

Exercise session 2 – Prob. & Stat. 2

Markov chains basics

In practice, we think of a Markov chain in terms of its transition graph, or its transition matrix (plus the initial condition). The transition matrix has all row sums equal to 1.

(Formally, Markov chain is a special type of a stochastic process. A stochastic process is a sequence $(X_t)_{t=0}^{\infty}$ of random variables with values in some set S . Here t represents the time and X_t the (random) state at time t . We say that a stochastic process is a (discrete time, time-homogeneous) *Markov chain* if “only the last value matters”, i.e., if for every $t \geq 0$ and every “history” $a_0, \dots, a_t = i, a_{t+1} = j \in S$ we have

$$P(X_{t+1} = j \mid X_t = i \& X_{t-1} = a_{t-1} \& \dots \& X_0 = a_0) = P(X_{t+1} = j \mid X_t = i) = p_{i,j},$$

for some collection of transition probabilities $p_{i,j}$. (The condition is only required when the conditional probabilities are defined, that is when $P(X_t = a_t \& \dots \& X_0 = a_0) > 0$.)

Also recall: On average, an event that has probability p happens after $1/p$ attempts.

1. (Matrix to graph) Draw a transition graph of the Markov chain $(X_t)_{t \geq 0}^{\infty}$ with states $\{1, 2, 3\}$ and transition matrix

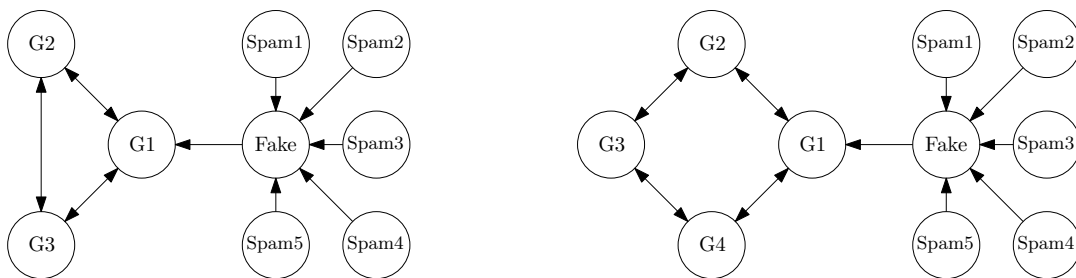
$$\begin{pmatrix} .4 & .4 & .2 \\ 0 & .8 & .2 \\ .9 & 0 & .1 \end{pmatrix}.$$

2. (Algebra) Consider the Markov chain from the previous problem.

- Find $P(X_4 = 3 \mid X_3 = 2)$.
- Find $P(X_3 = 1 \mid X_2 = 3)$.
- Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \& X_1 = 2)$.
- Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \& X_1 = 2 \& X_2 = 3)$.
- Suppose that $X_0 = 1$. What is $P(X_3 = 1)$?

3. (Random walk) A token starts at a node selected uniformly at random and it performs 100 steps, always selecting a uniformly random outgoing edge. Where does the token end?

- In the left diagram, handwave the approximate resulting probability distribution.
- In the right diagram, compute the exact resulting probability distribution. (Hint: Think what happens in the first few steps if the token starts at G1.)



4. (Washing machine) Each day, the washing machine breaks down with probability 20%. When it breaks down, it takes 3 days to fix it.

- Draw a transition graph of a Markov chain that models this situation.
- In the long term, what percentage of time does the machine work?

5. (Mouse) A mouse moves along a corridor with 4 tiles numbered 1-4 left to right. From tile $i \in \{2, 3\}$ it moves to either tile $i - 1$ or $i + 1$ with equal probability. From tile 1 (resp. 4), it moves to tile 2 (resp. 3) with probability 1. Each time the mouse moves left, an electronic device outputs a signal “L”, otherwise it outputs “R”.

- Is the position of the mouse a Markov chain?
- Can the generated sequence of signals L and R be described as a Markov chain with states L and R?
- Can the sequence be represented as a Markov chain with more states?

6. ($t \rightarrow 2t$) Suppose $(X_t)_{t=0}^\infty$ is a Markov chain. Show that $(X_{2t})_{t=0}^\infty$ is also a Markov chain. What is its transition matrix?
7. (iids) Show that any sequence of independent identically distributed random variables taking values in a finite set S is a Markov chain.
8. (Die rolls) A die is rolled repeatedly. Show that if we allow Markov chains to have infinitely many states, then all of the following are Markov chains. Also draw the transition graphs.
- (a) $(M_t)_{t=1}^\infty$, where M_t is the largest number that appeared during the first t rolls.
 - (b) $(N_t)_{t=1}^\infty$, where N_t is the total number of sixes during the first t rolls.
 - (c) $(A_t)_{t=1}^\infty$, where A_t is the number of rolls after the most recent six.
 - (d) (bonus) $(B_t)_{t=0}^\infty$, where B_t is the number of rolls *before* the next six.
-

Quiz

- The probability of winning a lottery is $1/1,000,000$. Then, the probability that I win it at least once, if I buy 100 tickets, is between $1/100,000$ and $1/10,000$.
- Let's say that the average salary in a country is 1,000 dollars. Then, the proportion of people who make $> 10,000$ dollars is $\leq 10\%$.
- Let T_1 be how many steps we need to finish a game of snakes & ladders. Let T_2 be how many steps we need if we play the variant without snakes / ladders. Say we have $E[T_1] \approx E[T_2] \approx 30$. Then $\text{Var}(T_1) > \text{Var}(T_2)$.
- Say we have a random variable X with $E[X] = \mu$ and $\text{Var}[X] = \sigma^2$. Then, it can't happen that $P(|X - \mu| > 10\sigma) > 0.99$, but it can happen that $P(|X - \mu| < \sigma/10) > 0.99$.

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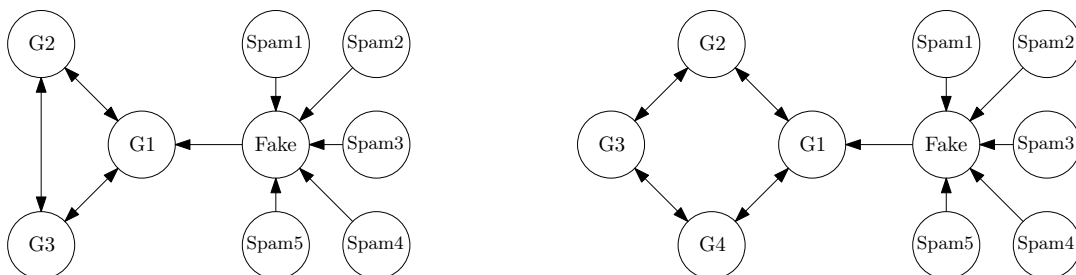
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