum and J. Wu in [14] in 2008.

In the following Theorem 4.26 (Selberg) of [13] the tuple $\mathcal{A}$ is an interval of integers and the notation of the last theorem is used.

Theorem 12 *We suppose that $\mathcal{A}$ consists of $N \in \mathbb{N}$ consecutive integers and that ($d \in \mathbb{N}$) the function $w(d) := |[0, d) \cap W(d)|$ satisfies conditions* (i) *and* (ii)*. Then for any $D \in \mathbb{N}$,*

$$S(\mathcal{A},\, P,\, z) \leq \frac{N + D^2 - 1}{\psi_f(D,\, z)}.$$

The next Theorem 4.27, stated in [13] without proof, was obtained in [14] like Theorem 11 and achieves an upper bound on $1/\psi_f(D, z)$. Besides the notation of Theorem 11 it requires some additional one. First of all, $w(d)$ is as in the last theorem. For $\kappa, u > 0$ let $\rho_\kappa(u)$ (the generalized Dickman function) be the continuous solution of the system

$$\rho_\kappa(u) = \frac{u^{\kappa-1}}{\Gamma(u)} \ \ldots\ u \in (0,\, 1] \wedge u\rho'_\kappa(u) + (1 - \kappa\rho_\kappa(u)) + \kappa\rho_\kappa(u-1) = 0 \ \ldots\ u > 1$$

where $\Gamma(\cdot)$ is Euler's Γ function. For $u \geq 0$ let (γ is Euler's constant)

$$\lambda_\kappa(u) = \mathrm{e}^{-\gamma\kappa} \int_u^{+\infty} \rho_\kappa(v)\,\mathrm{d}v \text{ and } j_\kappa(u) = 1 - \lambda_\kappa(u).$$

Let $H(z) = \prod_{p\leq z}\big(1 - \sum_{\nu=1}^{\infty} w(p^\nu/p^\nu)\big)$. Finally, besides the conditions (i) and (ii) of Theorem 11 we introduce condition (iii) ($\eta \in (0, \frac{1}{2})$, $\kappa > 0$ and $r \in \mathbb{N}$): $\vartheta(p^\nu) \geq \eta$ for any $p \in P$ and $\nu \in \mathbb{N}$,

$$\sum_p \sum_{\nu=1}^{r} \frac{w(p^\nu)^2 \log p}{p^{2\nu}} + \sum_p \sum_{\nu>r} \frac{w(p^\nu)}{p^{(1-\eta)\nu}} < \infty$$

and for any $t \geq y \geq 1$,

$$\sum_{\substack{y<p\leq t\\ \nu\in[r]}} \frac{w(p^\nu \log p)}{p^\nu} \leq \kappa \log(t/y) + O(1).$$

Theorem 13 *Let* $r, D \in \mathbb{N}$, $\kappa > 0$, $\eta \in (0, \frac{1}{2})$, $v = \min\big(\{\frac{\log D}{r\log z}, \frac{3\log D}{r\eta\log\log D}\}\big)$ *and* $\lambda_\kappa^+(v) = \lambda_\kappa(v)\cdot v\log(1+v)$. *If the conditions* (i) *and* (iii) *hold, then there exists a constant* B *such that we have uniformly for* $2 \leq z \leq D^{1/r}$ *that*

$$\frac{1}{\psi_f(D,\, z)} \leq \frac{H(z)}{j_\kappa(v)}\big(1 + O\big(\lambda_\kappa^+(v)\exp((Bv\log v)/\log z)/\log z\big)\big).$$

In the last eighth section of Chapter I.4 of [13] Selberg's sieve is applied to bound the number of natural numbers in an interval that are sums of two squares. The next Theorem 4.28 in [13] is the well known Euler's criterion of quadratic (non) residues. The symbol $\big(\frac{a}{p}\big)$ is Legendre's symbol.

Theorem 14 *For every* $a \in (\mathbb{Z}/p\mathbb{Z})^*$, *we have*

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}.$$

The next Theorem 4.29 (Girard–Fermat) in [13] is a classical result.

Theorem 15 *An odd prime number is the sum of two squares if, and only if, it is congruent to 1 modulo 4.*

The next Theorem 4.30 in [13] characterizes numbers expressible as sums of two squares; $v_p(n)$ is the p-adic order of n.

Theorem 16 *A positive integer is representable as a sum of two squares if, and only if, for any prime number* p *such that* $p \equiv 3 \pmod 4$, *we have* $v_p(n) \equiv 0 \pmod 2$.

Finally, the last Theorem 4.31 in Chapter I.4 in [13] gives upper bounds on the number of natural numbers in an interval that are sums of two squares. We set

$$K_0 = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \,(\mathrm{mod}\, 4)} \frac{1}{\sqrt{1-p^{-2}}} \approx 0.764$$

(this is so called Landau–Ramanujan constant).

Theorem 17 *There is an absolute constant $K > 0$ such that for any set $I \subset \mathbb{N}$ of $N \in \mathbb{N}$ consecutive numbers it holds that*

$$Z_N = |\{n \in I \mid n = l^2 + m^2 \text{ for some } l, m \in \mathbb{N}\}| \le KN \prod_{\substack{p \le N \\ p \equiv 3 \,(\mathrm{mod}\, 4)}} \Big(1 - \frac{1}{p}\Big).$$

For $N \to \infty$,

$$Z_N \le (\pi + o(1)) \frac{K_0 N}{\sqrt{\log N}}.$$

Nothing is said in [13] about the 1908 asymptotics [7] of E. Landau: for $x \ge 2$,

$$|\{n \le x \mid n = l^2 + m^2 \text{ for some } l, m \in \mathbb{N}\}| = \frac{K_0 x}{\sqrt{\log x}} + O\big(x/(\log x)^{3/2}\big).$$

See [1] for the function field analogue of Landau's result.

One still finds Theorem 4.32 (Iwaniec) in the Notes to Chapter I.4 in [13] but we do not quote it and instead mention the excellent and often witty book [3] on sieves written by J. Friedlander and H. Iwaniec.

Chapter I.5. Extremal orders

The following is Definition 5.1 in [13].

Definition 18 *Let $f, g\colon \mathbb{N} \to \mathbb{R}$ where g is non-decreasing and eventually positive. We say that g is a maximal (resp. minimal) order for f if*

$$\limsup_{n \to \infty} f(n)/g(n) = 1 \text{ (resp. } \liminf_{n \to \infty} f(n)/g(n) = 1\text{)}.$$

The following are Theorem 5.2, Corollary 5.3 and Theorem 5.4 in [13]. Recall that $\tau(n) = \sum_{lm=n} 1$ is the number of divisors of n. The notation $O_\varepsilon(\cdot)$ means that the implicit constant c in the $O(\cdot)$ (see the definition of O in the last lecture) may depend on ε.

Theorem 19 *If $f\colon \mathbb{N} \to \mathbb{C}$ is multiplicative and such that $\lim_{p^\nu \to \infty} f(p^\nu) = 0$ then $\lim_{n \to \infty} f(n) = 0$.*

The assumption on f says that if the sequence $(m_n) = (m_1 < m_2 < \ldots) \subset \mathbb{N}$ lists in the increasing order all prime powers, so that

$$(m_n) = (2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 11,\ 13,\ 16,\ 17,\ 19,\ 23,\ \ldots)\,,$$

then $\lim_{n\to\infty} f(m_n) = 0$. It is easy to see that then this limit holds for any permutation of (m_n).

Proof. Suppose that f is as stated and that an $\varepsilon \le 1$ is given. By $q \in \mathbb{N}$ we denote powers of primes. There is a $Q = Q(\varepsilon) > 0$ such that $q > Q \Rightarrow |f(q)| \le \varepsilon$. Let Q_1, resp. Q_2, be the set of $q \le Q$ with $|f(q)| \le 1$, resp. $|f(q)| > 1$, and let Q_3 be the set of $q > Q$. Then $Q_1 \cup Q_2 \cup Q_3$ is a partition of all q and every $n \in \mathbb{N}$ expresses uniquely as the product

$$n = n_1 n_2 n_3 \ \text{ where } \ n_i = \prod_{q\,\|\,n,\ q \in Q_i} q\,.$$

It is clear that the n_i are mutually coprime and $|f(n_1)| \le 1$. It is also clear that $|f(n_2)| \le A$ where $A > 0$ is an absolute constant (independent of ε) and that if $n_3 > 1$ then $|f(n_3)| \le \varepsilon$. Finally, let n_0 be such that $n > n_0 \Rightarrow n_3 > 1$. Then

$$n > n_0 \Rightarrow |f(n)| = |f(n_1)| \cdot |f(n_2)| \cdot |f(n_3)| \le 1 \cdot A \cdot \varepsilon = A\varepsilon\,.$$

This means that $\lim f(n) = 0$. □

Corollary 20 *For any $\varepsilon > 0$ we have $\tau(n) = O_\varepsilon(n^\varepsilon)$ (for $n \in \mathbb{N}$).*

Proof. We apply the previous theorem to the function $f(n) = \tau(n)/n^\varepsilon$. The assumption of the theorem is satisfied because

$$f(p^\nu) = \frac{\nu + 1}{(p^\nu)^\varepsilon} \le \frac{1 + 2\log(p^\nu)}{(p^\nu)^\varepsilon}$$

and $\lim_{p^\nu\to\infty} f(p^\nu) = 0$. □

Theorem 21 *A maximal order for* $\log(\tau(n))$ *is* $(\log 2)(\log n)/(\log\log n)$.

This theorem is essentially due to S. Ramanujan in [9, 10] in 1915.

The next Theorem 5.5 in [13] deals with the arithmetic functions $\omega(n) = \sum_{p\,|\,n} 1$ and $\Omega(n) = \sum_{p^\nu\,\|\,n} \nu$.

Theorem 22 *A maximal order for* $\omega(n)$ *is* $(\log n)/(\log\log n)$. *A maximal order for* $\Omega(n)$ *is* $(\log n)/(\log 2)$.

The next Theorem 5.6 in [13] deals with Euler's totient function $\varphi(n) = \sum_{\substack{m,\, m\le n \\ (m,n)=1}} 1$.

Theorem 23 *A maximal order for $\varphi(n)$ is n. A minimal order is*

$$\frac{\mathrm{e}^{-\gamma}n}{\log\log n}$$

where γ denotes Euler's constant.

The maximal order for $\varphi(n)$ is witnessed by primes. By [8] the result on the minimal order for $\varphi(n)$ is due to E. Landau in [6] in 1903.

Finally, the next last Theorem 5.7 in Chapter I.5 in [13] (with only sketched proof) deals with the functions $\sigma_\kappa(n) = \sum_{lm=n} l^\kappa$, $\kappa \in \mathbb{R}$.

Theorem 24 *The following hold on extremal orders for $\sigma_\kappa(n)$.*

1. *For $\kappa > 0$, a minimal order is n^κ.*
2. *For $\kappa > 1$, a maximal order is $\zeta(\kappa)n^\kappa$.*
3. *For $\kappa = 1$, a maximal order is $\mathrm{e}^\gamma n(\log\log n)$.*
4. *For $0 < \kappa < 1$, we have*

$$\sigma_\kappa(n) \le n^\kappa \exp\big((1+o(1))(1-\kappa)^{-1}(\log n)^{1-\kappa}/(\log\log n)\big)$$

and the opposite inequality is satisfied for infinitely many $n \in \mathbb{N}$.

We add two more theorems on maximal orders, these are not in [13]. Let $m(n)$ be the number of ordered factorizations

$$n = n_1 n_2 \dots n_k,\ n_i \ge 2\,,$$

of the number $n \in \mathbb{N}$. Thus $(m(n)) = (0, 1, 1, 2, 1, 3, 1, 3, 2, 3, 1, 8, 1, \dots)$ but let us change $m(1)$ to 1. Let $\rho = 1.72684\dots$ be the only positive solution of the equation $\zeta(s) = \sum_{n\ge1} n^{-s} = 2$. The appearance of this constant is explained by the formula

$$\sum_{n=1}^{\infty}\frac{m(n)}{n^s} = \sum_{k=0}^{\infty}(\zeta(s)-1)^k = \frac{1}{1-(\zeta(s)-1)} = \frac{1}{2-\zeta(s)}$$

for the formal Dirichlet series of $(m(n))$. The next theorem was obtained in [5].

Theorem 25 (Klazar and Luca, 2007) *For every $\varepsilon > 0$,*

$$m(n) < \frac{n^\rho}{\exp\big((\log n)^{1/\rho}/(\log\log n)^{1+\varepsilon}\big)}$$

holds for every $n > n_0 = n_0(\varepsilon)$. On the other hand, there is a constant $c > 0$ such that

$$m(n) > \frac{n^\rho}{\exp\big(c((\log n)/(\log\log n))^{1/\rho}\big)}$$

holds for infinitely many $n \in \mathbb{N}$.

These bounds on maximal orders for $m(n)$ were strengthened in [2]:

Theorem 26 (Deléglise, Hernane and Nicolas, 2008) *For some constants $A, B > 0$ one has*

$$\log(m(n)) \le \rho \log n - A(\log n)^{1/\rho}/(\log\log n)$$

for every large n and

$$\log(m(n)) \ge \rho \log n - B(\log n)^{1/\rho}/(\log\log n)$$

for infinitely many n.

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