Interval linear programming: A survey

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Abstract

Uncertainty is a common phenomenon in practice. Due to measurement errors we can hardly expect precise values in real-life linear programming problems. Using estimated quantities may lead to unsatisfactory results, so inexactness must be taken into account. Uncertainty can be handled in various manners, e.g. by stochastic programming, interval analysis or fuzzy numbers; each of them has some pros and cons. In this paper, we suppose that we are given lower and upper bounds on the quantities, and the quantities may perturb independently and simultaneously within these bounds. In this model we investigate the problems of optimal value range, basis stability, optimal solutions enclosures, duality etc. Complexity issues are discussed, too; some tasks are polynomially solvable while another are NP-hard.

This approach is more general and powerful than the standard sensitivity analysis. In sensitivity analysis, we consider variations of only one parameter, which is very restrictive. On the other hand, interval analysis based approach enables to handle simultaneously all required parameters. We present a brief exposition of the known results with new insights, and close the survey by some challenging problems.

Keywords: *Linear interval systems, linear programming, interval analysis, optimal value range, interval matrix, basis stability.*

1 Introduction

Many practical problems are solved by linear programming. Since real-life problems are subject to uncertainties due to errors, measurements and estimations, we have to reflect it in linear programming methodology and decision making. Inaccuracy is modelled in diverse ways; see overviews by Sahinidis [85], or Liu [41]. In stochastic approach, we handle inexact quantities as random variables, in fuzzy set theory as vague numbers with weighted membership function, and in interval analysis we assume that the quantities perturb simultaneously and independently within a priori known fixed bounds.

There are two basic approaches to deal with interval linear programming (ILP). In the first one, we resign on guaranteed bounds and enclosures covering all possibilities, aiming to satisficing solutions. These techniques reduce the problem into solving several real-valued (usually linear) programs. This family involves robust optimization, fuzzy programming and others.

Robust optimization is a methodology to process optimization problems with uncertain data. Herein, we seek for a solution that is robust (stable) under some data perturbation. From this viewpoint, ILP is reduced to one (possibly difficult) optimization problem the solution of which is considered to be good. One of the basic methods in robust optimization is the minimax regret method inspected e.g. in [2, 4, 5, 12, 29, 50].

There other methods designed in a different way. The resulting interval solutions are mostly some approximation sets, but represent an acceptable compromise for a decision maker. The techniques used are, for example, introducing a proper ordering for intervals [37, 38, 87], using a satisfaction function [52], or another reductions [25, 99]. Often, principles from fuzzy linear programming are used [27, 40, 44, 94].

Somewhere in between, there is *stochastic programming*. Interval values can be considered as random quantities with uniform distribution. Thus, ILP is a specific case of stochastic programming, and—in principle—any method of stochastic programming can be applied in ILP. Nevertheless, is too specific to be successfully processed by stochastic programming methods. Some stochastic-like methods to solve ILP are in [45, 100].

In the second approach, which is dealt with in this paper, the aim is different. We want to cover all possible scenarios and compute rigorous interval solutions containing all possibilities. The methods used are multiparametric programming, perturbation theory, and interval arithmetic and analysis.

ILP ca be viewed as a multiparametric linear programming with interval domains for parameters [13, 14, 51]. This approach can solve some questions arising in ILP, particularly special cases dealt with in Section 6, but cannot handle the general ILP problem.

Perturbation theorems in linear systems [69, 90] give raise to another possibility to handle interval uncertainties. This approach is usually less conservative and solves some particular tasks in ILP, but not applicable for all questions arising.

One of the fundamental methods is to use *interval arithmetic*, which was introduced to extend the basic functions and operation such as addition and multiplication to intervals [1, 53, 62]. Interval arithmetic always returns rigorous results, meaning, whatever scenario happens, the results are included within the calculated intervals. The drawback is that the resulting intervals are usually too conservative and overestimated. Interval arithmetic was applied in interval linear programming e.g. in [3, 30, 31, 32, 36, 49, 91].

More sophisticated methods rely on direct inspection of particular subproblems by means interval analysis methodology, among others. Results of interval linear algebra topics such as interval linear equation solving and interval matrix theory are often exploited. That is why we introduce some interval notion in the next sections before we introduce the main objectives in Section 1.2.

We use the following notation: I denotes the identity matrix (with convenient dimension), $A_{i,*}$ the ith row of a matrix A, $\mathrm{diag}(v)$ the diagonal matrix with entries $v_1, \dots, v_n$, and $\mathrm{sgn}(r)$ stands for the sign of a real r. Notice that the operators $|\cdot|$, $\mathrm{sgn}(\cdot)$ are used also for vectors and matrices with entrywise meaning. The spectral radius of a matrix A is denoted by $\rho(A)$.

1.1 Interval computing

An interval matrix is defined as

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A} \le \overline{A}$ are given matrices; n-dimensional interval vectors can be regarded as interval matrices n-by-1. By

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the center and the radius of $\boldsymbol{A}$, respectively. The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$. Standard arithmetic is extended to intervals in a natural way as follows [1, 53, 62]. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$, then

$$\begin{aligned}
\boldsymbol{a} \pm \boldsymbol{b} &:= [\underline{a} \pm \underline{b}, \overline{a} \pm \overline{b}],\\
\boldsymbol{a} \cdot \boldsymbol{b} &:= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}),\ \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})],\\
\boldsymbol{a}/\boldsymbol{b} &:= \begin{cases} [\min(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}),\ \max(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b})] & \text{if } 0 \notin \boldsymbol{b},\\ \text{undefined} & \text{otherwise.} \end{cases}
\end{aligned}$$

Interval arithmetic is defined such that the resulting intervals correspond to the image of the basic operations. Let $\circ$ be any of the basic operations. From this viewpoint, interval arithmetics reads

$$\boldsymbol{a} \circ \boldsymbol{b} = \{x \in \mathbb{R};\ \exists a \in \boldsymbol{a}\, \exists b \in \boldsymbol{b} : x = a \circ b\}.$$

Solving linear equations is a basic task in linear algebra, and solving interval linear equation is a basic task in interval computations. Let an interval system

$$\boldsymbol{A}x = \boldsymbol{b} \tag{1}$$

be given, where $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^n$. The solution set to a linear system is defined as a set of solutions of all scenarios of the interval data, that is, for (1)

$$\{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b\}.$$

A well-known description of the solution set was given by Oettli and Prager [66] (cf. [81]).

Theorem 1 (Oettli and Prager, 1964)**.** *The solution set to $\boldsymbol{A}x = \boldsymbol{b}$ is described by*

$$|A_c x - b_c| \le A_\Delta |x| + b_\Delta. \tag{2}$$

The Oettli and Prager system is non-linear, which makes difficulties in determining or sharp bounding of the solution set. If non-negativity of variables is incorporated then the problem becomes easy and the characterization becomes linear [81]:

$$\underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,$$

compare Theorem 5 and Remark 2.

The interval hull of a set is the smallest interval vector containing the set. Computing the interval hull of the solution set to (1) is NP-hard problem [83]. In many practical circumstances, one doesn't need to find the exact interval hull, but any sufficiently sharp superset (called *an enclosure*) is desirable, too. Such enclosures are much faster computed, and there are plenty of methods available; see e.g. [62, 64, 75, 79] and references therein.

Remark 1. It is a common phenomenon in interval analysis that non-negativity (or another sign restriction) of variables makes to problem easier; see Theorem 1 for instance. Provided that non-negativity is not given, one possibility to solve the problem is a decomposition of the space into orthants, where variables become sign restricted. Let $p \in \{\pm 1\}^n$. Then $\mathrm{diag}(p)\, x \geq 0$ determines one of the orthants of $\mathbb{R}^n$, and the problem of feasibility of (2) can be solved by decomposition into 2^n systems

$$|A_c x - b_c| \leq A_\Delta \,\mathrm{diag}(p)\, x + b_\Delta, \;\; \mathrm{diag}(p)\, x \geq 0,$$

or

$$(A_c - A_\Delta \,\mathrm{diag}(p))x \leq \overline{b}, \;\; (-A_c - A_\Delta \,\mathrm{diag}(p))x \leq -\underline{b}, \quad \mathrm{diag}(p)\, x \geq 0. \quad (3)$$

Now, (2) is feasible if and only if the linear system (3) is feasible for some $p \in \{\pm 1\}^n$. Not surprisingly, when s is the number of sign restricted variables and $n - s$ the number of the remaining, then it suffices to decompose only to 2^{n-s} subproblems.

Analogously, the problem solving can be accelerated when coefficients by some x_i are degenerated (have zero widths). Then the absolute value of x_i in (2) vanishes and we do not have to decompose along the sign of x_i. Any such case reduces in half the time effort [6].

We utilize this decomposition at several points, e.g. in Theorem 6 and in some places of Section 3. □

1.2 Interval linear programming

Consider a linear program

$$\min c^T x \;\text{ subject to }\; x \in \mathcal{M}(A, b), \quad (4)$$

where $\mathcal{M}$ is the feasible set characterized by a linear system. We say that it is *feasible* if the feasible set $\mathcal{M}(A, b)$ is not empty.

Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^m$ and $\boldsymbol{c} \in \mathbb{IR}^n$ be given. By an *interval linear programming (ILP)* problem we mean a family of linear programs (4), where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$. *A scenario* is a concrete realization of interval values, that is, any linear program (4) with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$.

Let us focus on the feasible set description for a while. In the linear programming theory, one of the following canonical forms

(A) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n;\ Ax = b,\ x \geq 0\}$,

(B) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n;\ Ax \leq b\}$,

(C) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n;\ Ax \leq b,\ x \geq 0\}$

is usually assumed. Any linear system can be rewritten to any of these canonical form by using a standard transformation. However, in interval linear programming, this is not the case. That is why we will discuss all the three systems separately in the remainder of the paper. Diverse systems are dealt with in a slightly different way, and even the computational complexity may differ.

Example 1. Consider a system of type (A), which can be transformed to type (B) by substitution $x = y - z$ as

$$Ay - Az \leq b,\ y, z \geq 0.$$

Thus an interval system $\boldsymbol{A}x \leq \boldsymbol{b}$ is transformable to a family of systems

$$Ay - Az \leq b,\ y, z \geq 0,$$

where $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. This family, however, differs from the family

$$\boldsymbol{A}y - \boldsymbol{A}z \leq \boldsymbol{b},\ y, z \geq 0$$

because of the double occurrence of the matrix $\boldsymbol{A}$. Such multiple occurrences are usually called *dependencies*. Relaxing such dependencies results in an overestimation. Thus the solution set to the former system lies within the solution set to the latter.

To be concrete, consider the interval system

$$[1, 2]x \leq 2$$

and the incorrect transformation into

$$[1, 2]y - [1, 2]z \leq 2,\ y, z \geq 0.$$

For the former system, the solution set (i.e., the union of solutions over all scenarios) is the interval $(-\infty, 2]$. Nonetheless, the latter has a real line in the backward transformation of the solution set. Using the scenario

$$y - 2z \leq 2,\ \ y, z \geq 0,$$

any real number r can be written as $r = y - z$ with $y = \max(2r, 0)$ and $z = |r|$. $\square$

Now, we are to set up the main goals. It is not possible to formulate one problem in ILP because there are several points that are studied and that the decision maker may ask for. The basic questions are:

- **Feasibility.** Is any (or some) scenario of ILP feasible?
- **(Un)boundedness.** Is any (or some) scenario of ILP (un)bounded?
- **Optimality.** Is there an optimal solution for each (or some) scenario of ILP?

The headline problems in ILP are:

- **Optimal value range.** What is the range of optimal values of (4) when data perturb within given intervals?
- **Basis stability.** Is there an optimal basis common to all scenarios of ILP?
- **Set of optimal solutions.** What is the set of all optimal solutions over all scenarios of ILP?

These questions set up a framework of the paper. Besides, we discuss what more can be said when some special cases appear. In particular, we focus on the cases with interval right hand side and interval objective function coefficients. We touch duality in ILP, too.

2 Basic questions

For every linear program, just one of the following situations happens: it is not feasible, it is unbounded or it has an optimal solution. We will inspect all the three possibilities within the interval context. We skip continuity issues [96], which require more space to expose.

2.1 Feasibility

Here, we address the questions concerning the problems of feasibility such as "Is every scenario feasible?" or "Is at least one scenario feasible?" Some questions are easy to answer while other problems are NP-hard, depending not only on the questions but also the linear system considered.

An interval linear system is *strongly feasible* if it is feasible for all scenarios, that is, each scenario has a solution. Similarly, an interval system is *weakly feasible* if it is feasible for at least one scenario. Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^m$. First we review the strong feasibility case.

Theorem 2 (Rohn, 1981)**.** *An interval system* $\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$ *is strongly feasible if and only if for each* $p \in \{\pm 1\}^m$ *the system*

$$(A_c - \operatorname{diag}(p)\, A_\Delta)x = b_c + \operatorname{diag}(p)\, b_\Delta,\ x \geq 0$$

is feasible.

Proof. See [72, 81]. □

Theorem 3 (Rohn & Kreslová, 1994)**.** *An interval system* $\boldsymbol{A}x \leq \boldsymbol{b}$ *is strongly feasible if and only if the system* $\overline{A}x^1 - \underline{A}x^2 \leq \underline{b},\ x^1 \geq 0,\ x^2 \geq 0$ *is feasible.*

Proof. See [81, 84]. □

Theorem 4. *An interval system* $\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$ *is strongly feasible if and only if the system* $\overline{A}x \leq \underline{b},\ x \geq 0$ *is feasible.*

Proof. See [49, 81, 84]. □

In the strong feasibility case, the "bad boy" is the system of equations as we have to check feasibility of 2^m systems. Indeed, it was shown [78, 81] that checking this property is NP-hard. The others are polynomially solvable since it suffices to test feasibility of a real-valued linear system.

Contrary in the weak feasibility case, the NP-hard problem is testing weak feasibility of a systems of inequalities [81]. Herein, the non-negativity condition is fundamental for polynomiality.

Theorem 5. *An interval system* $\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$ *is weakly feasible if and only if the system* $\underline{A}x \leq \overline{b},\ -\overline{A}x \leq -\underline{b}, x \geq 0$ *is feasible.*

Proof. See [81]. □

Theorem 6 (Gerlach, 1981)**.** *An interval system $\boldsymbol{A}x \le \boldsymbol{b}$ is weakly feasible if and only if the nonlinear system*

$$A_c x - A_\Delta |x| \le \overline{b}$$

is feasible, or, if and only if the linear system

$$(A_c - A_\Delta \operatorname{diag}(p))x \le \overline{b}$$

is feasible for some $p \in \{\pm 1\}^n$.

Proof. See [15, 81]. □

Theorem 7. *An interval system $\boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0$ is weakly feasible if and only if the system $\underline{A}x \le \overline{b},\ x \ge 0$ is feasible.*

Proof. See [49, 81]. □

Remark 2. Note that the above theorems give not only characterization of weak feasibility of particular interval systems, but also description of their solution sets. Thus, solution set for types (A) and (C) is a convex polyhedral set, whereas the solution set in case of (B) is a union of 2^n convex polyhedral sets. □

2.2 Unboundedness

It is known [67, 86] that the linear program (4) is unbounded if and only if it is feasible and the dual problem is not feasible. Thus testing unboundedness in interval linear programming (ILP) can be easily reduced to feasibility issues, which were already addressed in Section 2.1.

We say then ILP is *strongly unbounded* if it is unbounded for each scenario, and *weakly unbounded* if it is unbounded for at least one scenario. Similarly, it is *strongly bounded* if it is bounded (i.e., not unbounded) for any scenario, and *weakly bounded* if it is bounded for some scenario.

From the above argument we have that ILP is strongly unbounded if and only if it is strongly feasible and the dual not weakly feasible.

Type (A): $\boldsymbol{A}x = \boldsymbol{b},\ x \ge 0$

When the linear program has the form (A) from Section 1 then the dual problem reads

$$\max b^T y \ \text{ subject to } \ A^T y \le c.$$

Coming to ILP, it is strongly unbounded if and only if

$$\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0 \tag{5}$$

is strongly feasible and

$$\boldsymbol{A}^T y \leq \boldsymbol{c} \tag{6}$$

is not weakly feasible. This can be checked by Theorems 2 and 6. It is computationally expensive, which is not surprising in view of [35], where Koníčková proved its NP-hardness. In [35], also an alternative viewpoint on strong unboundedness was given.

Theorem 8 (Koníčková, 2006)**.** *ILP is strongly unbounded if and only if for each $p \in \{\pm 1\}^m$ the linear program*

$$\min \underline{c}^T x \quad \textit{subject to} \quad (A_c - \operatorname{diag}(p)\, A_\Delta)x = b_c + \operatorname{diag}(p)\, b_\Delta,\ x \geq 0$$

is unbounded.

For weak unboundedness only some sufficient conditions and necessary conditions are known. A complete sufficient and necessary condition is not known yet. ILP is weakly unbounded if (5) is strongly feasible and (6) is not strongly feasible, or, if (5) is weakly feasible and (6) is not weakly feasible. In both cases, there exists a scenario in which the primal problem is feasible and the dual one is not feasible, thus the unboundedness is attained. A necessary condition is that the primal problem is weakly feasible and the dual problem is not strongly feasible.

Boundedness is a complementary property to unboundedness, that is, one holds if and only if the second one does not hold. Thus strong boundedness is equivalent to a negation of weak unboundedness, and weak boundedness is equivalent to a negation of strong unboundedness. In the same manner, negations of sufficient conditions for one become necessary conditions to the other, and vice versa.

Type (B): $\boldsymbol{A}x \leq \boldsymbol{b}$

A dual counterpart to the primal system

$$\boldsymbol{A}x \leq \boldsymbol{b} \tag{7}$$

is

$$\boldsymbol{A}^T y = \boldsymbol{c},\ y \leq 0. \tag{8}$$

Questions concerning strong and weak unboundedness and boundedness are dealt with in a similar way as for the previous type. It suffices just to replace the system (5) by (7), (6) by (8), and to utilize the corresponding feasibility theorems from Section 2.1.

Type (C): $\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$

In this case, we can replace (5) by

$$\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$$

and (6) by

$$\boldsymbol{A}^T y \leq \boldsymbol{c},\ y \leq 0,$$

too. The methodology is the same as for type (A), but the complexity results may differ. For example, strong unboundedness is easily tested in polynomial time by the virtue of Theorems 4 and 7.

2.3 Optimality

The most important fundamental question asks for optimality. Does the interval linear program (ILP) have an optimal solution for each scenario? Or for some scenario? Naturally in the established manner, we called it "strong optimality" and "weak optimality", respectively.

From the linear programming theory we know that an optimal solution exists if and only if both the primal and dual problems are feasible. Thus we again employ the feasibility results to answer optimality questions.

ILP is strongly optimal if and only if the primal and the dual problems are strongly feasible. Hence, types (A) and (B) are handled via Theorems 2 and 3. Similar result for type (A) was obtained by Rohn [72]. Rohn [80] observed that testing strong optimality is an NP-hard problem for type (A), and the same holds probably for type (B) as well. Contrary, type (C) is the simplest one and Theorem 4 gives an efficient algorithm. More results on strong optimality are in Rohn [80] in the section devoted to the finite range problem, which is an equivalent point of view.

The problem is weakly optimal if and only if there is a scenario of interval data such that the primal and dual programs are feasible. This is difficult to test. Consider, for example, type (A). Herein, we have to find $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$ such that the system

$$Ax = b,\ x \geq 0,\ A^T y \leq b$$

is feasible. Due to the double appearance of the matrix A, it is not a standard interval linear system. Such systems with dependencies are very hard to characterize (cf. [19]) and to solve by a finite algorithm. So far, no finite characterization for weak optimality has been proposed, but we have two sufficient conditions. Weak optimality follows from strong feasibility of the primal problem and weak feasibility of the dual one, or vice versa. A necessary condition for weak optimality is that both primal and dual problems are weakly feasible.

Example 2. Consider type (B) problem

$$\min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x \leq \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{pmatrix} -[3,5] & [5,6] \\ [6,7] & -[7,8] \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [10,11] \\ [17,18] \\ -1 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} -[1,2] \\ -[2,3] \end{pmatrix}.$$

We inspect strong feasibility first. By Theorem 3, we test feasibility of the system

$$\begin{pmatrix} -3 & 6 & 5 & -5 \\ 7 & -7 & -6 & 8 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \leq \begin{pmatrix} 10 \\ 17 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Its solution is e.g. $x^1 = (0,1)^T$, $x^2 = (0,0)^T$. Thus the problem is strongly feasible, that is, each scenario is feasible.

Turning our attention to unboundedness, consider the dual system (8). It is easy to see that it is weakly feasible (Theorem 5), but not strongly feasible (Theorem 2). It means that the primal problem is not strongly unbounded, but it is weakly unbounded. In other words, some scenarios are unbounded and some are not. See illustration on Figure 1, where intersection and union of feasible sets over all scenarios is drawn.

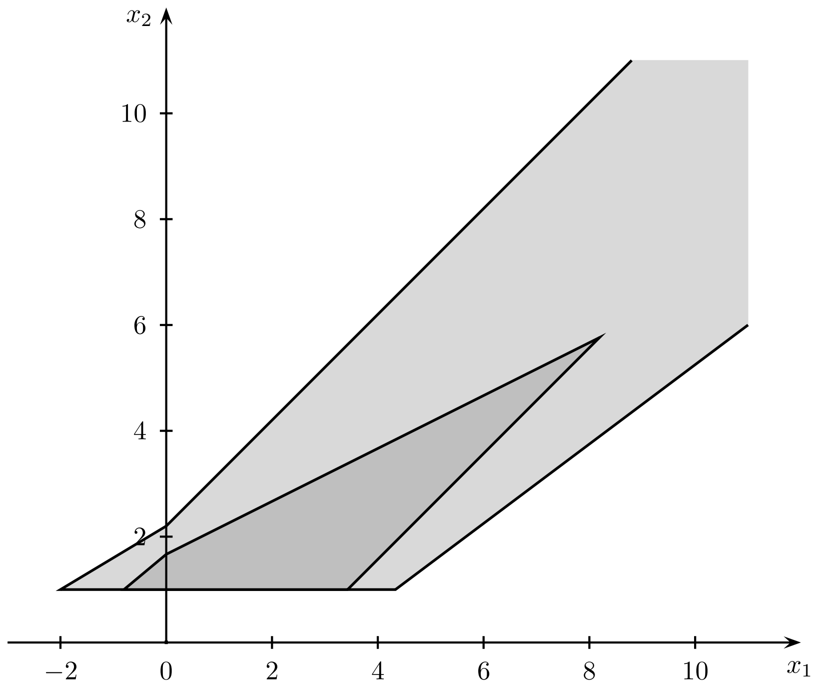

Figure 1: (Example 2): Intersection of all feasible sets in dark gray; union in light gray.

Optimality is deal with in a similar manner. We have already observed that the dual system is not strongly feasible, so the problem cannot be strongly optimal. However, the weak feasibility of (8) implies weak optimality. Therefore some scenarios have optimal solutions whereas the others do not have any. □

Example 3. Consider type (C) problem

$$\min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0$$

where

$$\boldsymbol{A} = \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} -[5,6] \\ -[1,2] \end{pmatrix}.$$

The system $\overline{A}x \leq \underline{b}$, $x \geq 0$ is feasible, so by Theorem 4 the problem is strongly feasible. Since the dual system $\boldsymbol{A}^T y \leq \boldsymbol{c}$, $y \leq 0$ is strongly feasible, too, the problem is strongly optimal. See Figure 2 on page 28. □

3 Optimal value range

A frequent problem in ILP is to compute the range of optimal values when the problem quantities vary within intervals [3, 6, 23, 30, 31, 32, 37, 49, 54, 55, 56, 57, 58, 59, 60, 73, 80]. A nice exposition for ILP in a general form, involving types (A)–(C), was given by Chinneck and Ramadan [6]. Denote by

$$f(A,b,c) := \min c^T x \ \text{ subject to } \ x \in \mathcal{M}(A,b)$$

the optimal value of the linear program. Notice that infinite values are allowed, too. The goal is to compute the lower and upper bound on the optimal value

$$\underline{f} := \inf f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},$$
$$\overline{f} := \sup f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

The lower and upper bound cases are sometimes called the best and the worst case, respectively.

Not all values in $\boldsymbol{f} = [\underline{f}, \overline{f}]$ are necessarily attained by some scenario. This is true when the range is finite (by continuity reasons), but when some limit is infinite then there may appear gaps. Calculating a real image of the optimal value function $f(A,b,c)$ has not been studied yet.

A unified approach for calculating the optimal value bounds was proposed by Hladík [23]. Denote by

$$\max b^T y \ \text{ subject to } \ y \in \mathcal{N}(A,c)$$

the dual problem, and by

$$\mathcal{M} := \{x \in \mathcal{M}(A,b);\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}\},$$
$$\mathcal{N} := \{y \in \mathcal{N}(A,c);\ A \in \boldsymbol{A},\ c \in \boldsymbol{c}\}$$

the solution sets to the primal and dual feasible sets, respectively. The solution sets are determined according to Remark 2. Now, we are ready to present an algorithm for calculating the optimal value bounds. As long as we are able to determine both solution set, we can compute the bounds for any ILP problem, not only types (A)–(C). Thus, in principle, we can also handle more general ILP problems with dependencies.

Algorithm 1.

1. Compute

$$\underline{f} := \inf c_c^T x - c_\Delta^T |x| \;\text{ subject to }\; x \in \mathcal{M}. \tag{9}$$

2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup b_c^T y + b_\Delta^T |y| \;\text{ subject to }\; y \in \mathcal{N}. \tag{10}$$

4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

Strong feasibility was discussed in Section 2.1. The optimization problems (9) and (10) are either linear programs or can be decomposed to at most respectively 2^n and 2^m linear programs; see Remark 1.

As simple consequences we obtain optimal value range formulae for particular types (A)–(C).

Type (A): $\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$

In this case, the primal solution set is described

$$M = \{x;\ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0\}$$

according to Theorem 5, and the dual solution set

$$N = \{y;\ A_c^T y - A_\Delta^T |y| \leq \overline{c}\}.$$

according to Theorem 6. Due to non-negativity of x we have

$$c_c^T x - c_\Delta^T |x| = c_c^T x - c_\Delta^T x = \underline{c}^T x,$$

so

$$\begin{aligned}
\underline{f} &= \inf \underline{c}^T x \ \text{ subject to } \ \underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\\
\overline{\varphi} &= \sup b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ A_c^T y - A_\Delta^T |y| \le \overline{c}.
\end{aligned}$$

Strong feasibility of $\boldsymbol{A}x = \boldsymbol{b},\ x \ge 0$ is equivalent (Theorem 2) to solvability of

$$(A_c - \operatorname{diag}(p)\, A_\Delta)x = b_c + \operatorname{diag}(p)\, b_\Delta,\ x \ge 0$$

for every $p \in \{\pm 1\}^m$.

Another result comes from Rohn [80]; the formula for $\underline{f}$ appeared already in [3, 49].

Theorem 9 (Rohn, 2006)**.** *We have*

$$\begin{aligned}
\underline{f} &= \inf \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0,\\
\overline{f} &= \sup_{p \in \{\pm 1\}^m} f(A_c - \operatorname{diag}(p)\, A_\Delta, b_c + \operatorname{diag}(p)\, b_\Delta, \overline{c}).
\end{aligned} \tag{11}$$

The lower bound $\underline{f}$ can be computed by a linear program in a polynomial time, whereas computing the upper bound $\overline{f}$ needs to solve 2^m linear programs. We cannot hope for a more efficient method since the latter problem is NP-hard [80]. Moreover, it is strongly NP-hard even in the specific case with intervals in the right-hand side only [11, 12].

Another method similar to that in Algorithm 1 was presented by Rohn [80].

Theorem 10 (Rohn, 2006)**.** *Let*

$$\overline{\varphi} := \sup b_c^T y + b_\Delta^T |y| \ \textit{ subject to } \ A_c^T y - A_\Delta^T |y| \le \overline{c}. \tag{12}$$

Then we have the following

1. *If $\overline{\varphi} > -\infty$ then $\overline{\varphi} = \overline{f}$.*

2. *If $\overline{\varphi} = -\infty$ then $\overline{\varphi} \in \{-\infty, \infty\}$.*

The theorem says that the upper bound is computable by solving a nonlinear programming problem. Thus we can employ some nonlinear programming method to (12), or to use a decomposition technique that splits the space into particular orthants, where the problem is reduced into 2^m linear programs (cf. [6, 23] and Remark 1). An algorithm based on a necessary condition was proposed by Mráz [59, 60].

As computing the upper bound $\overline{f}$ is computationally expensive, one can be interested in an estimation of the real value. The following one is due to Rohn [80]

Theorem 11. *If A_c has linearly independent rows and $\rho(A_\Delta|A_c^+|) < 1$ then*

$$\overline{\varphi} \le \overline{c}^T |A_c^+| (I - A_\Delta |A_c^+|)^{-1} (|b_c| + b_\Delta),$$

where $\overline{\varphi}$ comes from (12) *and A_c^+ denotes the Moore–Penrose pseudoinverse of A_c.*

More related results for type (A) are found in [60, 80]. For instance, the finite range case $-\infty < \underline{f} \le \overline{f} < \infty$ is discussed in Rohn [80].

Type (B): $\boldsymbol{A}x \le \boldsymbol{b}$

The corresponding dual problem is

$$\max b^T y \ \text{ subject to } \ A^T y = c,\ y \le 0.$$

Adapting Theorems 6 and 5, the solution sets read

$$\mathcal{M} = \{x;\, A_c x - A_\Delta |x| \le \overline{b}\},$$
$$\mathcal{N} = \{y;\, \overline{A}^T y \le \overline{c},\ -\underline{A}^T y \le -\underline{c},\ y \le 0\}.$$

Notice that non-positivity of variable y caused the opposite matrix limit in the description of $\mathcal{N}$. Strong feasibility is checked along Theorem 3 by verifying feasibility of $\overline{A}x^1 - \underline{A}x^2 \le \underline{b},\ x^1 \ge 0,\ x^2 \ge 0$. The optimization subproblems of Algorithm 1 take the form

$$\underline{f} = \inf c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ A_c x - A_\Delta |x| \le \overline{b}, \tag{13}$$
$$\overline{\varphi} = \sup \underline{b}^T y \ \text{ subject to } \ \overline{A}^T y \le \overline{c},\ -\underline{A}^T y \le -\underline{c},\ y \le 0. \tag{14}$$

Computing the lower bound is computationally expensive; it was proved by Gabrel et al. [12] that it is strongly NP-hard even in the class of problems with interval objective function coefficients and real constraint coefficients. The lower bound can be calculated along Remark 1. A similar approach by using 2^{n+1} sub-systems was proposed by Tong [94].

The upper bound computation requires just to solve one or two linear programs (one for calculating $\overline{\varphi}$, and one possibly for testing strong feasibility).

Type (C): $\boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0$

Here, the dual problem is

$$\max b^T y \ \text{ subject to } \ A^T y \le c,\ y \le 0.$$

By Theorem 7, the primal and dual solution sets are respectively described

$$M = \{x;\, \underline{A}x \le \overline{b},\ x \ge 0\}, \quad N = \{y;\, \overline{A}^T y \le \overline{c},\ y \le 0\}.$$

By Theorem 4, strong feasibility is equivalent to feasibility of

$$\overline{A}x \le \underline{b}.$$

Hence

$$\underline{f} = \inf \underline{c}^T x \ \text{ subject to } \ \underline{A}x \le \overline{b},\ x \ge 0,$$
$$\overline{\varphi} = \sup \underline{b}^T y \ \text{ subject to } \ \overline{A}^T y \le \overline{c},\ y \le 0.$$

Since all the subproblems are linear, Algorithm 1 calculates the optimal value bounds in polynomial time.

For this specific case, stronger results may be derived and the condition on strong feasibility can be removed [49, 94]. Due to non-negativity of variables, the worst case for the objective function coefficients is $c := \overline{c}$. Similarly,

$$Ax \le \overline{A}x \le \underline{b} \le b$$

for every scenario $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and nonnegative x. So the worst case for the constraint coefficients is $A := \overline{A}$ and $b := \underline{b}$; the feasible set lies in intersection of all feasible sets over all scenarios. Hence we have with no assumption

$$\underline{f} = \inf \underline{c}^T x \ \text{ subject to } \ \underline{A}x \le \overline{b},\ x \ge 0, \tag{15}$$
$$\overline{f} = \inf \overline{c}^T x \ \text{ subject to } \ \overline{A}x \le \underline{b},\ x \ge 0. \tag{16}$$

Remark 3. One can be interested in which scenarios the extremal optimal values are attained [6, 12, 80, 88]. Considering the lower bound, let x^* be an optimal solution of (9); if the optimization problem is unbounded then x^* denotes an unbounded direction. From

$$c_c^T x^* - c_\Delta^T |x^*| = (c_c - \operatorname{diag}(\operatorname{sgn}(x^*))\, c_\Delta)^T x^*$$

we obtain the vector of objective function coefficients
$c := c_c - \mathrm{diag}(\mathrm{sgn}(x^*))\, c_\Delta$. The other coefficients are calculated according to the particular form of ILP. In case of type (A), we have [80]

$$\begin{aligned} A &:= A_c - \mathrm{diag}(p)\, A_\Delta, \\ b &:= b_c + \mathrm{diag}(p)\, b_\Delta, \\ c &:= \underline{c}, \end{aligned}$$

where $p \in [-1, 1]^m$ is defined entrywise as

$$p_i = \begin{cases} \frac{(A_c x^* - b_c)_i}{(A_\Delta x^* + b_\Delta)_i} & \text{if } (A_\Delta x^* + b_\Delta)_i > 0, \\ 1 & \text{if } (A_\Delta x^* + b_\Delta)_i = 0. \end{cases}$$

In (B) we assign

$$A := A_c - A_\Delta\, \mathrm{diag}(\mathrm{sgn}(x^*)), \quad b := \overline{b},$$

and eventually for type (C) we have $A := \underline{A}$, $b := \overline{b}$ and $c := \underline{c}$. Notice that the above assignments are not unique and other ways are possible. Analogously we proceed for the upper bound $\overline{f}$, where we come from the dual problem representation. □

Example 4 (Example 2 continued)**.** The lower bound on the optimal value is calculated in view of (13) and by decomposing into particular orthants (cf. Remark 1). We get $\underline{f} = -\infty$, which is in correspondence with the weak unboundedness recognized in Example 2. By (14) we compute $\overline{\varphi} = -19.7143$. We already know that the primal problem is strongly feasible, so we have $\overline{f} = \overline{\varphi}$. Thus the optimal value range is $[\underline{f}, \overline{f}] = [-\infty, -19.7143]$.

Now, let us determine for which scenarios the extremal optimal values are attained. The problem (13) is unbounded in direction of $x^* = (1, 1)^T$. According to Remark 3, the lower bound is achieved in the setting

$$\begin{aligned} A &:= A_c - A_\Delta\, \mathrm{diag}(\mathrm{sgn}(x^*)) = \begin{pmatrix} -5 & 5 \\ 6 & -8 \\ 0 & -1 \end{pmatrix}, \\ b &:= \overline{b} = (11, 18, -1)^T, \\ c &:= c_c - \mathrm{diag}(\mathrm{sgn}(x^*))\, c_\Delta = (-2, -3)^T. \end{aligned}$$

The optimal solution to (14) is $y^* = (-1, -0.5714, 0)^T$. Since the upper bound is finite, we can consider the dual ILP problem

$$\max \boldsymbol{b}^T y \ \text{ subject to } \ \boldsymbol{A}^T y = \boldsymbol{c}, \ y \leq 0,$$

or, substituting $z := -y$

$$-\min \boldsymbol{b}^T z \quad \text{subject to} \quad \boldsymbol{A}^T z = -\boldsymbol{c},\ z \geq 0.$$

It has the form of type (A). We compute $p = (-1, -1)^T$ by formula

$$p_i = \begin{cases} \frac{(-A_c^T y^* + c_c)_i}{(-A_\Delta^T y^* + c_\Delta)_i} & \text{if } (-A_\Delta^T y^* + c_\Delta)_i > 0, \\ 1 & \text{if } (-A_\Delta^T y^* + c_\Delta)_i = 0, \end{cases}$$

and the wanted scenario draws

$$\begin{aligned} A &:= A_c + A_\Delta \operatorname{diag}(p) = \begin{pmatrix} -3 & 6 \\ 7 & -7 \\ 0 & -1 \end{pmatrix}, \\ b &:= \underline{b} = (10, 17, -1)^T, \\ c &:= c_c - \operatorname{diag}(p)\, c_\Delta = (-1, -2)^T. \end{aligned}$$

□

Example 5 (Example 3 continued)**.** The optimal value bounds are computed by formulae (15) and (16) as $\underline{f} = -33.6364$ and $\overline{f} = -21.2727$. These extremal values are achieved for scenarios given by (15) and (16), so no extra calculation is needed. □

4 Set of optimal solutions

In ILP, it is not obvious at the first sight what does “an optimal solution” mean. We do not discuss a plenty of satisficing solutions developed in robust or fuzzy optimization here. Instead we focus particularly on the set of all possible optimal solution.

Determining the set of all optimal solutions over all scenarios is not only one of the most challenging problems in ILP, but also among the most difficult ones. Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to (4) for a given scenario. By a set of optimal solutions to an ILP problem we mean

$$\mathcal{S} := \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c).$$

This set is hard to determine unless some basis stability conditions hold true. Moreover, $\mathcal{S}$ needn't be a polyhedron. In practice, we do not have to determine the solution set; often, it is sufficient to find a (tight) enclosure. *An interval hull* of $\mathcal{S}$, denoted by $\square\mathcal{S}$, is defined as the smallest interval vector (with respect to inclusion) containing $\mathcal{S}$. An interval superset to $\square\mathcal{S}$ is usually referred to as *an enclosure*, and an subset to $\square\mathcal{S}$ is called *inner enclosure*. Note that an inner enclosure needn't be a subset of $\mathcal{S}$. Enclosures are much more important, but inner enclosure may be useful, too, for example to check quality and accuracy of enclosures [49]. Another kind of approximation was discussed in [100].

As long as the problem is basis stable (see Section 5) then we know the structure of $\mathcal{S}$, and the set itself, its intervall hull and diverse inner approximations can be efficiently calculated. Otherwise, the situation is less optimistic.

Enclosures can be computed by replacing the standard operations by interval arithmetic [3, 30, 36, 49, 91], but the resulting intervals are usually too overestimated.

A different direction to solve the problem is via linear programming duality. Consider type (A), for instance. It is know from duality theory that x and y are optimal solutions to primal and dual problem, respectively, if and only if they solve the linear system

$$Ax = b,\ x \ge 0,\ A^T y \le c,\ c^T x = b^T y.$$

The interval counterpart is

$$\boldsymbol{A}x = \boldsymbol{b},\ x \ge 0,\ \boldsymbol{A}^T y \le \boldsymbol{c},\ \boldsymbol{c}^T x = \boldsymbol{b}^T y.$$

Due to the dependencies, the solution set to this interval system is not equal to the set of optimal solutions, but it is the superset. In view of Theorems 1, 5 and 6, the solution set is described by

$$\underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\ A_c^T y - A_\Delta^T |y| \le \overline{c},\ |c_c^T x - b_c^T y| \le c_\Delta^T x + b_\Delta^T |y|. \tag{17}$$

Therefore, any enclosure to x-solution of (17) is also an enclosure to $\mathcal{S}$.

Another fundamental problem is as follows. Given $x^* \in \mathbb{R}^n$, is it optimal for each scenario? Is it optimal for some scenario? The answer for the former is mostly "no" since one point can hardly be optimal for any data perturbation; this may be true only in some special cases (see Section 6). The latter is an open problem, too.

Example 6 (Example 3 continued)**.** Rewrite the problem into type (A) as follows

$$\min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x + Iy = \boldsymbol{b}, \ x, y \geq 0.$$

Using the above method and decomposing (17) to 2^m linear sub-problems we obtain an enclosure to the optimal solution set as follows

$$\mathcal{S} \subseteq ([2.4356, 4.9091], [1.0909, 3.7001])^T.$$

Compared to the exact interval hull (Example 7), the overestimation is not so tremendous. Surprisingly, one limit is exact. □

5 Basis stability

If a linear program has an optimal solution then it possess an optimal basic solution; simplex methods always converge to optimal basic solutions. A natural question in interval linear programming is whether there is some basis that is optimal for some or for each scenario of interval data. The first issue was investigated e.g. by McKeown and Minch [51] for the case of interval objective function coefficients. Based on multiparametric programming, the authors proposed an enumeration algorithm to compute all bases that are optimal for some scenario. Multi-parametric approach to uncertainties in the right-hand side and in the objective function was treated in [14]. The general case ILP has not been inspected yet. The second issue was addressed e.g. in [3, 34, 36, 59, 77]. Below, we discuss it below more deeply because it is important in several aspects. It is not only a stability criterion, but also enables to efficiently determine set of optimal solutions and optimal value range.

Consider type (A) linear program, where the primal problem reads

$$\min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0. \tag{18}$$

By a basis we mean an index set $B \subseteq \{1, \ldots, n\}$ such that A_B is nonsingular, where A_B denotes the restriction of A to the columns indexed by B. Analogously, $N := \{1, \ldots, n\} \setminus B$ stands for nonbasic variables and as an subscript it denotes the restriction to the nonbasic indices.

Let a basis B be given. The ILP problem is called B-stable if B is optimal basis for each scenario of interval values. ILP is called [unique] non-degenerate B-stable if each scenario has a [unique] non-degenerate optimal basic solution with the basis B.

In the sequel, we review B-stability for a candidate basis B. The basis B can be computed by an interval version of simplex method [3, 30, 32, 49], or estimated by solving a suitable scenario (e.g. taking the midpoint values).

5.1 B-stability

We describe the method by Hladík [24]. Remind that a basis B is optimal in a real-valued linear programming problem (18) if and only if three conditions simultaneously hold:

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Extension to interval data leads to the following characterization of B-stability. The basis B is optimal for each scenario if and only if conditions C1 to C3 hold for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$.Below, we discuss the three conditions in detail.

An interval matrix $\boldsymbol{M} \in \mathbb{IR}^{n\times n}$ is called *regular* if every $M \in \boldsymbol{M}$ is non-singular. It was proved by Poljak and Rohn [68] that checking regularity is NP-hard problem, so the first condition cannot be answered efficiently. However, there is a plenty of diverse methods for testing regularity; see e.g. a review paper by Rohn [82]. Moreover, there are several sufficient conditions that can be employed as well [70]. For instance, a broadly used one is that if the spectral radius of $|(M_c)^{-1}|M_\Delta$ is less than 1 then $\boldsymbol{M}$ is regular, which gives rise to the following.

Theorem 12. *If* $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$ *then* $\boldsymbol{A}_B$ *is regular.*

Turning to the second point C2, the inequality $A_B^{-1}b \geq 0$ holds for every $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ if and only if the solution set to the interval system $\boldsymbol{A}_B x_B = \boldsymbol{b}$ lies in the non-negative orthant. The simple but exponential method is to compute the exact interval hull of the solution set and check for non-negativity. Another way is to utilize some solver for interval equations to get an enclosure of the solution set, and again to check its non-negativity. This leads to a fast sufficient condition.

In the third point C3, the condition is equivalent to feasibility of

$$c_N^T \geq y^T A_N,\ A_B^T y = c_B. \tag{19}$$

The interval counterpart takes the form

$$\boldsymbol{A}_N^T y \leq \boldsymbol{c}_N, \ \boldsymbol{A}_B^T y = \boldsymbol{c}_B.$$

First, we derive a simple sufficient condition. Let $\boldsymbol{y}$ be an enclosure to the solution set of $\boldsymbol{A}_B^T y = \boldsymbol{c}_B$. If

$$\overline{(\boldsymbol{A}_N^T)\boldsymbol{y}} \leq \underline{c}_N \tag{20}$$

then in each scenario the solution to the equation system solves also the whole system (19) and thus the strong feasibility is valid. Note that the left-hand side of (20) is an upper limit of the interval matrix product calculation.

A sufficient and necessary characterization to the third condition was given by Hladík [24].

Theorem 13. *The third condition C3 holds true if and only if for each* $q \in \{\pm 1\}^m$ *the polyhedral set described by*

$$((A_c)_B^T - (A_\Delta)_B^T \operatorname{diag}(q))y \leq \overline{c}_B, \tag{21a}$$

$$-((A_c)_B^T + (A_\Delta)_B^T \operatorname{diag}(q))y \leq -\underline{c}_B, \ \operatorname{diag}(q)\, y \geq 0 \tag{21b}$$

lies inside the polyhedral set

$$((A_c)_N^T + (A_\Delta)_N^T \operatorname{diag}(q))y \leq \underline{c}_N, \ \operatorname{diag}(q)\, y \geq 0. \tag{22}$$

Decomposing to particular arthants, the polyhedral sets (21) and (22) become convex. In this way, testing for C3 requires 2^m inclusion tests, each of which can be done in polynomial time. A polyhedral set described $Vx \leq v$ lies inside $Wx \leq w$ if and only if for each i the following is true

$$w_i \geq \max W_{i,*}x \ \text{ subject to } \ Vx \leq v.$$

5.2 Non-degenerate B-stability

The method from the previous section can be adapted for unique and non-degenerate B-stability, too. When we consider strict inequality in the second condition then we obtain a method for testing non-degenerate B-stability. Independently, considering strict inequality in the third condition we have only a sufficient (but strong) characterization of unique B-stability. Thus, (20) with strict inequality and Theorem 13 with the strict inclusion test give sufficient, but not necessary characterization of unique B-stability.

Another approaches were utilized in [77, 34]. Rohn [77] proposed the following reduction to 2^{2m} linear programs. The subsequent sufficient condition is due to Koníčková [34], and it is related to the approach presented in Section 5.1.

Theorem 14 (Rohn, 1993)**.** *Let a basis B be given. ILP is [unique] non-degenerate B-stable if and only if it is true for the following finite set of scenarios*

$$\begin{aligned} &\min\ (c_c + \operatorname{diag}(q)\, c_\Delta)^T x \\ &\ \textit{subject to}\ \ (A_c - \operatorname{diag}(p)\, A_\Delta \operatorname{diag}(q))x = b_c + \operatorname{diag}(p)\, b_\Delta, x \geq 0, \end{aligned}$$

where $p \in \{\pm 1\}^n$ and $q \in \{q \in \mathbb{R}^n;\ |q_j| = 1\ \forall j \in B,\ q_j = 1\ \forall j \notin B\}$.

Theorem 15 (Koníčková, 2001)**.** *Let B be a basis, $\boldsymbol{x}_B$ the interval hull of the solution set to $\boldsymbol{A}_B x_B = \boldsymbol{b}$, and $\boldsymbol{y}$ the interval hull to $\boldsymbol{A}_B^T y = \boldsymbol{c}_B$. Suppose that $\boldsymbol{A}_B$ is regular, and $\underline{x}_B > 0$. If the inequality*

$$\overline{(\boldsymbol{A}_N^T)\boldsymbol{y}} \leq \underline{c}_N \tag{23}$$

holds then ILP is non-degenerate B-stable with a basis B. If (23) *holds strictly then it is unique non-degenerate B-stable.*

Note that the sufficient condition presented in Theorem 15 is not very efficient as it requires also an exponential number of operations. Replacing $\boldsymbol{x}_B$ and $\boldsymbol{y}$ by enclosures to their interval hulls we obtain more useful sufficient condition, related to that described in Section 5.1.

B-stability of ILP is very important because it enables to describe the set of all possible optimal solutions [3, 34, 59] and to calculate the optimal value range (cf. Section 3). Under the assumption of unique B-stability, the set of all optimal solutions is equal to the solution set of the interval system $\boldsymbol{A}_B x_B = \boldsymbol{b}$, $x_B \geq 0$, $x_N = 0$. By Theorem 5, the set represents a convex polyhedral set described by

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0. \tag{24}$$

When the ILP problem is B-stable, but not unique B-stable, then each scenario of ILP has at least one optimal solution in this set, and, conversely, each solution of the set is an optimal solution of some scenario.

B-stability also implies that the optimal value ranges within an interval $\boldsymbol{f} = [\underline{f}, \overline{f}]$, where

$$\underline{f} = \min \underline{c}_B^T x \text{ subject to } \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$
$$\overline{f} = \max \overline{c}_B^T x \text{ subject to } \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Thus the upper bound $\overline{f}$ is much more easy to compute than in the general case.

Remark 4. In this section, we treated type (A). For type (B) and (C), few results are known, but we can transform them to type (A). Type (B) is transformed to type (A) by taking the dual problem. Here, we must be sure that the duality gap is always zero. Strong feasibility of the primal or the dual problem implies zero duality gap (cf. Section 7), so if it is the case we are done.

In type (C), the constraints $\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$ are transformed into equality constrains

$$\boldsymbol{A}x + Iy = \boldsymbol{b},\ x, y \geq 0.$$

Since there are no dependencies, both systems are equivalent. In principle, this reduction of (C) to (A) can be used in any topic discussed in this work. However, when doing it we loose some information and it can be on account of complexity. For example, compare performance of testing strong feasibility for types (A) and (C). □

Example 7 (Example 6 continued)**.** We again use the transformation of type (C) to type (A)

$$\min \boldsymbol{c}^T x \text{ subject to } \boldsymbol{A}x + Iy = \boldsymbol{b},\ x, y \geq 0.$$

For the scenario with midpoint values, the optimal basis is $B = (1, 2, 3)$. Let us check whether it is optimal for any other scenario.

C1. By Theorem 12, we calculate the spectral radius 0.0909, so the interval matrix $\boldsymbol{A}_B$ is regular.

C2. By the Hansen–Bliek–Rohn method (see [64, 76, 79]) we compute an enclosure to the solution set of $\boldsymbol{A}_B x_B = \boldsymbol{b}$ to be

$$\boldsymbol{x}_B = ([3.7913, 4.9455], [1.5268, 2.8546], [0.0545, 20.2637])^T.$$

The lower limit is non-negative, so the second point is satisfied.

C3. By the Hansen–Bliek–Rohn method we compute an enclosure to the solution set of $\boldsymbol{A}_B^T y = \boldsymbol{c}_B$ to be

$$\boldsymbol{y} = ([-0.0001, 0.0001], [-0.5001, -0.2499], [-3.8864, -2.3863])^T.$$

Now, the relation (20) reads

$$\overline{(\boldsymbol{A}_N^T)\boldsymbol{y}} = (-0.2500, -2.3864)^T \leq \underline{c}_N = (0, 0)^T.$$

Hence the sufficient condition is fulfilled and the problem is B-stable. Moreover, since the above inequalities are strict, it follows from Theorem 15 that the problem is unique non-degenerate B-stable. So we can determine the exact description of the solution set by (24); its projection into the (x_1, x_2) subspace reads

$$-7x_1 + 4x_2 \leq -18,\ 6 \leq x_1 + x_2 \leq 7,\ 6x_1 - 5x_2 \leq 19,$$

and its interval hull $\square\mathcal{S} = ([3.8181, 4.9091], [1.5454, 2.8182]$. See illustration in Figure 2. $\square$

6 Special cases

In specific cases, sometimes stronger results may be developed. In practical problems not all input quantities must be subject to inaccuracy. The typical situation is that some of them are proper intervals and some of them are degenerate ones (real numbers). If it is the case, some of the exponential method presented in this work can be accelerated. These exponential algorithms are based on a decomposition along the signs of variables x_i, $i = 1, \dots, n$ (Remark 1). However, if the coefficients by some x_i are altogether degenerate then we needn't decompose along the sign of x_i, and save the time; see Remark 1.

Since uncertainty in the objective function and in the right-hand side are the most common situations in practice, we focus on these situations in this section. We leave aside specific problems that can be formulated as linear programs, but for which more effective algorithms exist, such as transportation problems [33, 87] or minimum cost flow problems [17]. Their interval extensions must be treated in a specific way.

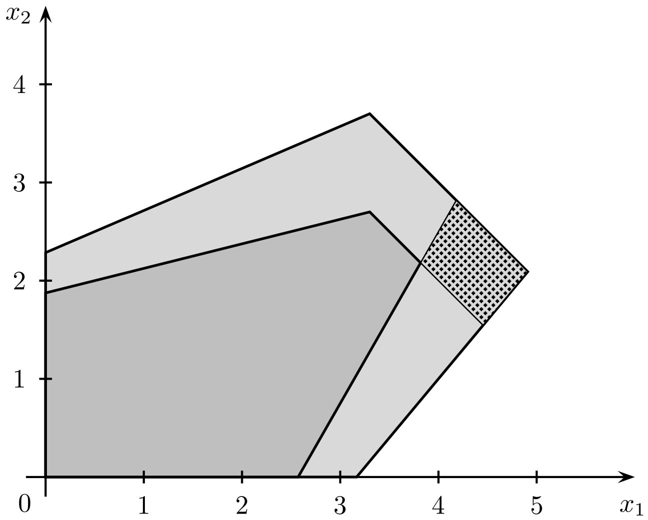

Figure 2: (Example 7): Intersection of all feasible sets in dark gray; union in light gray; set of optimal solutions in dotted area.

6.1 Interval right-hand side

Interval right-hand side was studied e.g. by Li & Wang [39], and Gabrel et al. [11, 12]. Let

$$\min c^T x \quad \text{subject to} \quad x \in \mathcal{M}(A, b)$$

be a family of linear programs with $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ given, and b perturbing within an interval vector $\boldsymbol{b} \in \mathbb{IR}^m$. Gabrel et al. [11, 12] investigated the optimal value range problem for this specific case. For type (A) with equality constraints, they presented several methods for calculating the lower bound on the optimal value and showed that computing the upper bound is NP-hard. Anyway, type (A) cannot be solved much more effectively than the general method from Section 3. This is not true for type (B) and (C) with inequality constraints, where the minimal optimal value is achieved for $b := \overline{b}$, and the maximal one for $b := \underline{b}$.

Let us look closer to basis stability. There is no occurrence of b in conditions C1 and C3, so it is enough to inspect C2. Condition C2 is simplified to

$$\underline{A_B^{-1}\boldsymbol{b}} \geq 0 \tag{25}$$

since the matrix A_B^{-1} contains no proper interval. Thus the problem is B-stable if and only if B is optimal basis for some scenario and (25) holds true. For non-degenerate B-stability we have the same with strict inequality in (25). Therefore, basis stability testing is tractable in this case.

6.2 Interval objective function coefficients

ILP problems with interval objective function coefficients were studied by Gabrel and Murat [12], Hladík [22, 18], Inuiguchi and Sakawa [28, 29], and McKeown and Minch [51], among others. Consider a family of linear programs

$$\min c^T x \;\; \text{subject to} \;\; x \in \mathcal{M}(A, b),$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given, and c perturbs within an interval vector $\boldsymbol{c} \in \mathbb{IR}^n$. Let $x \in \mathcal{M}(A, b)$ be a feasible solution. It is called *possibly optimal* if it is optimal for some $c \in \boldsymbol{c}$, and *necessarily optimal* if it is optimal for each $c \in \boldsymbol{c}$. This notion was adopted from [28].

Three algorithms to compute all possibly optimal solutions were presented by Steuer [89]. Here, we show how to check possibly optimality of a given feasible solution $x^* \in \mathcal{M}(A, b)$; the method is obviously polynomial. Without loss of generality, let a tangent cone to $\mathcal{M}(A, b)$ at x^* be described by $Dx \leq 0$. Then x^* is optimal if and only if the linear inequality system

$$Dx \leq 0,\; c^T x \leq -1 \tag{26}$$

has no solution. Thus x^* is optimal for some $c \in \boldsymbol{c}$ if and only if the system (26) is infeasible for some $c \in \boldsymbol{c}$, or, equivalently, if and only if it is not true that (26) is feasible for all $c \in \boldsymbol{c}$. Strong feasibility is characterized by Theorem 3, which implies the following assertion. A similar result was derived in [28].

Proposition 1. *A point $x^* \in \mathcal{M}(A, b)$ is possibly efficient if and only if there is no solution to the linear system*

$$D(x^1 - x^2) \leq 0,\; \overline{c}^T x^1 - \underline{c}^T x^2 \leq -1,\; x^1, x^2 \geq 0.$$

Remark 5 (Tangent cone)**.** Remind some basic properties on tangent cones. Let x^* be a feasible solution to a convex polyhedral set $\mathcal{M}(A, b)$. In type (A), where the feasible set is described by $Ax = b$, $x \geq 0$, the tangent cone at x^* reads

$$Ax = 0, \; x_i \geq 0 \quad \forall i \in I(x^*),$$

where $I(x^*) := \{i;\, x^*_i = 0\}$ is the set of active indices. In type (B), with constraints $Ax \leq b$, the active set is defined analogously as $I(x^*) := \{i;\, A_{i*}x^* = b_i\}$ and the tangent cone is described by

$$A_{i*}x \leq 0, \quad \forall i \in I(x^*).$$

Finally, type (C) is easily transformed to type (B) by incorporating the non-negativity constraints into the principal inequality system. □

A minmax regret characterization of necessarily optimal solutions was given in [29], a heuristic in [50], an exponential algorithm in [28], and complexity results in [2]. The method from Section 5.1 on basis stability is adapted and simplified in the following way. Conditions C1 and C2 hold trivially. In C3, testing strong feasibility to

$$A_N^T y \leq \boldsymbol{c}_N, \; A_B^T y = \boldsymbol{c}_B$$

is equivalent to

$$A_N^T A_B^{-T} c_B \leq c_N, \quad c \in \boldsymbol{c},$$

by substitution $y := A_B^T c_B$. Due to the special structure, it si strongly feasible if and only if

$$\overline{A_N^T A_B^{-T} \boldsymbol{c}_B} \leq \underline{c}_N. \tag{27}$$

This is easily checked by interval arithmetic. Thus B-stability is equivalent to (27). As long as x^* is a basic solution corresponding to a basis B then (27) is sufficient and necessary condition for necessary optimality of x^*.

Some results are related to inequality constrained type (B). In [18] it was shown that testing necessarily optimality of x is an NP-hard problem. However, restriction to a class of problems where x is a non-degenerate basic solution makes the problem polynomial. It is because we can efficiently determine the normal cone to $\mathcal{M}(A, b)$ at x^*, and x^* is optimal for each scenario if and only if the box $-\boldsymbol{c}$ lies within the normal cone. The following result is adapted from [18].

Theorem 16. *Let x be a non-degenerate basic solution corresponding to a basis B. It is necessarily optimal if and only of*

$$\overline{A_B^T \boldsymbol{c}} \leq 0.$$

7 Duality

Various aspects of duality in ILP were investigated e.g. by Rohn [71], Serafini [88], and Gabrel et al. [10, 12]. Basically, linear programming duality is straightforwardly extended to interval linear programming. Consider, for example, a pair of respectively primal and dual linear programs

$$f(A, b, c) := \min c^T x \quad \text{subject to} \quad Ax = b, \ x \geq 0,$$

and

$$g(A, b, c) := \max b^T y \quad \text{subject to} \quad A^T y \leq c.$$

The following considerations hold for other types as well. As long as at least one of the problems is feasible then strong duality holds, that is, $f(A, b, c) = g(A, b, c)$, where $\min \emptyset = \infty$ and $\max \emptyset = -\infty$ by convention. Now, consider a family of primal dual programs over $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$. First, we have to verify that in each scenario at least one of the primal dual problems is feasible. So far, there is no algorithm known that gives answer in every case, but there are some useful sufficient conditions. For instance, the property is valid if the primal (or dual) ILP problem is strongly feasible (see Section 2.1).

Suppose that zero duality gap is ensured for all scenarios. Then the lower bound on the optimal value for the primal ILP is equal to the lower bound for the dual ILP. In other words,

$$\min_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} f(A, b, c) = \min_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} g(A, b, c) \tag{28}$$

and likewise for the upper bounds

$$\max_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} f(A, b, c) = \max_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} g(A, b, c). \tag{29}$$

The left-hand side of (28) is a basic problem of computation the lower bound on the optimal value in type (A), whereas the right-hand side corresponds to the upper bound in type (B). Similarly, the left-hand side of (29) corresponds to the upper bound in type (A) and the right-hand side to the lower bound in type (B). Not surprisingly, each pair consists of problems with the same complexity: polynomial in the first case, and NP-hard in the second case.

Provided that $\underline{f}$ is finite, the relation (28) takes an alternative form [80]

$$\underline{f} = \max\left(\min_{b\in\boldsymbol{b}} b^T y\right) \quad \text{subject to} \quad A^T y \le c \text{ for each } A \in \boldsymbol{A}, c \in \boldsymbol{c}.$$

Considering uncertainties in the constraint matrix only, the relation reads [88]

$$\begin{aligned} &\min c^T x \quad \text{subject to} \quad x \in \bigcup_{A\in\boldsymbol{A}} \{x;\ Ax = b,\ x \ge 0\} \\ =&\max b^T y \quad \text{subject to} \quad y \in \bigcap_{A\in\boldsymbol{A}} \{y;\ A^T y \le c\}. \end{aligned}$$

Analogously, when $\underline{f}$ is finite then (29) has an equivalent form [80]

$$\overline{f} = \max\left(\max_{b\in\boldsymbol{b}} b^T y\right) \quad \text{subject to} \quad A^T y \le c \text{ for some } A \in \boldsymbol{A}, c \in \boldsymbol{c}.$$

8 Conclusion

8.1 Applications

Since uncertainty is common in many scientific disciplines, we find applications in diverse fields such as economics, sociology, or logistic. In economics, portfolio selection problem was studied in [5, 21, 38]. An application to network topology of transmission systems is to be found in [65]. The feedmix problem was considered in [94, 89], and the related diet problem in [30, 88]. In the area of logistics, environmental management and planning, ILP was applied e.g. in air quality management [40], water resources and quality management [46, 47, 95, 99], solid waste management planning [25, 26, 43, 92], long-term hydropower planning [48], and inventory management [4, 12].

Game theory is engaged in competing and strategic interaction among subjects in social sciences, biology, engineering, and others. The intrinsic uncertainty can be modelled by interval estimates, as was done in interval matrix game works by Collins and Hu [7, 8], Liu and Kao [42], or Nayak and Pal [61]. Since zero sum matrix games are equivalent to linear programming, interval matrix games are solvable by ILP methodology, in principle.

In essence, ILP can help in global optimization methods based on interval analysis and branch & bound framework [9, 16, 63]. Often, the objective function is linearized, and knowledge of optimal value range may improve lower and upper bounds of the objective function over sub-boxes.

On a theoretical basis, Tigan and Stancu-Minasian [93] and Rohn [74] applied ILP to introduce sensitivity coefficients of linear programs.

ILP methodology is a suitable tool for sensitivity analysis in linear programming. In traditional sensitivity analysis, one studies behaviour of optimal value and basis stability with respect to one parameter perturbation. This is a simplified approach since it doesn't take into account dependencies with other coefficients. Simultaneous and independent variations of rim coefficients were investigated by Hladík [20], Ward and Wendell [97] and Wendell [98], for instance. Sensitivity analysis of simultaneous perturbations of arbitrary coefficients can be performed just by ILP techniques.

8.2 Inverse problems

The above mentioned sensitivity analysis closely relates with inverse problems. In inverse problems, we are usually given a linear program, and we have to maximally extend the reals to intervals such that some kind of invariancy is satisfied. For example, Hladík [21] touched the inverse optimal value range problem. Therein, one is given a linear program and bounds on the optimal value, and the aim is to extend it to an ILP problem such that the optimal value range lies within the prescribed bounds. In [21], an efficient algorithm is proposed that computes a Pareto optimal extension to ILP in most of the cases.

8.3 Open problems

More then thirty years of research in ILP have brought many interesting results and closed many questions. Nevertheless, there are still some open questions remaining. We accomplish our survey with a list of open problems concerning ILP.

- A sufficient and necessary condition for weak unboundedness, strong boundedness and weak optimality.

- A method to check if a given $x^* \in \mathbb{R}^n$ is an optimal solution for some scenario.
- A method for determining the image of the optimal value function.
- A sufficient and necessary condition for duality gap to be zero for each scenario.
- A method to test if a basis B is optimal for some scenario.
- Characterization of basis stability for types (B) and (C).
- Characterization of the set of optimal solutions and its interval hull.

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