Weak and strong solvability of interval linear systems of equations and inequalities

Milan Hladík*

Abstract

We consider weak and strong solvability of general interval linear systems consisting of mixed equations and inequalities with mixed free and sign-restricted variables. We generalize the well-known weak solvability characterizations by Oettli–Prager (for equations) and Gerlach (for inequalities) to a unified framework. In the same manner, we extend strong solvability theorems to general interval linear systems. Next, we propose a sufficient condition for checking strong solvability. We give an application in linear programming with interval data. By means of weak and strong solvability we determine limits of the optimal values for any form of the problem setting.

Keywords: *Interval matrix, interval analysis, interval linear system.*

1 Introduction

Solving interval linear systems is a fundamental problem in interval analysis. Intervals naturally appear in many situations when dealing

*Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800, Prague, Czech Republic, e-mail: `milan.hladik@matfyz.cz`

with inexact values, error measurements or when performing sensitivity analysis. Square interval linear systems of equations are the leading reasearch subject in this topic, but non-square systems and inequalities are of interest, too. Various results exist separately for interval equations and inequalities, but a unified framework is still missing.

In this paper, we consider the general case of mixed interval linear equation and inequalities with both non-negative and free variables, and generalize the results developed for specific systems. We characterize the so called weak and strong solvability. First we deal with weak solvability, which is quite easy to handle due to its nature. Contrary, strong solvability is more attractive to extend to the general framework. Furthermore, we propose a new sufficient condition for strong solvability. Eventually, we apply our approach in interval linear programming by showing how to determine the optimal value range for the general linear program with interval coefficients. Beside interval linear programming, it can be utilized in linearization techniques in constraint programming, or when performing a Lyapunov-like function relaxation in finding a basin of attraction for polynomial ordinary differential equations [10].

Let us introduce some notation. An interval matrix is defined as

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \leq \overline{A}$, are given matrices. Interval vectors are defined similarly. The set of all m-by-n interval matrices will be denoted by $\mathbb{IR}^{m\times n}$ and the set of all n-dimensional interval vectors by $\mathbb{IR}^n$. By

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the center and the radius of $\boldsymbol{A}$, respectively. The relation $\boldsymbol{A} \leq \boldsymbol{B}$ is defined as $\overline{A} \leq \underline{B}$. For the standard interval arithmetic see e.g. the books [1, 7, 8].

The symbol T_s stands for the diagonal matrix with entries given by the vector s, and $\operatorname{sgn}(r)$ for the sign of a real r, i.e., $\operatorname{sgn}(r) = 1$

if $r \geq 0$ and $\operatorname{sgn}(r) = -1$ otherwise. The convex hull of a set $\mathcal{S}$ is denoted by $\operatorname{conv}\mathcal{S}$, and the vector of ones is denoted by e. An equation, inequality, sign and absolute value applied on matrices and vectors are understood entrywise.

2 Solvability of a general interval linear system

Let $\boldsymbol{A} \in \mathbb{IR}^{k\times m}$, $\boldsymbol{B} \in \mathbb{IR}^{k\times n}$, $\boldsymbol{C} \in \mathbb{IR}^{\ell\times m}$, $\boldsymbol{D} \in \mathbb{IR}^{\ell\times n}$, $\boldsymbol{b} \in \mathbb{IR}^{k}$, and $\boldsymbol{d} \in \mathbb{IR}^{\ell}$. Consider the general interval linear system

$$\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{b},\ \boldsymbol{C}x + \boldsymbol{D}y \leq \boldsymbol{d},\ x \geq 0 \tag{1}$$

which is a shortage for the family of linear systems

$$Ax + By = b,\ Cx + Dy \leq d,\ x \geq 0 \tag{2}$$

with $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $D \in \boldsymbol{D}$, $b \in \boldsymbol{b}$, and $d \in \boldsymbol{d}$. By *a realization* we mean a concrete setting (2).

Each interval linear system can be transformed into the form of (1). However, the standard reduction to an system of inequalities, or to a system of equations with non-negative variables is not possible here due to dependencies that would emerge. For example, transforming $\boldsymbol{A}x = \boldsymbol{b}$ to $\boldsymbol{A}x \leq \boldsymbol{b}$, $\boldsymbol{A}x \geq \boldsymbol{b}$ leads to a non-equivalent interval system since values of interval are chosen independently and the correlation of entries is lost. Contrary, the transformation of $\boldsymbol{A}x \leq \boldsymbol{b}$ to $\boldsymbol{A}x + z = \boldsymbol{b}$, $z \geq 0$ is possible, but, for the purpose of checking strong solvability, inequalities are much more convenient. Therefore, one has to study interval systems in the general form of (1).

2.1 Weak solutions

First, we deal with weak solutions, usually addressed to as *solutions*. A weak solution to (1) is any $(x, y) \in \mathbb{R}^{m+n}$ satisfying (2) for some

$A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $D \in \boldsymbol{D}$, $b \in \boldsymbol{b}$, and $d \in \boldsymbol{d}$. An interval system is called weakly solvable if it has a weak solution. For interval systems of equations $\boldsymbol{M}z = \boldsymbol{v}$, the set of all weak solutions is described by the Oettli–Prager theorem [9]

$$|M_c z - v_c| \leq M_\Delta |z| + v_\Delta.$$

For interval inequalities $\boldsymbol{M}z \leq \boldsymbol{v}$, Gerlach's theorem [3] does the job by the description

$$M_c z \leq M_\Delta |z| + \overline{v}.$$

Checking weak solvability of both cases is NP-hard [6, 15]. Provided variables are known a priori to be non-negative, the description is simplified to ordinary linear constraints (cf. [15]); for equations we get $\underline{M}z \leq \overline{v}$, $\overline{M}z \geq \underline{v}$, $z \geq 0$ and for inequalities we have $\underline{M}z \leq \overline{v}$, $z \geq 0$. The above results can be put together to give rise a characterization of weak solutions of the general interval system.

Theorem 1. *The set of weak solutions to* (1) *is described by*

$$\underline{A}x + B_c y \leq B_\Delta |y| + \overline{b}, \tag{3a}$$

$$-\overline{A}x - B_c y \leq B_\Delta |y| - \underline{b}, \tag{3b}$$

$$\underline{C}x + D_c y \leq D_\Delta |y| + \overline{d},\ \ x \geq 0. \tag{3c}$$

Proof. By the Oettli–Prager theorem, to be a weak solution to $\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{b}$, $x \geq 0$, any $(x, y) \in \mathbb{R}^{m+n}$ has to fulfill

$$|A_c x + B_c y - b_c| \leq A_\Delta |x| + B_\Delta |y| + b_\Delta = A_\Delta x + B_\Delta |y| + b_\Delta,$$

that is,

$$A_c x + B_c y - b_c \leq A_\Delta x + B_\Delta |y| + b_\Delta,\ \ -A_c x - B_c y + b_c \leq A_\Delta x + B_\Delta |y| + b_\Delta.$$

By Gerlach's theorem, weak solutions to $\boldsymbol{C}x + \boldsymbol{D}y \leq \boldsymbol{d}$, $x \geq 0$, are described

$$C_c x + D_c y \leq C_\Delta x + D_\Delta |y| + \overline{d},\ \ x \geq 0.$$

Thus, any solution to (1) must satisfy both condition, which results in the characterization by (3). ∎

Since $|y| = T_{\mathrm{sgn}(y)}y$ for any $y \in \mathbb{R}^n$, we can linearize the absolute value in the description of weak solutions and obtain a characterization by means of 2^n linear systems.

Corollary 1. *A pair $(x, y) \in \mathbb{R}^{m+n}$ is a weak solution to (1) if and only if there is $s \in \{\pm 1\}^n$ such that*

$$\begin{aligned} \underline{A}x + (B_c - B_\Delta T_s)y &\le \overline{b}, \\ -\overline{A}x - (B_c y + B_\Delta T_s)y &\le -\underline{b}, \\ \underline{C}x + (D_c - D_\Delta T_s)y &\le \overline{d},\ \ x \ge 0. \end{aligned}$$

Given a weak solution $(x, y) \in \mathbb{R}^{m+n}$, one may ask for a realization (2) of the interval system having (x, y) as a solution.

Theorem 2. *Let $(x, y) \in \mathbb{R}^{m+n}$ be a weak solution to (1). Then (x, y) solves (2) with*

$$\begin{aligned} &A = A_c - T_u A_\Delta,\ \ B = B_c - T_u B_\Delta T_{\mathrm{sgn}(y)},\ \ b = b_c + T_u b_\Delta, \\ &C = \underline{C},\ \ D = D_c - D_\Delta T_{\mathrm{sgn}(y)},\ \ d = \overline{d}, \end{aligned}$$

where $u \in [-1, 1]^k$ is defined

$$u_i = \begin{cases} \frac{(A_c x + B_c y - b_c)_i}{(A_\Delta x + B_\Delta |y| + b_\Delta)_i} & \text{if } (A_\Delta x + B_\Delta |y| + b_\Delta)_i > 0, \\ 1 & \text{otherwise,} \end{cases} \qquad i = 1, \dots, k.$$

Proof. Realizations concerning equations follow from [15, Theorem 2.9]. Realizations concerning inequalities easily follow from $|y| = T_{\mathrm{sgn}(y)}y$ and the inequality

$$\underline{C}x + D_c y \le D_\Delta |y| + \overline{d}. \qquad \square$$

2.2 Strong solutions

The general interval system (1) is called strongly solvable if each realization is solvable. A strong solution to (1) is any $(x, y) \in \mathbb{R}^{m+n}$ such that it solves (2) for every realization $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$,

$D \in \boldsymbol{D}$, $b \in \boldsymbol{b}$, and $d \in \boldsymbol{d}$. Clearly, existence of a strong solution implies strong solvability, but not converse (contrary to a pure interval system of inequalities; see [15, Theorem 2.24]).

Theorem 3. *A pair $(x, y) \in \mathbb{R}^{m+n}$ is a strong solution to* (1) *if and only if it satisfies*

$$b_\Delta = 0,\ A_\Delta x = 0,\ B_\Delta|y| = 0,\ A_c x + b_c y = b_c,\ \overline{C}x + D_c y + D_\Delta|y| \le \underline{d},\ x \ge 0.$$

Proof. To be $(x, y) \in \mathbb{R}^{m+n}$ a strong solution, it should satisfy

$$Ax + By - b = 0,\ Cx + Dy - d \le 0, x \ge 0$$

for every realization. Equivalently,

$$\boldsymbol{A}x + \boldsymbol{B}y - \boldsymbol{b} = 0,\ \boldsymbol{C}x + \boldsymbol{D}y - \boldsymbol{d} \le 0, x \ge 0,$$

where the left-hand sides of equations and inequalities are calculated by interval arithmetic. Two intervals are identical if and only if they have the same centers and radii. So the interval condition $\boldsymbol{A}x+\boldsymbol{B}y-\boldsymbol{b} = 0$ is equivalent to the equality of their centers, i.e. $A_c x + B_c y = b_c$, and equality of their radii, i.e. $A_\Delta x + B_\Delta|y| + b_\Delta = 0$. The latter can be written as $b_\Delta = 0$, $A_\Delta x = 0$, $B_\Delta|y| = 0$. The condition $\boldsymbol{C}x + \boldsymbol{D}y - \boldsymbol{d} \le 0$ holds true if and only if the upper limit of the left-hand side is non-positive, that is, $\overline{\boldsymbol{C}x} + \overline{\boldsymbol{D}y} - \underline{d} \le 0$. Since $\overline{\boldsymbol{C}x} = \overline{C}x$, and $\overline{\boldsymbol{D}y} = D_c y + D_\Delta|y|$, we have the characterization in question. □

2.3 Strong solvability

Due to the presence of equations in the general interval systems, a strong solution is not likely to exist. Nevertheless, strong solvability may still hold true.

An interval system $\boldsymbol{M}z \le \boldsymbol{v}$ is strongly solvable if and only if $\overline{M}z^1 - \underline{M}z^2 \le \underline{v},\ z^1 \ge 0,\ z^2 \ge 0$, is solvable, see [15, 17]. For an interval system $\boldsymbol{M}z \le \boldsymbol{v}$, $z \ge 0$, strong solvability is equivalent to solvability of $\overline{M}z \le \underline{v}$. Thus, checking strong solvability for inequalities in a tractable problem. Contrary, checking strong solvability of equations is NP-hard, even with non-negative variables [12, 15]. An interval system $\boldsymbol{M}z = \boldsymbol{v}$ of k equations is strongly solvable if and only if

$$(M_c + T_s M_\Delta)z^1 - (M_c - T_s M_\Delta)z^2 = v_c - T_s v_\Delta,\ z^1, z^2 \geq 0,$$

is solvable for each $s \in \{\pm 1\}^k$, see [13, 15]. Strong solvability of $\boldsymbol{M}z = \boldsymbol{v}$, $z \geq 0$, is equivalent to solvability of

$$(M_c + T_s M_\Delta)z = v_c - T_s v_\Delta,\ z \geq 0$$

for each $s \in \{\pm 1\}^k$, see [11, 15]. The following theorem extends all the above mentioned results to a unified characterization of strong solvability of the general interval system (1).

Theorem 4. *The system* (1) *is strongly solvable if and only if the system*

$$(A_c + T_s A_\Delta)x + (B_c + T_s B_\Delta)y^1 - (B_c - T_s B_\Delta)y^2 = b_c - T_s b_\Delta, \tag{4a}$$

$$\overline{C}x + \overline{D}y^1 - \underline{D}y^2 \leq \underline{d}, \tag{4b}$$

$$x, y^1, y^2 \geq 0. \tag{4c}$$

is solvable for every $s \in \{\pm 1\}^k$.

Proof. By negation, suppose that (2) is not solvable for some realization of interval data. By Farkas theorem, there are $p \in \mathbb{R}^k$ and $q \in \mathbb{R}^\ell$, $q \geq 0$, such that

$$A^T p + C^T q \geq 0,\ B^T p + D^T q = 0,\ b^T p + d^T q \leq -1.$$

That is, (p, q) forms a weak solution to the interval system

$$\boldsymbol{A}^T p + \boldsymbol{C}^T q \geq 0,\ \boldsymbol{B}^T p + \boldsymbol{D}^T q = 0,\ \boldsymbol{b}^T p + \boldsymbol{d}^T q \leq -1,\ q \geq 0$$

and by Corollary 1 there is $s \in \{\pm 1\}^k$ such that

$$\begin{aligned}
-(A_c + T_s A_\Delta)^T p - \overline{C}^T q &\le 0,\\
(B_c - T_s B_\Delta)^T p + \underline{D}^T q &\le 0,\\
-(B_c + T_s B_\Delta)^T p - \overline{D}^T q &\le 0,\\
(b_c - T_s b_\Delta)^T p + \underline{d}^T q &\le -1,\ \ q \ge 0.
\end{aligned}$$

Again, we utilize Farkas theorem, saying that there is no non-negative x, y^1, y^2, z satisfying

$$\begin{aligned}
-(A_c + T_s A_\Delta)x + (B_c - T_s B_\Delta)y^2 - (B_c + T_s B_\Delta)y^1 + (b_c - T_s b_\Delta)z &= 0,\\
-\overline{C}x + \underline{D}y^2 - \overline{D}y^1 + \underline{d}z &\ge 0,\\
-z &= -1.
\end{aligned}$$

This is equivalent to (4). □

Provided we check strong solvability by Theorem 4, we can also output a bounded set in which each realization (2) has at least one solution; cf. [15].

Theorem 5. *If the system* (1) *is strongly solvable then each realization has a solution in the set*

$$\operatorname*{conv}_{s \in \{\pm 1\}^k} \{(x_s, y_s^1 - y_s^2)\},$$

where x_s, y_s^1, y_s^2 *is a solution to* (4), $s \in \{\pm 1\}^k$.

Proof. Let $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $D \in \boldsymbol{D}$, $b \in \boldsymbol{b}$, and $d \in \boldsymbol{d}$. Then for any convex combination

$$x = \sum_{s \in \{\pm 1\}^k} \lambda_s x_s, \quad y = \sum_{s \in \{\pm 1\}^k} \lambda_s (y_s^1 - y_s^2),$$

with $\sum_{s\in\{\pm 1\}^k} \lambda_s = 1$ and $\lambda_s \geq 0$, $s \in \{\pm 1\}^k$, we have

$$\begin{aligned} Cx + Dy &= \sum_{s\in\{\pm 1\}^k} \lambda_s C x_s + \sum_{s\in\{\pm 1\}^k} \lambda_s D(y_s^1 - y_s^2) \\ &\leq \sum_{s\in\{\pm 1\}^k} \lambda_s \overline{C} x_s + \sum_{s\in\{\pm 1\}^k} \lambda_s (\overline{D} y_s^1 - \underline{D} y_s^2) \\ &\leq \sum_{s\in\{\pm 1\}^k} \lambda_s \underline{d} = \underline{d} \leq d. \end{aligned}$$

Thus it suffices to find an appropriate convex combination satisfying the equations. That is, we want to show that the system

$$\sum_{s\in\{\pm 1\}^k} \lambda_s (A x_s + B(y_s^1 - y_s^2)) = b, \quad \sum_{s\in\{\pm 1\}^k} \lambda_s = 1,\ \lambda_s \geq 0,\ s \in \{\pm 1\}^k$$

has a solution with respect to λ_s, $s \in \{\pm 1\}^k$. Equivalently, by Farkas theorem, for each $(p, q) \in \mathbb{R}^{k+1}$ it should hold that

$$\big(\forall s \in \{\pm 1\}^k : p^T(A x_s + B(y_s^1 - y_s^2)) + q \geq 0\big) \quad \Rightarrow \quad p^T b + q \geq 0.$$

Let $(p, q) \in \mathbb{R}^{k+1}$ and put $s := \operatorname{sgn}(p)$. Now,

$$\begin{aligned} 0 &\leq p^T(A x_s + B(y_s^1 - y_s^2)) + q \\ &\leq p^T((A_c + T_s A_\Delta) x_s + (B_c + T_s B_\Delta) y_s^1 - (B_c - T_s B_\Delta) y_s^2) + q \\ &= p^T(b_c - T_s b_\Delta) + q \leq p^T b + q, \end{aligned}$$

which closes the proof. $\square$

2.4 A sufficient condition for strong solvability

In view of intractability of checking strong solvability, a sufficient condition is welcome. We propose the following method:

1. Solve a suitable linear program.
2. Based on the previous result, transform the equations to a square system.

3. Compute an enclosure $\boldsymbol{x}, \boldsymbol{y}$ to the square system.

4. Check whether $\boldsymbol{x}, \boldsymbol{y}$ fulfills the inequalities.

Now, we describe particular steps in detail. In Step 1, we solve the linear program

$$\max \alpha \ \text{ subject to } \ A_c x + B_c y = b_c,\ C_c x + D_c y + \alpha e \leq \underline{d},\ x \geq 0.$$

In case there are no inequalities, we solve

$$\max \alpha \ \text{ subject to } \ A_c x + B_c y = b_c, \quad x \geq \alpha e.$$

instead. The idea is to determine a sufficiently robust feasible solution. Let x^*, y^* be an optimal solution. If the linear program is infeasible, then the strong solvability is not valid. If the linear program is unbounded, then we take any solution on its unbounded edge.

In Step 2, we transform the inequality system $A_c x + B_c y = b_c$ to a square one. Suppose $k < m + n$. For each i such that $x_i^* = 0$ we remove the ith column of $\boldsymbol{A}$ and $\boldsymbol{C}$, and the ith entry of x^*. Thus, the number of variables decreases. If there are more vanishing entries of x^* than the value of $m + n - k$, then we remove only $m + n - k$ columns. If after the dimension reduction the system is not square yet, we add a suitable number of artificial equations $A'x + B'y = b'$. The natural choice for $(A'\ B')$ is the orthogonal basis of the null space to $(A_c\ B_c)$, and the right-hand side is calculated $b' := A'x^* + B'y*$.

In Step 3, we solve the interval linear system of equations

$$\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{b}, \quad A'x + B'y = b'.$$

This is a standard problem in interval analysis and many methods exist to calculate interval vectors $\boldsymbol{x}, \boldsymbol{y}$ enclosing all weak solutions [7, 8, 15]. Even though $\boldsymbol{x}, \boldsymbol{y}$ do not contain all weak solutions of $\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{b}$, they do contain at least one solution of any realization $Ax + By = b$ with $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$ and $b \in \boldsymbol{b}$.

In Step 4, we check whether $\boldsymbol{x} \geq 0$ and $\boldsymbol{C}\boldsymbol{x}+\boldsymbol{D}\boldsymbol{y} \leq \boldsymbol{d}$. If yes, then each realization of interval data has at least one solution fulfilling the constraints and we output that the system is strongly solvable. If no, then we cannot decide.

The method requires solving one linear program, orthogonal basis of the null space and enclosure to the solution set of interval linear equations, so the overall computational cost is very low. Moreover, it is easy to implement it in a reliable way. Since the linear program and orthogonal basis play role of a heuristic only, they needn't be computed verified. The other computation is done by using interval arithmetic. In our example, we employed INTLAB v6 , the interval toolbox for Matlab, see [18], and the verification package VERSOFT v10, see [16].

Next, the method gives a new and strong sufficient condition for strong solvability of specific cases of interval linear systems, namely for $A_c x = b_c$, $x \geq 0$ and for $B_c y = b_c$.

Example 1. Let

$$A = \begin{pmatrix} -3 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 4 \end{pmatrix}, \ C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ D = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}, \ d = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

For a given parameter $\delta \geq 0$, we enlarge real quantities to intervals having radii δ. Thus, $\boldsymbol{A}$ is defined as $A := [A - \delta ee^T, A + \delta ee^T]$ and analogously for $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{b}, \boldsymbol{d}$. The maximal value of δ, for which our method confirm strong solvability of (1), is $\delta \approx 0.6929$. The maximal value of δ, for which (1) is really strong solvable, is $\delta \approx 0.7391$.

3 Application to interval linear programming

We apply the developed results in interval linear programming [2, 4, 14]; for a survey of recent development see [5]. Consider a linear programming problem in the general form

$$f(M, u, v) = \min u^T z \ \text{ subject to } \ z \in \mathcal{P}(M, v) \tag{5}$$

where $\mathcal{P}(M, v)$ is a linearly constraint set with the constraint matrix M and the right hand side v. Suppose that input data vary within given intervals, that is, $M \in \boldsymbol{M}$, $u \in \boldsymbol{u}$ and $v \in \boldsymbol{v}$. One of the basic problems in interval linear programming is to calculate the optimal value range $\boldsymbol{f} = [\underline{f}, \overline{f}]$, where

$$\underline{f} := \inf \{f(M, u, v);\ M \in \boldsymbol{M}, u \in \boldsymbol{u}, v \in \boldsymbol{v}\},$$
$$\overline{f} := \sup \{f(M, u, v);\ M \in \boldsymbol{M}, u \in \boldsymbol{u}, v \in \boldsymbol{v}\}.$$

In [4], the author proposed a general algorithm to determine the optimal value range, which we remind now. Let

$$g(M, u, v) = \min v^T p \ \text{ subject to } \ p \in \mathcal{D}(M, u)$$

be the corresponding dual problem to (5), where $\mathcal{D}(M, u)$ is the dual feasible set with constraint matrix M and the right-hand side u. Denote by $\mathcal{P}$ and $\mathcal{D}$ the sets of weak feasible solutions of the primal and dual problem, respectively. Algorithm 1 exhibits particular steps to compute $\boldsymbol{f}$.

Algorithm 1 (Optimal value range $[\underline{f}, \overline{f}]$ for (5))**.**

1. Compute
$$\underline{f} := \inf \{u_c^T z - u_\Delta^T |z|;\ z \in \mathcal{P}\}.$$
2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute
$$\overline{\varphi} := \sup \{v_c^T p + v_\Delta^T |p|;\ p \in \mathcal{D}\}.$$
4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If $\mathcal{P}(M, v)$, $M \in \boldsymbol{M}$, $v \in \boldsymbol{v}$, is strongly solvable, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

We utilize the general scheme of Algorithm 1 to the family of linear programs

$$f(A,B,C,D,a,b,c,d) = \min a^T x + c^T y \ \text{ subject to } \ (2), \tag{6}$$

where $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, $C \in \boldsymbol{C}$, $D \in \boldsymbol{D}$, $a \in \boldsymbol{a}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, and $d \in \boldsymbol{d}$. Each linear program with interval data and no dependencies can be transformed into this form, so it generalizes the classical results from [2, 5, 14] and elsewhere.

By Theorem 1, the set of all possible feasible solutions $\mathcal{P}$ is described by (3). The dual problem to (6) reads

$$\max b^T p + d^T q \ \text{ subject to } \ A^T p + C^T q \le a, \ B^T p + D^T q = c, \ q \le 0.$$

By Theorem 1, the set $\mathcal{D}$ of all dual solutions is described

$$A_c^T p + \overline{C}^T q \le A_\Delta^T |p| + \overline{a}, \tag{7a}$$

$$B_c^T p + \overline{D}^T q \le B_\Delta^T |p| + \overline{c}, \tag{7b}$$

$$-B_c^T p - \underline{D}^T q \le B_\Delta^T |p| - \underline{c}, \ q \le 0. \tag{7c}$$

Theorem 4 characterizes strong solvability of the primal feasible set. Thus, we determined all the ingredients needed in Algorithm 1 to calculate the optimal value range. Algorithm 2 gives an exposition of the particular steps adapted to (6)

Algorithm 2 (Optimal value range $[\underline{f}, \overline{f}]$ for (6))**.**

1. Compute

$$\underline{f} := \inf \ \underline{a}^T x + c_c^T y - c_\Delta^T |y| \ \text{ subject to } \ (3).$$

2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup \ b_c^T p + b_\Delta^T |p| + \underline{d}^T q \ \text{ subject to } \ (7).$$

4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If (1) is strongly solvable (e.g. by Theorem 4), then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

The optimization problems in steps 1 and 3 and nonlinear in general. Using the same approach as in Corollary 1, we can linearize the absolute values giving rise to 2^n and 2^k linear programs, respectively.

Conclusion

We proposed a unified way to deal with interval linear systems. We extended the known characterizations of weak and strong solvability of interval linear equation and inequalities to common forms. Thus, each particular result is a special case of our general approach. For checking strong solvability, which is NP-hard problem, we proposed a sufficient condition. As an important application of weak and strong solvability we showed how to determine the optimal value range in interval linear programming. We needn't focus on a particular canonical form of interval linear programming since our algorithm works for any of them.

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