Exercise session 3 – Prob. & Stat. 2

Classification of states. For states i, j and $k \in \mathbb{N}$ let $r_{i,j}(k) = P(\text{we get from } i \text{ to } j \text{ in } k \text{ steps})$. We also say that:

1. A state j is *accessible* from i, denoted $j \in A(i)$ or $i \to j$, if starting at i, the probability of ever visiting j is positive. (Alternatively: in the diagram, there is a directed path from i to j.)

2. States i, j are in the same *communicating class*, denoted $i \leftrightarrow j$, if $j \in A(i)$ and $i \in A(j)$. (Alternatively: in the diagram, i and j are in the same strongly connected component.)

3. A MC is *irreducible* if there is only 1 communicating class. (Alternatively: the diagram is strongly connected as a directed graph.)

4. A state i has *period* $d_i := \gcd\{t : r_{i,i}(t) > 0\}$. It is called *periodic* if $d_i > 1$ and aperiodic if $d_i = 1$.

5. A state i is *recurrent* if we return to it with probability 1, that is if $f_{i,i} := P(\exists t > 0 : X_t = i \mid X_0 = i) = 1$. States that are not recurrent are called *transient*.

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1. (Classification) Consider the Markov chain $(X_t)_{t\geq 0}$ with the below diagram.

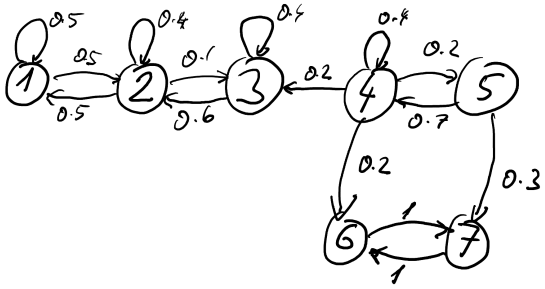
(a) Determine its communicating classes.

(b) Which states are transient? Which states are recurrent?

(c) From among the recurrent states, which ones are periodic?

(d) Suppose that $X_0 = 5$. Find the probability distributions of X_1 and X_2.

(e) Suppose that $X_0 = 5$. Prove that $P(X_n = 5) \leq 0.7^n$. Use this to compute $P(X_{20} = 5)$ to the nearest %.



2. (Joined cycles) Consider a Markov chain with states $1, 2, \ldots, 9$ and the following transition probabilities: $p_{1,2} = p_{1,7} = 1/2$, $p_{i,i+1} = 1$ for $i \neq 1, 6, 9$ and $p_{6,1} = p_{9,1} = 1$.

(a) Determine which states are recurrent/transient/periodic.

(b) What if there was an extra state 10 with $p_{9,10} = p_{10,1} = 1$?

3. (Towards the stationary distribution) In the above diagram, consider the Markov chain $(X_t)_{t\geq 0}$ with states 1, 2, 3 only. Suppose that we do 100 steps starting from say $X_0 = 2$.

(a) Denote $x_1 = P(X_{100} = 1)$, $x_2 = P(X_{100} = 2)$, $x_3 = P(X_{100} = 3)$. Write $y_3 := P(X_{101} = 3)$ using x_1, x_2, x_3.

(b) Do the same for $y_1 = P(X_{101} = 1)$ and $y_2 = P(X_{101} = 2)$.

(c) Asuming that $x_1 \approx y_1$, $x_3 \approx y_3$, $x_3 \approx y_3$, solve for (approximate) x_1, x_2 and x_3.

(d) What would change if instead we had $X_0 = 3$?

4. (Spider vs fly) A spider and a fly simultaneously move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is 100 units. When the spider and the fly land in the same position, the spider captures the fly. (They may "exchange the positions" though.)

(a) Construct a Markov chain that describes the distance between the spider and the fly.

(b) Identify the transient and the recurrent states.

5. (Shuffling cards) Consider the following ways to shuffle a deck of $n \geq 3$ cards:

A: Repeatedly take the top card and put it at the bottom.

B: Repeatedly take the top k cards, where k is random, uniform over $1 \leq k \leq n-1$, and put them at the bottom.

C: Repeatedly take the top card and put it at a random position (incl., possibly, back at the top).

Model those ways to shuffle as Markov chain M_A, M_B, M_C with $n!$ states.

(a) Which M_i are (ir)reducible?

(b) Which states of which M_i are (a)periodic?

6. Let C be a communicating class of a Markov chain, that is, an equivalence class of $\leftrightarrow$. Prove that:

(a) Either all states in C are recurrent, or they are all are transient.

(b) All states in C have the same period.

7. Consider a finite irreducible Markov chain $(X_t)_{t\geq 0}$ with at least two states and the following two properties:

(a) $(X_t)_{t\geq 0}$ is aperiodic.

(b) There is n such that for every i, j we have $r_{i,j}(n) > 0$.

Show that (b) implies (a). Think for a while about the other implication (but a proper proof is somewhat lengthy).

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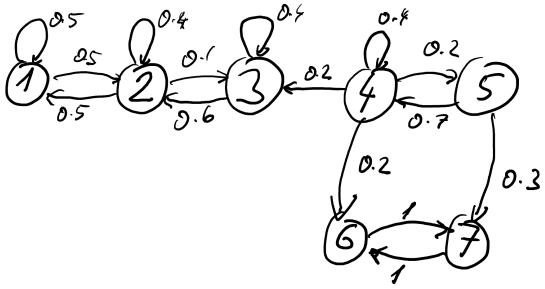

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