Revision set for Probability and Statistics 1 — 16 February 2026

Discrete probability spaces

A *discrete probability space* $(\Omega, 2^\Omega, \mathbb{P})$ consists of

- a (finite or countable) set of *outcomes*, labelled Ω;
- the set of subsets of Ω, labelled 2^Ω, with each subset called an *event*;
- and a function $\mathbb{P} : \Omega \to [0,1]$ such that

$$\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$$

For any event $A \subseteq \Omega$, we define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$$

We say an event A is *impossible* if $\mathbb{P}[A] = 0$, and it is *certain* if $\mathbb{P}[A] = 1$.
A discrete probability space is *uniform* if, for all $\omega \in \Omega$, we have

$$\mathbb{P}[\omega] = \frac{1}{|\Omega|}$$

For events $B_1, \ldots, B_n$, we have $\mathbb{P}[B_1 \cup \ldots \cup B_n] \leq \mathbb{P}[B_1] + \ldots + \mathbb{P}[B_n]$. This is known as *Boole's inequality* and equality holds when the B_i are pairwise disjoint.

If the B_i are not disjoint, then the probability of their union can be computed via the *inclusion-exclusion principle*:

$$\mathbb{P}[B_1 \cup \ldots \cup B_n] = \sum_{1 \leq i_1 \leq n} \mathbb{P}[B_{i_1}] - \sum_{1 \leq i_1 < i_2 \leq n} \mathbb{P}[B_{i_1} \cap B_{i_2}] + \ldots + (-1)^{n-1} \mathbb{P}[B_1 \cap \ldots \cap B_n]$$

1. Justify why the probability of any event A satisfies $0 \leq \mathbb{P}[A] \leq 1$.

2. Give an expression for the probability of an event A in a uniform discrete probability space.

3. For the following experiments, describe a suitable discrete probability space. Is the space finite? Is it uniform?

- rolling a six-sided die three times;
- rolling a six-sided die until the sum of the rolls is at least 3;
- rolling a six-sided die until the average of the rolls is at least 3.

4. An integer between 1 and 100 is chosen uniformly at random. What is the probability that the number is divisible by 3, 4 or 5?

Conditional probability, independent events

Let $(\Omega, 2^\Omega, \mathbb{P})$ be a probability space and let $B \subseteq \Omega$ be an event with $\mathbb{P}[B] \neq 0$. The probability that an event A has occurred, given the knowledge that B has occurred, is called the *conditional probability* of A given B, and it can be calculated as

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

If $B_1, \ldots, B_n$ are pairwise disjoint events such that $B_1 \cup \ldots \cup B_n = \Omega$, then we can use them to split another event A into cases and compute

$$\mathbb{P}[A] = \mathbb{P}[A|B_1]\mathbb{P}[B_1] + \ldots + \mathbb{P}[A|B_n]\mathbb{P}[B_n]$$

This is referred to as the *law of total probability.*

By comparing the expressions for $\mathbb{P}[A|B]$ and $\mathbb{P}[B|A]$ and using the law of total probability, we have *Bayes' theorem:*

$$\mathbb{P}[B_i|A] = \frac{\mathbb{P}[A|B_i]\mathbb{P}[B_i]}{\mathbb{P}[A|B_1]\mathbb{P}[B_1] + \ldots + \mathbb{P}[A|B_n]\mathbb{P}[B_n]}$$

We say events A, B are *independent* if the knowledge of B having occurred does not alter the probability of A having occurred, i.e. $\mathbb{P}[A|B] = \mathbb{P}[A]$. Substituting this into the definition of $\mathbb{P}[A|B]$ and rearranging yields the equivalent condition $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. In general, the events $A_1, \ldots, A_n$ are independent if, for all $I \subseteq \{1, \ldots, n\}$, we have

$$\mathbb{P}[\bigcap_{i \in I} A_i] = \prod_{i \in I} \mathbb{P}[A_i]$$

5. For $(\Omega, 2^\Omega, \mathbb{P})$ and B as above, check that $(B, 2^B, \tilde{\mathbb{P}})$ is a discrete probability space where $\tilde{\mathbb{P}}$ is given by $\tilde{\mathbb{P}}[A \cap B] = \mathbb{P}[A|B]$.

6. A six-sided die is rolled twice.

(a) What is the probability that the result of the second roll is divisible by the result of the first roll?

(b) Given that the result of the second roll is divisible by the result of the first roll, what is the probability that the result of the first roll was 1?

7. A coin is flipped twice. For these sets of events, which of them are independent?

- A : the first flip is heads, B : the second flip is tails;
- A : the first flip is heads, B : two flips were performed;
- A : the first flip is heads, B : both flips are tails;
- A : the first flip is heads, B : the second flip is tails, C : exactly one of the flips is heads.

Problems

8. A bag contains m ordinary playing cards, with a face and a back, and n misprinted playing cards, with both sides showing a back. All of the backs are identical. The cards are shuffled and one of them is drawn uniformly at random, with one of the sides facing up. If the side facing up is a face, it is returned to the bag and the cards are shuffled again, if not, it is flipped over. Player A wins if the other side is a face, player B wins if the other side is a back. What is the probability of either player winning?

9. A six-sided die is rolled 10 times.

- What is the probability that there is at least one number that is never rolled?
- What is the probability that each number is rolled at least once?

10. Let us choose a permutation σ of $\{1, 2, \ldots, 10\}$ uniformly at random. For each $i \in \{1, 2, \ldots, 10\}$, let $A_{i,j}$ denote the event $\sigma(i) = j$.

(a) For $i \neq j$, are the events $A_{i,i}$ and $A_{j,j}$ independent?
(b) For $i \neq j$, are the events $A_{i,j}$ and $A_{j,i}$ independent?
(c) What is the probability that there is no $i \in \{1, 2, \ldots, 10\}$ such that $\sigma(i) = i$?

11. On a game show, the contestant is presented with a set of three doors and asked to pick one. Each door has an equal chance of having either nothing or a cash prize of 525000 Kč behind it, independently of what is behind the other doors. Once the contestant has made their pick, the host (who knows what is behind each door) chooses to open one of the doors the contestant did not pick and reveals nothing behind it. The host then asks whether the contestant wishes to stick with their initial choice or switch to the other unopened door.

- Determine the probability the contestant wins the cash prize if they stick with their initial choice.
- Determine the probability the contestant wins the cash prize if they switch to the other unopened door.

12. There are $n \geq 4$ people at a party. Any two of the people are acquaintances with one another with probability $\frac{1}{2}$, independently of the other pairs. Choosing four people uniformly at random, what is the probability that either they all are acquaintances with each other or that none of them are acquaintances with each other?