

Exercise session 1 – Prob. & Stat. 2

Chance means probability. *Random* means uniformly random (and all events are independent). *Rolling an n -die* gives a random number from 1 to n . A (regular) die is a 6-die. (Plural of one die is multiple *dice*.) *Flipping a p -coin* gives heads (aka H aka 1) with probability p and tails (aka T aka 0) with probability $1 - p$. A (fair) coin is a $\frac{1}{2}$ -coin.

1. (Time until a 5) Repeatedly rolling a die, how many rolls, on average, until you get a number 5?
2. (A 6 before a prime) Repeatedly rolling a die, what is the chance that you get a 6 before getting a prime?
3. (Rabbits) There are 2 black and 8 white rabbits in a hat. A magician pulls three of them out one by one.
 - (a) What is the chance that the first rabbit is black?
 - (b) What is the chance that the second rabbit is black?
 - (c) What is the chance that the third rabbit is black?
4. (Voucher) You got a voucher to play the following game: you keep rolling a 3-die, for every 1 you get \$1, for every 2 you get \$2, and once you roll 3 you stop and take all the money you won so far. How much is the voucher worth?
5. (Time until a pattern) Repeatedly flipping a coin, how many flips, on average, until you see...
 - (a) a pattern HT?
 - (b) a pattern HH?
 - (c) (Bonus) a pattern HHTH?(Btw it can be shown that for any binary pattern of any length, the average is always an integer!)
6. (Catch) In a game of “Catch”, 3 people (Alice, Bob, and Charlie) stand around a circle with Alice initially having a disk, and the person with the disk always passes to another random person. Let p_n be the probability that after n passes Alice again has the disk.
 - (a) Find p_2 . Is it more or less than $1/3$?
 - (b) Find p_3 . Is it more or less than $1/3$?
 - (c) For each $n \geq 2$, express p_{n+1} in terms of p_n .
 - (d) Assuming that $\lim_{n \rightarrow \infty} p_n$ exists, find it.
7. (Penney’s game) Alice and Bob each have a favorite pattern of Hs and Ts, and they keep flipping a coin until one of those two patterns occurs for the first time, at which point that person is the winner.
 - (a) Suppose $A = \text{HHH}$ and $B = \text{THH}$. What is the chance that Alice wins?
 - (b) Suppose $A = \text{THH}$ and $B = \text{TTH}$. What is the chance that Alice wins?
8. (Distance per turn) In certain board games (e.g. *Člověče*), in each turn you roll a die and move your token by the number rolled.
 - (a) By how much do you move your token per turn, on average?
 - (b) Suppose that anytime you roll a 6, you immediately roll again (and you end your turn only after you have rolled a non-6). By how much do you move your token in total, on average?
 - (c) Imagine instead of a 6 you can pick your own “lucky” number. Which lucky number allows you to move your token the furthest per turn, on average?
9. (Coupon collector) How many rolls of a die, on average, until you see each number from 1 to 6 at least once?
10. (Light switches) There are ten light switches on a wall next to each other, each controlling a different light bulb. Initially, all light bulbs are off. In each step, you pick a switch at random and flip it. On average, how many steps until all light bulbs are off again?
11. (Bonus bonus) In question 5., how would you proceed for an arbitrary pattern?

Exercise session 1 – Prob. & Stat. 2

Chance means probability. *Random* means uniformly random (and all events are independent). *Rolling an n -die* gives a random number from 1 to n . A (regular) die is a 6-die. (Plural of one die is multiple *dice*.) *Flipping a p -coin* gives heads (aka H aka 1) with probability p and tails (aka T aka 0) with probability $1 - p$. A (fair) coin is a $\frac{1}{2}$ -coin.

1. (Time until a 5) Repeatedly rolling a die, how many rolls, on average, until you get a number 5?
2. (A 6 before a prime) Repeatedly rolling a die, what is the chance that you get a 6 before getting a prime?
3. (Rabbits) There are 2 black and 8 white rabbits in a hat. A magician pulls three of them out one by one.
 - (a) What is the chance that the first rabbit is black?
 - (b) What is the chance that the second rabbit is black?
 - (c) What is the chance that the third rabbit is black?
4. (Voucher) You got a voucher to play the following game: you keep rolling a 3-die, for every 1 you get \$1, for every 2 you get \$2, and once you roll 3 you stop and take all the money you won so far. How much is the voucher worth?
5. (Time until a pattern) Repeatedly flipping a coin, how many flips, on average, until you see...
 - (a) a pattern HT?
 - (b) a pattern HH?
 - (c) (Bonus) a pattern HHTH?(Btw it can be shown that for any binary pattern of any length, the average is always an integer!)
6. (Catch) In a game of “Catch”, 3 people (Alice, Bob, and Charlie) stand around a circle with Alice initially having a disk, and the person with the disk always passes to another random person. Let p_n be the probability that after n passes Alice again has the disk.
 - (a) Find p_2 . Is it more or less than $1/3$?
 - (b) Find p_3 . Is it more or less than $1/3$?
 - (c) For each $n \geq 2$, express p_{n+1} in terms of p_n .
 - (d) Assuming that $\lim_{n \rightarrow \infty} p_n$ exists, find it.
7. (Penney’s game) Alice and Bob each have a favorite pattern of Hs and Ts, and they keep flipping a coin until one of those two patterns occurs for the first time, at which point that person is the winner.
 - (a) Suppose $A = \text{HHH}$ and $B = \text{THH}$. What is the chance that Alice wins?
 - (b) Suppose $A = \text{THH}$ and $B = \text{TTH}$. What is the chance that Alice wins?
8. (Distance per turn) In certain board games (e.g. *Člověče*), in each turn you roll a die and move your token by the number rolled.
 - (a) By how much do you move your token per turn, on average?
 - (b) Suppose that anytime you roll a 6, you immediately roll again (and you end your turn only after you have rolled a non-6). By how much do you move your token in total, on average?
 - (c) Imagine instead of a 6 you can pick your own “lucky” number. Which lucky number allows you to move your token the furthest per turn, on average?
9. (Coupon collector) How many rolls of a die, on average, until you see each number from 1 to 6 at least once?
10. (Light switches) There are ten light switches on a wall next to each other, each controlling a different light bulb. Initially, all light bulbs are off. In each step, you pick a switch at random and flip it. On average, how many steps until all light bulbs are off again?
11. (Bonus bonus) In question 5., how would you proceed for an arbitrary pattern?