Non-three-colorable common graphs exist *

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Abstract

A graph H is called *common* if the total number of copies of H in every graph and its complement asymptotically minimizes for random graphs. A former conjecture of Burr and Rosta, extending a conjecture of Erdős asserted that every graph is common. Thomason disproved both conjectures by showing that K_4 is not common. It is now known that in fact the common graphs are very rare. Answering a question of Sidorenko and of Jagger, Šťovíček and Thomason from 1996 we show that the 5-wheel is common. This provides the first example of a common graph that is not three-colorable.

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1 Introduction

A natural question in extremal graph theory is how many monochromatic subgraphs isomorphic to a graph H must be contained in any two-coloring of the edges of the complete graph K_n. Equivalently, how many subgraphs isomorphic to a graph H must be contained in a graph and its complement?

Goodman [Goo59] showed that for $H = K_3$, the optimum solution is essentially obtained by a typical random graph. The graphs H that satisfy this property are called *common*. Erdős [Erd62] conjectured that all complete graphs are common. Later, this conjecture was extended to all graphs by Burr and Rosta [BR80]. Sidorenko [Sid89] disproved Burr and Rosta's conjecture by showing that a triangle with a pendant edge is not common. Later Thomason [Tho89] disproved Erdős's conjecture by showing that for $p \geq 4$, the complete graphs K_p are not common. It is now known that in fact the common graphs are very rare. For example, Jagger, Šťovíček and Thomason [JŠT96] showed that every graph that contains K_4 as a subgraph is not common. If we work with k-edge-colorings of K_n rather than 2-edge-colorings we get the notion of a k-*common* graph. Cummings and Young [CY] recently proved that no graph containing the triangle K_3 is 3-common, a counterpart of the result of Jagger, Šťovíček and Thomason above.

There are some classes of graphs that are known to be common. Sidorenko [Sid89] showed that cycles are common. A conjecture due to Erdős and Simonovits [ES84] and Sidorenko [Sid91, Sid93] asserts that for every bipartite graph H, among graphs of given density random graphs essentially contain the least number of subgraphs isomorphic to H. It is not hard to see that every graph H with the latter property is common, therefore this conjecture would imply that all bipartite graphs are common. The Erdős-Simonovits-Sidorenko conjecture has been verified for a handful of graphs [Sid93, Sid96, Hat10, CFS10], and hence there are various classes of bipartite graphs that are known to be common. In [JŠT96] and [Sid96] some graph operations are introduced that can be used to "glue" common graphs in order to construct new common graphs. However none of these operations can increase the chromatic number to a number larger than three, and as a result, all of the known common graphs are of chromatic number at most 3. With these considerations Jagger, Šťovíček and Thomason [JŠT96] state "We regard the determination of the commonality of W_5 [the wheel with 5 spokes] as the most interesting open problem in the area."

We will prove in Theorem 3.1 that W_5 (see Figure 1) is common. This

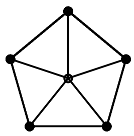

Figure 1: The 5-wheel.

will also answer a question of Sidorenko [Sid96]. He showed [Sid96, Theorem 8] that every graph that is obtained by adding a vertex of full degree to a bipartite graph of average degree at least one satisfying the Erdős-Simonovits-Sidorenko conjecture is common. Sidorenko further asked whether in this theorem both conditions of being bipartite and having average degree at least one are essential in order to obtain a common graph. Our result answers his question in the negative, as W_5 is obtained by adding a vertex of full degree to a non-bipartite graph.

The proof of Theorem 3.1 is a rather standard Cauchy-Schwarz calculation in flag algebras [Raz07], and is generated with the aid of a computer using semi-definite programming. A similar approach was successfully applied for example in [Raz10, HKN09, BT11, Grz11, HHK+11].

2 Preliminaries

We write vectors with bold font, e.g. $\mathbf{a} = (\mathbf{a}(1), \mathbf{a}(2), \mathbf{a}(3))$ is a vector with three coordinates. For every positive integer k, $[k]$ denotes the set $\{1, \ldots, k\}$.

All graphs in this paper are finite and simple (that is, loops and multiple edges are not allowed). For every natural number n, let $\mathcal{M}_n$ denote the set of all simple graphs on n vertices up to an isomorphism. For a graph G, let $V(G)$ and $E(G)$, respectively denote the set of the vertices and the edges of G. The complement of G is denoted by G^*.

The *homomorphism density* of a graph H in a graph G, denoted by $t(H; G)$, is the probability that a random map from the vertices of H to the vertices of G is a *graph homomorphism*, that is it maps every edge of H to an edge of G. If $H \in \mathcal{M}_\ell$, $G \in \mathcal{M}_n$, and $\ell \leq n$, then $t_0(H; G)$ denotes the probability that a random **injective** map from $V(H)$ to $V(G)$ is a graph homomorphism, and $p(H, G)$ denotes the probability that a random set of ℓ vertices of G induces a graph isomorphic to H.

We have the following chain rule (cf. [Raz07, Lemma 2.2]):

$$t_0(H;G) = \sum_{F\in\mathcal{M}_\ell} t_0(H;F)p(F,G), \tag{2.1}$$

where $|V(H)| \leq \ell \leq |V(G)|$.

Definition 2.1. *A graph H is called* common *if*

$$\liminf_{n\to\infty} \min_{G\in\mathcal{M}_n} (t(H;G) + t(H;G^*)) \geq 2^{1-|E(H)|}. \tag{2.2}$$

It is easy to see that as $n \to \infty$, for a random graph G on n vertices, we have, with high probability, $t(H;G) + t(H;G^*) = 2^{1-|E(H)|} \pm o(1)$. Thus, H is common if the total number of copies of H in every graph and its complement asymptotically minimizes for random graphs. Note also that since $t(H;G)$ and $t_0(H;G)$ are asymptotically equal (again, as $n \to \infty$), one could use $t_0(H;G)$ in place of $t(H;G)$ in (2.2), and this is what we will do in our proof.

2.1 Flag algebras

We assume certain familiarity with the theory of flag algebras from [Raz07]. However, for the proof of the central Theorem 3.1 only the most basic notions are required. Thus, instead of trying to duplicate definitions, we occasionally give pointers to relevant places in [Raz07].

In our application of the flag algebras calculus we work exclusively with the theory of simple graphs (cf. [Raz07, §2]). As in [Raz07], flags of type σ and size k are denoted by $\mathcal{F}_k^\sigma$. The flag algebra generated by all flags of type σ is denoted by $\mathcal{A}^\sigma$ (cf. [Raz07, §2]). Apart from already defined model $W_5 \in \mathcal{M}_6$ we need to introduce the following models, types, and flags.

We shall work with five types $\sigma_0, \sigma_1, \ldots, \sigma_4$ of size four which are illustrated in Figure 2. For a type σ of size k and a set of vertices $V \subseteq [k]$ in σ, let F_V^σ denote the flag $(G,\theta) \in \mathcal{F}_{k+1}^\sigma$ in which the only unlabeled vertex v is connected to the set $\{\theta(i) \, : \, i \in V\}$. We further define $f_V^\sigma \in \mathcal{A}^\sigma$ by

$$f_V^\sigma \stackrel{\text{def}}{=} F_\emptyset^\sigma - \frac{1}{|\mathrm{Aut}(\sigma)|} \cdot \sum_{\eta\in\mathrm{Aut}(\sigma)} F_{\eta(V)}^\sigma.$$

These elements form a basis (for $V \neq \emptyset$ and with repetitions) in the space spanned by those $f \in \mathcal{A}_{k+1}^\sigma$ that are both $\mathrm{Aut}(\sigma)$-invariant and asymptotically vanish on random graphs; other than that, our particular choice of elements with this property is more or less arbitrary.

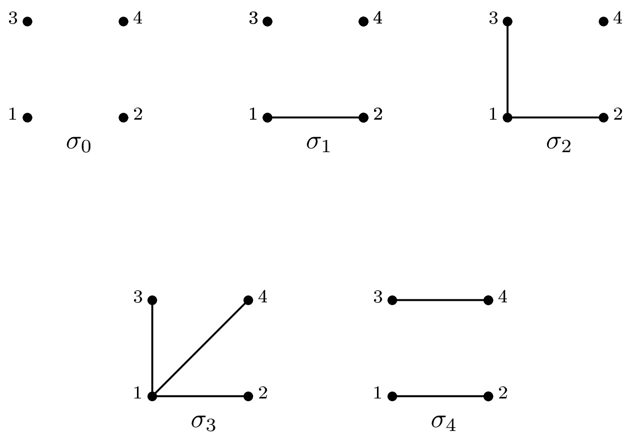

Figure 2: Types.

Recall that in [Raz07, §2.2] a certain "averaging operator" $[\![\cdot]\!]$ was introduced. This operator plays a central role in the flag algebra calculus.

Let $* \in \mathrm{Aut}(\mathcal{A}^0)$ be the involution that corresponds to taking the complementary graph. That is, we extend $*$ linearly from $\bigcup_n \mathcal{M}_n$ to $\mathcal{A}^0$.

3 Main result

We can now state the main result of the paper.

Theorem 3.1. *The* 5*-wheel* W_5 *is common.*

Proof. Let $\widehat{W}_5 \in \mathcal{A}^0$ be the element that counts the injective homomorphism density of the 5-wheel, that is

$$\widehat{W}_5 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{M}_6} t_0(W_5, F)F.$$

We shall prove that

$$\widehat{W}_5 + \widehat{W}_5^* \geq 2^{-9}\ , \tag{3.1}$$

where the inequality $\leq$ in the algebra $\mathcal{A}^0$ is defined in [Raz07, Definition 6]. An alternate interpretation of this inequality [Raz07, Corollary 3.4] is that

$$\liminf_{n\to\infty} \min_{G \in \mathcal{M}_n} (p(\widehat{W}_5, G) + p(\widehat{W}_5^*, G)) \geq 2^{-9}.$$

Since $p(\widehat{W}_5, G) = \sum_{F\in\mathcal{M}_6} t_0(\widehat{W}_5; F)p(F;G) = t_0(\widehat{W}_5; G)$ by (2.1), and, likewise, $p(\widehat{W}_5^*, G) = p(\widehat{W}_5, G^*) = t_0(\widehat{W}_5; G^*)$, (3.1) implies Theorem 3.1.

We now give a proof of (3.1). To this end we work with suitable quadratic forms $Q^{+/-}_{\sigma_i}$ defined by symmetric matrices $M^{+/-}_{\sigma_i}$ and vectors $\mathbf{g}_i^{+/-}$ in the algebras $\mathcal{A}^{\sigma_i}$. The numerical values of the matrices $M^{+/-}_{\sigma_i}$ and vectors $\mathbf{g}_i^{+/-}$ are given in the appendix. It is essential that all the matrices $M^{+/-}_{\sigma_i}$ are positive definite which can be verified using any general mathematical software. Next we define

$$R := \left(\sum_{i=0}^{4} [\![Q^{+}_{\sigma_i}(\mathbf{g}_i^{+})]\!]_{\sigma_i}\right) + [\![Q^{-}_{\sigma_1}(\mathbf{g}_1^{-})]\!]_{\sigma_1} + [\![Q^{-}_{\sigma_4}(\mathbf{g}_4^{-})]\!]_{\sigma_4}\ .$$

We claim that

$$\widehat{W}_5 + \widehat{W}_5^* = 2^{-9} + R + R^*. \tag{3.2}$$

All the terms in (3.2) can be expressed as linear combinations of graphs from $\mathcal{M}_6$ and thus checking (3.2) amounts to checking the coefficients of the 156 flags from $\mathcal{M}_6$. We offer a `C`-code available at `http://kam.mff.cuni.cz/~kral/wheel` that verifies the equality (3.2).

By [Raz07, Theorem 3.14], we have

$$\left(\sum_{i=0}^{4} [\![Q^{+}_{\sigma_i}(\mathbf{g}_i^{+})]\!]_{\sigma_i}\right) + [\![Q^{-}_{\sigma_1}(\mathbf{g}_1^{-})]\!]_{\sigma_1} + [\![Q^{-}_{\sigma_4}(\mathbf{g}_4^{-})]\!]_{\sigma_4} \geq 0\ .$$

Therefore, (3.2) implies (3.1). □

Theorem 3.1 shows that a typical random graph $G = G_{n,\frac{1}{2}}$ asymptotically minimizes the quantity $t(W_5; G) + t(W_5; G^*)$. Extending our method, we convinced ourselves that $G_{n,\frac{1}{2}}$ is essentially the only minimizer of $t(W_5; G) + t(W_5; G^*)$. In terms of flag algebras this means that the homomorphism $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ (see [Raz07, Definition 5]) satisfying $\phi(\widehat{W}_5 + \widehat{W}_5^*) = 2^{-9}$ is unique.

The outline of the argument is as follows. Let $\rho \in \mathcal{M}_2$ denote a graph consisting of a single edge, let $C_4 \in \mathcal{M}_4$ denote the cycle of length 4, and, as before, let

$$\widehat{C}_4 \stackrel{\mathrm{def}}{=} \sum_{F\in\mathcal{M}_4} t_0(C_4; F)F.$$

The Erdős-Simonovits-Sidorenko conjecture is known for C_4 [Sid91], and it implies that $\widehat{C}_4 \geq \rho^4$ and $\widehat{C}_4^* \geq (1-\rho)^4$ in $\mathcal{A}^0$. Therefore, $C_4 + C_4^* \geq 1/8$ (i.e., C_4 is common), and, moreover, every $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ attaining equality must satisfy $\phi(\rho) = 1/2$ and $\phi(\widehat{C}_4) = 1/16$.

On the other hand, is is shown in [CGW89] that the density of edges and the density cycles of length 4 characterize quasi-random graphs, implying that the homomorphism ϕ satisfying $\phi(\widehat{C}_4 + \widehat{C}_4^*) = 1/8$ is unique (and corresponds to quasi-random graphs). Therefore, to verify the uniqueness of the homomorphism ϕ satisfying $\phi(\widehat{W}_5 + \widehat{W}_5^*) = 2^{-9}$ it suffices to show that

$$\widehat{W}_5 + \widehat{W}_5^* \geq 2^{-9} + \frac{1}{100}\left(\widehat{C}_4 + \widehat{C}_4^* - 1/8\right). \tag{3.3}$$

We have used a computer program to verify (3.3), and it is telling us that this inequality holds with quite a convincing level of accuracy 10^{-10}. But we have not converted the floating point computations into a rigorous proof.

4 Conclusion

In this paper we have exhibited the first example of a common graph that is not three-colorable. This naturally gives rise to the following interesting question: do there exist common graphs with arbitrarily large chromatic number?

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A The matrices $M_i^{+/-}$ and the vectors $\mathbf{g}_i^{+/-}$

Here, we list the numerical values of the matrices $M_i^{+/-}$ and the vectors $\mathbf{g}_i^{+/-}$.

The vectors $\mathbf{g}_i^{+}$ are given by the tuples

$$\begin{aligned}
\mathbf{g}_0^+ &\stackrel{\text{def}}{=} (f^{\sigma_0}_{\{1\}}, f^{\sigma_0}_{\{1,2\}}, f^{\sigma_0}_{\{1,2,3\}}, f^{\sigma_0}_{\{1,2,3,4\}})\\
\mathbf{g}_1^+ &\stackrel{\text{def}}{=} (f^{\sigma_1}_{\{1\}}, f^{\sigma_1}_{\{3\}}, f^{\sigma_1}_{\{1,2\}}, f^{\sigma_1}_{\{1,3\}}, f^{\sigma_1}_{\{3,4\}}, f^{\sigma_1}_{\{1,2,3\}}, f^{\sigma_1}_{\{1,3,4\}}, f^{\sigma_1}_{\{1,2,3,4\}})\\
\mathbf{g}_2^+ &\stackrel{\text{def}}{=} (f^{\sigma_2}_{\{1\}}, f^{\sigma_2}_{\{2\}}, f^{\sigma_2}_{\{4\}}, f^{\sigma_2}_{\{1,2\}}, f^{\sigma_2}_{\{1,4\}}, f^{\sigma_2}_{\{2,3\}}, f^{\sigma_2}_{\{2,4\}}, f^{\sigma_2}_{\{1,2,3\}},\\
&\qquad f^{\sigma_2}_{\{1,2,4\}}, f^{\sigma_2}_{\{2,3,4\}}, f^{\sigma_2}_{\{1,2,3,4\}})\\
\mathbf{g}_3^+ &\stackrel{\text{def}}{=} (f^{\sigma_3}_{\{1\}}, f^{\sigma_3}_{\{2\}}, f^{\sigma_3}_{\{1,2\}}, f^{\sigma_3}_{\{2,3\}}, f^{\sigma_3}_{\{1,2,3\}}, f^{\sigma_3}_{\{2,3,4\}}, f^{\sigma_3}_{\{1,2,3,4\}})\\
\mathbf{g}_4^+ &\stackrel{\text{def}}{=} (f^{\sigma_4}_{\{1\}}, f^{\sigma_4}_{\{1,2\}}, f^{\sigma_4}_{\{1,3\}}, f^{\sigma_4}_{\{1,2,3\}}, f^{\sigma_4}_{\{1,2,3,4\}}),
\end{aligned}$$

and the vectors $\mathbf{g}_i^{-}$ are given by

$$\begin{aligned}
\mathbf{g}_1^- &\stackrel{\text{def}}{=} (F^{\sigma_1}_{\{3\}} - F^{\sigma_1}_{\{4\}}, F^{\sigma_1}_{\{1,3,4\}} - F^{\sigma_1}_{\{2,3,4\}}, F^{\sigma_1}_{\{1,3\}} - F^{\sigma_1}_{\{2,3\}}, F^{\sigma_1}_{\{1,3\}} - F^{\sigma_1}_{\{2,4\}},\\
&\qquad F^{\sigma_1}_{\{1,3\}} - F^{\sigma_1}_{\{3,4\}})\\
\mathbf{g}_4^- &\stackrel{\text{def}}{=} (F^{\sigma_4}_{\{1,2\}} - F^{\sigma_4}_{\{3,4\}}, F^{\sigma_4}_{\{1,3\}} - F^{\sigma_4}_{\{2,3\}}, F^{\sigma_4}_{\{1,3\}} - F^{\sigma_4}_{\{2,4\}}, F^{\sigma_4}_{\{1,3\}} - F^{\sigma_4}_{\{3,4\}}).
\end{aligned}$$

The matrices $M_i^{+/-}$ are listed on the next three pages.

$$M_0^+ \stackrel{\text{def}}{=} \frac{1}{2\cdot 10^8} \times \begin{pmatrix} 104133330 & -67645847 & -126443014 & -53041562\\ -67645847 & 58559244 & 68999274 & 28961030\\ -126443014 & 68999274 & 166581934 & 69653308\\ -53041562 & 28961030 & 69653308 & 29368489 \end{pmatrix}$$

$$M_3^+ \stackrel{\mathrm{def}}{=} \frac{1}{24\cdot 10^8} \times \begin{pmatrix} 1770465360 & -40788068 & 770354664 & -280179622 & -1109635560 & -593033461 & -1434435065 \\ -40788068 & 503182008 & -377074674 & -65682192 & -316936632 & 337167432 & -405260664 \\ 770354664 & -377074674 & 942288720 & -5442408 & -584215338 & -635915808 & -299584920 \\ -280179622 & -65682192 & -5442408 & 90869472 & 187091280 & -48623352 & 356458176 \\ -1109635560 & -316936632 & -584215338 & 187091280 & 1325422128 & 196268064 & 1280101992 \\ -593033461 & 337167432 & -635915808 & -48623352 & 196268064 & 706802676 & -31363774 \\ -1434435065 & -405260664 & -299584920 & 356458176 & 1280101992 & -31363774 & 1763018404 \end{pmatrix}$$

$$M_4^+ \stackrel{\mathrm{def}}{=} \frac{1}{12\cdot 10^8} \times \begin{pmatrix} 6589068 & -137160 & 60408 & -3635796 & -5354976 \\ -137160 & 3975070 & -399180 & -720636 & -1388043 \\ 60408 & -399180 & 3506988 & -1778640 & -3413616 \\ -3635796 & -720636 & -1778640 & 5107716 & 3969708 \\ -5354976 & -1388043 & -3413616 & 3969708 & 12276592 \end{pmatrix}$$

$$M_1^- \stackrel{\mathrm{def}}{=} \frac{1}{48\cdot 10^8} \times \begin{pmatrix} 1871684759 & 828164352 & 153135600 & 2205677647 & 32494800 \\ 828164352 & 647325323 & 122226960 & 1702274830 & 23569680 \\ 153135600 & 122226960 & 32894794 & 317036160 & 988560 \\ 2205677647 & 1702274830 & 317036160 & 4533494520 & 62236800 \\ 32494800 & 23569680 & 988560 & 62236800 & 7445060 \end{pmatrix}$$

$$M_4^- \stackrel{\mathrm{def}}{=} \frac{1}{24\cdot 10^8} \times \begin{pmatrix} 371929992 & -665160 & 31885344 & 6896381 \\ -665160 & 4952616 & 15347271 & -425892 \\ 31885344 & 15347271 & 420643536 & 5244336 \\ 6896381 & -425892 & 5244336 & 1704738 \end{pmatrix}$$

$$M_2^+ \stackrel{\text{def}}{=} \frac{1}{24 \cdot 10^8} \times$$

$$\times \begin{pmatrix}
4114457904 & -2123660510 & 578302533 & 2402100408 & 1609339896 & -4979381511 & -1073916061 & -711542544 & -108075291 & -311854200 & -1172726832 \\
-2123660510 & 4697332052 & -146727648 & -2893487330 & -831349224 & 5132020824 & 1140828192 & -2533278088 & -3120849612 & 586989168 & -2130186959 \\
578302533 & -146727648 & 2842930424 & -2377739616 & 2453284752 & -1134538157 & 949692648 & -2122945241 & 799767696 & -646840455 & -1452441435 \\
2402100408 & -2893487330 & -2377739616 & 5029589784 & -1305679056 & -3694198620 & -1628657160 & 2987352093 & -17138568 & 174993936 & 1346820763 \\
1609339896 & -831349224 & 2453284752 & -1305679056 & 2899169976 & -3008866416 & 227603736 & -2158976640 & 1272333144 & -824389152 & -1468496784 \\
-4979381511 & 5132020824 & -1134538157 & -3694198620 & -3008866416 & 9045922946 & 1585531176 & -492543642 & -2720802624 & 1167719184 & 119548200 \\
-1073916061 & 1140828192 & 949692648 & -1628657160 & 227603736 & 1585531176 & 1198933584 & -594013398 & -787158072 & -14360286 & -864511462 \\
-711542544 & -2533278088 & -2122945241 & 2987352093 & -2158976640 & -492543642 & -594013398 & 4445640792 & 1152146526 & 408353664 & 3139157376 \\
-108075291 & -3120849612 & 799767696 & -17138568 & 1272333144 & -2720802624 & -787158072 & 1152146526 & 4353119928 & -778415544 & 2410765872 \\
-311854200 & 586989168 & -646840455 & 174993936 & -824389152 & 1167719184 & -14360286 & 408353664 & -778415544 & 430490652 & 217228440 \\
-1172726832 & -2130186959 & -1452441435 & 1346820763 & -1468496784 & 119548200 & -864511462 & 3139157376 & 2410765872 & 217228440 & 3407087808
\end{pmatrix}$$

$$
M_1^+ \stackrel{\text{def}}{=} \frac{1}{24 \cdot 10^8} \times
$$

$$
\times \begin{pmatrix}
3376427096 & -550659377 & 1175122309 & -274818336 & -1951510989 & 133242698 & -2978772360 & -1118255328 \\
-550659377 & 3579306230 & -2818779263 & 254758382 & 1853810147 & -3593215008 & 1149060744 & -2243131164 \\
1175122309 & -2818779263 & 2446135762 & -153160723 & -1883990616 & 2571244464 & -1644918408 & 1392930672 \\
-274818336 & 254758382 & -153160723 & 259013952 & 207245488 & -524428416 & 59129384 & -87439632 \\
-1951510989 & 1853810147 & -1883990616 & 207245488 & 2026568566 & -1339529064 & 2075124696 & -196178016 \\
133242698 & -3593215008 & 2571244464 & -524428416 & -1339529064 & 4383894552 & -474279456 & 2753404296 \\
-2978772360 & 1149060744 & -1644918408 & 59129384 & 2075124696 & -474279456 & 2987175794 & 578705400 \\
-1118255328 & -2243131164 & 1392930672 & -87439632 & -196178016 & 2753404296 & 578705400 & 2302497768
\end{pmatrix}
$$