Problem set for Probability and Statistics 1 — 4 May 2026

Summary

- **Bias:** $\mathbb{E}_\theta[\hat{\theta} - \theta]$ where θ is the true parameter, $\hat{\theta}$ is our estimate (random variable as it depends on observed data).
- An estimator is **unbiased:** $bias = 0$ for all $\theta \in \Theta$.
- An estimator is **asymptotically unbiased:** bias converges to 0, i.e., $\mathbb{E}_\theta[\hat{\theta}] \to \theta$ for all $\theta \in \Theta$.
- An estimator is **consistent**, denoted $\hat{\theta} \xrightarrow{P} \theta$, if for all $\varepsilon > 0$ and all $\theta \in \Theta$, $\mathbb{P}[|\hat{\theta} - \theta| > \varepsilon] \to 0$.
- The **mean squared error (MSE)** of an estimator: $\mathbb{E}_\theta((\hat{\theta} - \theta)^2)$
- Theorem: $\mathsf{MSE} = bias(\hat{\theta})^2 + \mathrm{var}(\hat{\theta})$.
- **Method of moments (MoM) estimation** solves the equation $\mathbb{E}_{\hat{\theta}}[X] = \frac{1}{n}\sum_{i=1}^{n} X_i$ for the unknown $\hat{\theta}$.
- Or a system of equations $\mathbb{E}_{\hat{\theta}}[X^r] = \frac{1}{n}\sum_{i=1}^{n} X_i^r$ for $r = 1, 2, \ldots$ when we are estimating r parameters.
- The **likelihood** of observed data dependent of parameter θ is given by $L(\theta; x_1, \ldots, x_n) = \mathbb{P}[X_1 = x_1 \wedge \cdots \wedge X_n = x_n]$ for discrete distributions, or by $L(\theta; x_1, \ldots, x_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ for continuous distributions.
- $l(\theta; x_1, \ldots, x_n) = \log L(\ldots)$ ...for easier computations.
- **Maximum likelihood estimation (MLE)** searches for θ for which $L(\theta; x_1, \ldots, x_n)$, or equivalently, $l(\theta; x_1, \ldots, x_n)$ is maximised. Usually done using derivatives of L or l.

Mean squared error

1. Consider a sample of random variables: $X_1, X_2, \ldots, X_n$, where $n > 10$, $E[X_i] = \mu$, $\mathrm{var}[X_i] = \sigma^2 > 0$ and the estimator $\hat{\mu}_n = \frac{1}{n-10} \sum\limits_{i=11}^{n} X_i$. Then calculate:

(a) The bias of $\hat{\mu}_n$.
(b) The variance of $\hat{\mu}_n$.
(c) The MSE of $\hat{\mu}_n$.

Method of moments, maximum likelihood estimation

2. In a TV show the host picks n random real numbers independently and uniformly from $[0, \theta]$ (where θ is known only to the host) and reveals them to the players. Based on the sample, the players have to guess θ. The first player guesses θ using method of moments estimation whereas the second player guesses θ using maximum likelihood estimation.

(a) Express the first player's guess in terms of the sample $X_1, \ldots, X_n$.
(b) Express the second player's guess in terms of the sample $X_1, \ldots, X_n$.
(c) For each of them, determine whether it is unbiased and consistent.
(d) Compute the mean squared error (MSE) for each of them.
(e) Which estimate is better? Can you think of an even better one?

3. For a random sample $X_1, \dots, X_n \sim \mathsf{Geom}(\frac{1}{\theta})$,

(a) propose a point estimate θ using the method of moments;
(b) propose a point estimate θ using the method of maximum likelihood;
(c) for each of them, determine whether it is unbiased and consistent;
(d) compute the mean squared error (MSE).

4. For a random sample $X_1, \dots, X_n \sim \mathsf{Exp}(\frac{1}{\theta})$,

(a) propose a point estimate θ using the method of moments;
(b) propose a point estimate θ using the method of maximum likelihood;
(c) for each of them, determine whether it is unbiased and consistent;
(d) compute the mean squared error (MSE).

5. We have a random sample $X_1, \dots, X_n \sim \mathsf{Exp}(\lambda)$. We are interested in the probability p that $X > 1$ for $X \sim \mathsf{Exp}(\lambda)$. (Recall that $p = e^{-\lambda \cdot 1}$.)

(a) Propose a point estimate for p (using any method).
(b) Investigate its properties.

6. We have a random sample $X_1, \dots, X_n \sim \mathsf{Poi}(\lambda)$.

(a) Propose a point estimate for λ using the method of moments.
(b) Propose a point estimate for λ using the method of maximum likelihood.
(c) Calculate the mean squared error (MSE).

7. * For a random sample $X_1, \dots, X_n \sim \mathsf{Exp}(\theta)$.

(a) Propose a point estimate θ using the method of moments.
(b) Propose a point estimate θ using the method of maximum likelihood.
(c) For each of them, determine whether it is unbiased and consistent.
(d) Compute the mean squared error (MSE).