Three-coloring triangle-free graphs on surfaces I. Extending a coloring to a disk with one triangle

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Abstract

Let G be a plane graph with with exactly one triangle T and all other cycles of length at least 5, and let C be a facial cycle of G of length at most six. We prove that a 3-coloring of C does not extend to a 3-coloring of G if and only if C has length exactly six and there is a color x such that either G has an edge joining two vertices of C colored x, or T is disjoint from C and every vertex of T is adjacent to a vertex of C colored x. This is a lemma to be used in a future paper of this series.

1 Introduction

This is the first paper in a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. All graphs in this paper are simple, with no loops or parallel edges.

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The subject of coloring graphs on surfaces goes back to 1890 and the work of Heawood [20], who proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24\gamma + 1})/2 \rfloor$. Here and later γ is the *Euler genus of* Σ, defined as $\gamma = 2g$ when $\Sigma = S_g$, the orientable surface of genus g, and $\gamma = k$ when $\Sigma = N_k$, the non-orientable surface with k cross-caps. Incidentally, the assertion holds for the sphere as well, by the Four-Color Theorem [3, 4, 6, 23]. Ringel and Youngs (see [22]) proved that the bound is best possible for all surfaces except the Klein bottle, for which the correct bound is 6. Dirac [10] and Albertson and Hutchinson [2] improved Heawood's result by showing that every graph in Σ is actually $(H(\Sigma) - 1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

For triangle-free graphs there does not seem to be a similarly nice formula, but Gimbel and Thomassen [16] gave very good bounds: they proved that the maximum chromatic number of a triangle-free graph drawn in a surface of Euler genus γ is at least $c_1\gamma^{1/3}/\log\gamma$ and at most $c_2(\gamma/\log\gamma)^{1/3}$ for some absolute constants c_1 and c_2.

In this series we adopt a more modern approach to coloring graphs on surfaces, following the seminal work of Thomassen [25, 28, 30]. The basic premise is that while Heawood's formula is best possible for all surfaces except the Klein bottle, only relatively few graphs attain the bound or even come close. To make this assertion more precise let us recall that a graph G is called *k-critical*, where $k \geq 1$ is an integer, if every proper subgraph of G is $(k-1)$-colorable, but G itself is not. It follows easily from Euler's formula that if Σ is a fixed surface, and G is a sufficiently big graph drawn in Σ, then G has a vertex of degree at most six. It follows that for every $k \geq 8$ the graph G is not k-critical, and hence there are only finitely many k-critical graphs that can be drawn in Σ. It is not too hard to extend this result to $k = 7$. In fact, it can be extended to $k = 6$ by the following deep theorem of Thomassen [28].

Theorem 1. *For every surface Σ there are only finitely many 6-critical graphs that can be drawn in Σ.*

The lists of 6-critical graphs are explicitly known for the projective plane [2], the torus [25] and the Klein bottle [9, 21]. An immediate consequence is that for every surface Σ there is a polynomial-time (in fact, linear-time) algorithm to test whether an input graph drawn in Σ is 5-colorable. Theorem 1 does not hold for 5-critical graphs, because of an elegant construction of Fisk [14]. For 3-colorability an algorithm as above does not exist, unless $P = NP$, because testing 3-colorability is NP-hard even for planar graphs [15]. It is an open problem whether there is an algorithm for testing 4-colorability of graphs in Σ when Σ is a fixed non-planar surface. The techniques currently available do not give much hope for a positive resolution in the near future.

How about triangle-free graphs? Similarly as above, if G is a sufficiently large triangle-free graph in a fixed surface Σ, then G has a vertex of degree at most four. Thus G is not 6-critical, and the argument can be strengthened to show that G is not 5-critical. Thus for a fixed integer $k \geq 4$ testing k-colorability of triangle-free graphs drawn in a fixed surface can be done in linear time, as before. That brings us to testing 3-colorability of triangle-free graphs in a fixed surface, the subject of this series of papers. The question has been raised by Gimbel and Thomassen [16] and we resolve it later in this series, after we develop some necessary theory.

Historically the first result in this direction is the following classical theorem of Grötzsch [17].

Theorem 2. *Every triangle-free planar graph is* 3*-colorable.*

Thomassen [26, 27, 29] found three reasonably simple proofs, and extended Theorem 2 to other surfaces. Recently, two of us, in joint work with Kawarabayashi [11] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 2. The statement of Theorem 2 cannot be extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical graphs that can be drawn in Σ. For instance, the graphs obtained from an odd cycle of length five or more by applying Mycielski's construction [7, Section 8.5] have that property. Thus an algorithm for testing 3-colorability of triangle-free graphs on a fixed surface will have to involve more than just testing the presence of finitely many obstructions.

The situation is different for graphs of girth at least five by another deep theorem of Thomassen [30], the following.

Theorem 3. *For every surface Σ there are only finitely many* 4*-critical graphs of girth at least five that can be drawn in Σ.*

Thus the 3-colorability problem on a fixed surface has a polynomial-time algorithm for graphs of girth at least five, but the presence of cycles of length four complicates matters. Let us remark that there are no 4-critical graphs of girth at least five on the projective plane and the torus [26] and on the Klein bottle [24].

The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [34], Gimbel and Thomassen [16] obtained the following elegant characterization. A graph drawn in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

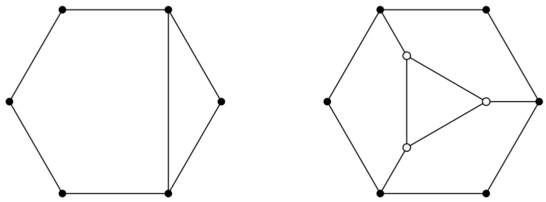
Theorem 4. *A triangle-free graph drawn in the projective plane is* 3-*colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

For other surfaces there does not seem to be a similarly nice characterization, but in a later paper of this series we will present a polynomial-time algorithm to decide whether a triangle-free graph in a fixed surface is 3-colorable. The algorithms naturally breaks into two steps. The first is when the graph is a quadrangulation, except perhaps for a bounded number of larger faces of bounded size, which will be allowed to be precolored. In this case there is a simple topological obstruction to the existence of a coloring extension based on the so-called "winding number" of the precoloring. Conversely, if the obstruction is not present and the graph is highly "locally planar", then we can show that the precoloring can be extended to a 3-coloring of the entire graph. This can be exploited to design a polynomial-time algorithm. With additional effort the algorithm can be made to run in linear time.

The second step covers the remaining case, when the graph has either many faces of size at least five, or one large face, and the same holds for every subgraph. In that case we show that the graph is 3-colorable. That is a consequence of the following theorem, which will form the cornerstone of this series.

Theorem 5. *There exists an absolute constant K with the following property. Let G be a graph drawn in a surface Σ of Euler genus γ with no separating cycles of length at most four, and let t be the number triangles in G. If G is* 4-*critical, then $\sum |f| \leq K(t+\gamma)$, where the summation is over all faces f of G of length at least five.*

If G has girth at least five, then $t = 0$ and every face has length at least five. Thus Theorem 5 implies Theorem 3, and, in fact, improves the bound given by the proof of Theorem 3 in [30]. The fact that our bound in Theorem 5 is linear in the number of triangles is needed in our solution of a problem of Havel [19], as follows.



(a) (b)

Figure 1: Critical graphs with a precolored 6-cycle and one triangle.

Theorem 6. *There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

In this paper we confine ourselves to proving a lemma that will be needed in a later paper of the series, the following. The two outcomes are illustrated in Figure 1.

Theorem 7. *Let G be a plane graph with a facial cycle R of length at most six, let T be a triangle in G, and assume that every cycle in G other than T and R has length at least five. Let ϕ be a 3-coloring of R that does not extend to a 3-coloring of G. Then C has length exactly six and either*

(a) *$\phi(u) = \phi(v)$ for two distinct vertices $u, v \in V(C)$ that are adjacent in G, or*

(b) *$\phi(u_1) = \phi(u_2) = \phi(u_3)$ for three pairwise distinct vertices $u_1, u_2, u_3 \in V(C)$, where each u_i is adjacent to a different vertex of T.*

Finally, let us mention a related interesting conjecture due to Steinberg [31], who conjectured that every planar graph without 4- and 5-cycles is 3-colorable. This conjecture is still open. Currently the best result of Borodin, Glebov, Montassier and Raspaud [8] shows that excluding cycles of lengths 4, 5 and 7 suffices to guarantee 3-colorability. (A proof of the same result by Xu [33] is disputed in [8].)

2 Auxiliary results

In this short section we present several results that will be needed later. Let G be a graph, and let R be a subgraph of G. We say that G is *R-critical* if

$G \neq R$ and for every proper subgraph G' of G that includes R as a subgraph there exists a 3-coloring of R that extends to a 3-coloring of G', but not to one of G. Theorem 2 admits the following strengthening.

Theorem 8. *There is no R-critical triangle-free plane graph G, where R is a cycle in G of length at most five.*

This result was later strengthened in several ways.
Gimbel and Thomassen [16] extended this to cycles of length six:

Theorem 9. *Let G be a plane triangle-free graph with a facial cycle R of length six. If G is R-critical, then all faces of G distinct from R have length four.*

For graphs of girth at least five the above results can be strengthened, as shown by Thomassen [30] and Walls [32].

Theorem 10. *Let H be a plane graph of girth at least five, and let C be a facial cycle in H of length $k \leq 11$. If H is C-critical, then*

(a) $k \geq 8$, $V(H) = V(C)$ and C is not induced, or

(b) $k \geq 9$, $H - V(C)$ is a tree with at most $k-8$ vertices, and every vertex of $V(H) - V(C)$ has degree three in G, or

(c) $k \geq 10$ and $H - V(C)$ is a connected graph with at most $k - 5$ vertices containing exactly one cycle, and the length of this cycle is five. Furthermore, every vertex of $V(H) - V(C)$ has degree three in H.

Proof. Since H is C-critical, there exists a 3-coloring of C that does not extend to a 3-coloring of G. By [30, Theorem 2.5] or [32, Theorem 3.0.2] there exists a subgraph H' of H such that C is a subgraph of H' and H' satisfies one of (a)–(c). We claim that $H = H'$. Indeed, otherwise the C-criticality of H implies that there exists a 3-coloring of C that does not extend to a 3-coloring of H, but extends to a 3-coloring ϕ of H'. By applying [30, Theorem 2.5] or [32, Theorem 3.0.2] to every face of H' we deduce that ϕ extends to every face of H', and hence extends to a 3-coloring of H, a contradiction. □

We will need a version of Theorem 8 that allows the existence of a triangle. Such a result was attempted by Grünbaum [18], but his proof is not correct. A correct proof was found by Aksionov [1]. The result of Aksionov together with Theorem 9 gives the following characterization of critical graphs with a triangle and a precolored face of length at most five.

Theorem 11. *Let G be a plane graph with a facial cycle R of length at most five and at most one triangle T distinct from R. If G is R-critical, then R has length exactly five, T shares at least one edge with R and all faces of G distinct from T and R have length exactly four.*

3 Graphs with one triangle

To prove Theorem 7 we prove, for the sake of the inductive argument, the following slightly more general result. Theorem 7 will be an immediate corollary.

Theorem 12. *Let G be a plane graph with outer cycle R of length at most six and assume that*

($*$) *there exists a face f_0 of G such that every cycle in G of length at most four bounds a closed disk containing f_0.*

If G is R-critical, then R has length exactly six and G is isomorphic to one of the graphs depicted in Figure 1.

Proof. Let G be as stated, and suppose for a contradiction that it is not isomorphic to either of the two graphs depicted in Figure 1. By Theorems 8 and 9 the graph G has a triangle T. We may assume that G is minimal in the sense that the theorem holds for every graph with fewer vertices. A vertex $v \in V(G) - V(R)$ will be called *internal*. The R-criticality of G implies that

(1) *every internal vertex of G has degree at least three.*

If C is a cycle in G, then by $\mathrm{ins}(C)$ we denote the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C.

(2) *For every cycle C in G that does not bound a face, $\mathrm{ins}(C)$ is a C-critical graph.*

To prove (2) let C be as stated, let G' be obtained from G by deleting every vertex and edge of G drawn in the open disk bounded by C, and let J be a proper subgraph of $\mathrm{ins}(C)$ that includes C. Then some 3-coloring of R extends to a 3-coloring ϕ of $G' \cup J$, but does not extend to a 3-coloring of G. It follows that the restriction of ϕ to C extends to J, but not to $\mathrm{ins}(C)$, as desired. This proves (2).

It follows from $(*)$, (2) and Theorem 11 that

(3) *T is the only cycle in G of length at most four.*

Next we constrain cycles in G of length at most seven:

(4) *Let $C \neq R$ be a cycle in G of length at most seven that does not bound a face. Then C has length at least six, and the closed disk bounded by C includes T.*

To prove (4) let C be as stated. By the minimality of G and (2) we deduce that C has length at least six. If T is not contained in the closed disk Δ bounded by C, then $(*)$ implies that $\mathrm{ins}(C)$ has girth at least five, contrary to (2) and Theorem 10. Thus T is contained in Δ, and (4) follows.

It follows that T bounds the face f_0. The same argument implies the following two claims. To prove the second one, Theorem 10 is applied to a graph obtained from G by splitting repeated vertices of C so that C will become a cycle in the new graph.

(5) *Let $C \neq R$ be a cycle in G of length six that does not bound a face. Then $\mathrm{ins}(C)$ is isomorphic to one of the graphs depicted in Figure 1.*

(6) *Let $C \neq R$ be a closed walk in G of length $k \leq 11$ bounding an open disk Δ disjoint from T, and let H be the subgraph of G consisting of vertices and edges drawn in the closure of Δ. Then H satisfies the conclusion of Theorem 10.*

From (3) and Theorem 11 it follows that

(7) *R has length six,*

and since every cycle of length at most five bounds a face by (4) we deduce that

(8) *the graph G has no subgraph H with outer face R such that H is isomorphic to either of the two graphs depicted in Figure 1; in particular, R is induced.*

Next we claim that

(9) *every internal vertex has at most one neighbor in $V(R)$.*

To prove (9) suppose for a contradiction that an internal vertex v_2 has two neighbors $v_1, v_3 \in V(R)$. Let P denote the path $v_1v_2v_3$, and let R, C_1, C_2 be the three cycles of $R \cup P$. By (3) either one of C_1, C_2 is T and the other has length seven, or C_1, C_2 both have length five. In either case it follows from (4) that C_1, C_2 both bound faces of G, and hence v_2 has degree two, contrary to (1). This proves (9).

(10) *The cycle T is disjoint from R.*

To prove (10) suppose for a contradiction that $v \in V(T) \cap V(R)$. By (8) and (9) v is the only vertex of $T \cap R$. The graph $T \cup R$ has a face bounded by a walk C of length nine. By (6) at least one of the vertices of $V(T) \setminus V(R)$ has degree two, contrary to (1). This proves (10).

Let us fix an orientation of the plane, and let $T = t_1t_2t_3$ and $R = r_1r_2 \dots r_6$ be numbered in clockwise cyclic order according to the drawing of G.

(11) *G has at most one edge joining T to R.*

To prove (11) suppose that say $t_1r_1, t_2r_i \in E(G)$ for some $i \in \{1, \dots, 6\}$. By (3) we have $3 \le i \le 5$. Let $C_2 = r_1t_1t_3t_2r_ir_{i+1} \dots r_6$. As t_3 has degree at least three, C_2 does not bound a face; thus C_2 has length at least eight by (4), and we conclude that $i = 3$. Thus C_2 has length exactly eight, and hence by (6) $\mathrm{ins}(C_2)$ consists of C_2 and at most one chord. Since t_3 has degree at least three, this chord exists and joins t_3 with r_5, and hence G has a subgraph isomorphic to the graph depicted in Figure 1(b), contrary to (8). This proves (11).

(12) *G does not contain a 5-face incident only with internal vertices of degree three.*

To prove (12) suppose for a contradiction that G contains such a 5-face $C = v_1v_2v_3v_4v_5$. For $1 \le i \le 5$, let x_i be the neighbor of v_i not belonging to C (each v_i has such a neighbor, because T and C bound faces by and

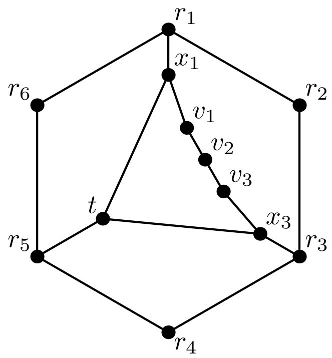


Figure 2: A configuration obtained in the proof of (12).

each vertex of C has degree three). Since T is disjoint from R by (10) and G contains no 4-cycles by (3), it follows that at most three of the vertices $x_1, \ldots, x_5$ belong to R. Without loss of generality we may assume that x_1 is internal. Note also that $x_1 \notin \{x_3, x_4\}$, as G does not contain a 4-cycle. By the symmetry between x_3 and x_4 we may assume that if x_3 is adjacent to a vertex of R, then so is x_4. Let G' be the graph obtained from $G - V(C)$ by adding the edge x_1x_3. Observe that every 3-coloring of G' extends to a 3-coloring of G: given a 3-coloring of G', every vertex in C has a list of two available colors, and the lists of v_1 and v_3 are different.

Our next objective is to show that G' satisfies $(*)$. To that end let $K' \neq T$ be a cycle in G' of length at most four. Then K' includes the edge x_1x_3 by (3). Consider the cycle K in G obtained from K' by replacing the edge x_1x_3 by the path $P = x_1v_1v_2v_3x_3$. Note that K has length at most seven, and that it does not bound a face (since one face of K includes the vertex x_2 and the other face includes v_5). Thus by (4) T is a subgraph of $\mathrm{ins}(K)$. We conclude that G' satisfies $(*)$, as desired.

Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R that extends to a 3-coloring of G'' extends to a 3-coloring of G'. Then $G'' \neq R$, for otherwise every 3-coloring of R extends to a 3-coloring of G'', and hence to a 3-coloring of G', and therefore to one of G, contrary to the R-criticality of G. We conclude that G'' is R-critical. The minimality of G implies that G'' is isomorphic to one of the graphs depicted in Figure 1. But R is an induced subgraph of G by (8) and x_1 is internal, and hence G'' is isomorphic to the graph of Figure 1(b). Let L' be the triangle of G''. By (11) we have $L' \neq T$, and hence x_1x_3 is an edge of L'. Let t be the third vertex of L'. We may assume that x_1 is adjacent to r_1, x_3 is adjacent to r_3 and t is adjacent to r_5, where the adjacencies take place in G, G' and G''. Let D' be the face boundary of the 5-face of G'' incident with the edge x_1x_3, and let D be the 8-cycle of G obtained from D' by replacing the edge x_1x_3 by the path P (see Figure 2). Let L be the 6-cycle in G obtained from L' by replacing the edge x_1x_3 by the path P. By (4) T lies in the closed disk bounded by L, and since t is adjacent to r_5 it follows that $\mathrm{ins}(D)$ includes no cycle of length at most four. By (6) no vertex of G lies in the open disk bounded by D, and hence v_4 and v_5 lie in the open disk bounded by L. Since G has no 4-cycles we deduce that $x_4 \notin \{x_1, t\}$, and $x_3 \neq x_4$, for otherwise $T = x_3v_4v_3$ and the cycle $x_3tx_1v_1v_5v_4$ includes the edge v_5x_5 in its inside but not T, contrary to (4). Since x_3 is adjacent to r_3, the choice of x_3 implies that x_4 is adjacent to a vertex of R, contrary to the planarity of G. This proves (12).

(13) *The distance between R and T is at least two.*

To prove (13) suppose for a contradiction that the distance between R and T is at most one. Then it is exactly one by (10), and so we may assume that say $t_1r_1 \in E(G)$. Let C be the closed walk in G of length 11 obtained by traversing $R \cup T$ and the edge t_1r_1 twice, and let H be the subgraph of G consisting of all vertices and edges of G drawn in the closure of open disk bounded by C. By (6) the graph H satisfies (a), (b) or (c) of Theorem 10. If it satisfies (a), then by (1) applied to t_2 and t_3 the graph H consists of $R \cup T$ and two edges, one incident with t_2 and the other with t_3. It follows that the graph of Figure 1(b) is isomorphic to a subgraph of G, contrary to (8). If H satisfies (b), then $H - V(C)$ is a tree X with at most three vertices, each of degree three. Both t_2 and t_3 have a neighbor in X, and hence $X \cup \{t_2, t_3\}$ includes the vertex-set of a 5-cycle, contrary to (12). Finally, H cannot satisfy (c), because the cycle referenced in (c) would contradict (12). This proves (13).

(14) *No two vertices of degree two are adjacent in G.*

To prove (14) suppose for a contradiction that G has two adjacent vertices of degree two. By (1) they belong to R, and so we may assume that say r_2 and r_3 have degree two. The edge r_2r_3 is not contained in any 5-cycle, as otherwise R would have a chord or an internal vertex would have two neighbors in R, contrary to (8) and (9). Let G' be the graph obtained from G by contracting the edge r_2r_3, and let R' be the corresponding outer cycle of G'. Then G' has no cycle of length at most 4 distinct from T. Furthermore, every 3-coloring ψ of R can be modified to a 3-coloring ψ' of R' such that $\psi(r_i) = \psi(r_i')$ for $i \in \{1, 4, 5, 6\}$, and ψ extends to G if and only if ψ' extends to a 3-coloring of G'. It follows that G' is R'-critical, contrary to Theorem 11. This proves (14).

(15) *For every path $v_1v_2v_3v_4$ with v_2 and v_3 internal and $v_1, v_4 \in V(R)$ there exists $r \in V(R)$ such that $v_1v_2v_3v_4r$ bounds a 5-face.*

To prove (15) consider a path $P = v_1v_2v_3v_4$ with v_2 and v_3 internal and $v_1, v_4 \in V(R)$, and let C_1 and C_2 be the cycles of $R \cup P$ other than R such that T lies in the closed disk bounded by C_1. Since T is disjoint from R, C_1 does not bound a face, and hence it has length at least six by (4). Thus C_2 has length at most six, and hence bounds a face by (4), and therefore has length at most five by (14). This proves (15).

(16) *All faces of G distinct from R and T have length exactly five.*

To prove (16) consider a face $v_1v_2 \dots v_k$ of length $k \geq 6$ in G. By (9) we may assume without loss of generality that v_2 and v_3 are internal. Furthermore, if $k = 6$, then not all of v_1, v_4, v_5 and v_6 may belong to R, by (14), and hence, by symmetry, we may assume that either v_4 or v_6 is internal. Let $W = \{v_2, v_k\}$ if $k > 6$ and $W = \{v_2, v_4, v_6\}$ if $k = 6$. Let G' be the graph obtained from G by identifying the vertices of W to a new vertex w and deleting all resulting parallel edges. Thus $E(G') \subseteq E(G)$. By (1) and (4) the vertices of W are pairwise non-adjacent; thus the identifications created no loops. Observe that every 3-coloring ψ of G' gives rise to a 3-coloring of G (color the vertices of W using $\psi(w)$). It follows that some 3-coloring of R does not extend to a 3-coloring of G'. Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R that extends to a 3-coloring of G'' also extends to a 3-coloring of G'; then G'' is R-critical.

Next we show that G'' satisfies $(*)$. As a first step we prove that G'' does not have a triangle other than T. To that end let $K' \neq T$ be a triangle in G''. Recall that $E(G'') \subseteq E(G)$. Two of the edges of K' are incident in G with distinct vertices $w_1, w_2 \in W$. Let K be the corresponding 5-cycle in G, obtained from K' by replacing w with the two-edge path between w_1 and w_2 with internal vertex in $\{v_1, v_3, v_5\}$. Observe that K does not bound a face in G, contrary to (4). Therefore, G'' does not have a triangle distinct from T. Consider now a 4-cycle L' in G''. The corresponding cycle L in G (constructed in the same way as K) has length six. As L does not bound a face we can apply (5) to the cycle L. By the first result of this paragraph it follows that T is contained in the closed disk bounded by L', and hence G'' satisfies $(*)$.

Since G'' has fewer vertices than G, G'' is one of the graphs drawn in Figure 1. Furthermore, the first result of the previous paragraph implies that T is the unique triangle of G''. However, this implies that the distance between T and R in G is at most one, contradicting (13). This proves (16).

(17) *Every* 5*-face incident with four internal vertices of degree three is incident with an edge of* T.

To prove (17) suppose for a contradiction that G contains a 5-face $f = v_1v_2v_3v_4v_5$, where v_1, v_3, v_4 and v_5 are internal vertices of degree three, and that f does not share an edge with T. By (12) the degree of v_2 is at least four: this follow directly from (12) if v_2 is internal; otherwise v_2 has two neighbors in R and two internal neighbors on f. Let x_1, x_3, x_4 and x_5 be the neighbors of v_1, v_3, v_4 and v_5, respectively, outside of $\{v_1, v_2, \ldots, v_5\}$. If $v_2 \in V(R)$, then x_3 is internal since v_3 has only one neighbor in R by (9), and x_4 and x_5 are internal by (15) and (1). Also, not all of x_1, x_3, x_4 and x_5 belong to R, as T does not share an edge with f. Thus we may assume that at least one of x_3 and x_4 and at least one of v_2 and x_5 is internal. As f does not share an edge with T, the vertices v_2, x_3, x_4 and x_5 are distinct and pairwise non-adjacent by (1) and (4). Let G' be the graph obtained from $G - \{v_1, v_3, v_4, v_5\}$ by identifying v_2 with x_5 to a new vertex w_1 and x_3 with x_4 to a new vertex w_2. Note that any coloring ψ of G' extends to a coloring of G: Give v_2 and x_5 the color $c_1 = \psi(w_1)$ and x_3 and x_4 the color $c_2 = \psi(w_2)$. If $c_1 = c_2$, then color the vertices of $V(F) \setminus \{v_2\}$ in the order v_1, v_5, v_4 and v_3. Similarly, if $c_1 = \psi(x_1)$, then color the vertices v_3, v_4, v_5 and v_1 in order. Finally, if $\psi(x_1) \neq c_1 \neq c_2$, then color v_1 by c_2, v_3 by $\psi(x_1)$, v_4 by c_1 and choose a color for v_5 distinct from c_1 and c_2. It follows that some 3-coloring of R does not extend to a 3-coloring of G'. Let G'' be a minimal subgraph of G' such that R is a subgraph of G'' and every 3-coloring of R

that extends to a 3-coloring of G'' also extends to a 3-coloring of G'. Then G'' is R-critical.

Next we show that G'' satisfies $(*)$. Consider a cycle K' of G'' of length at most four distinct from T, and let $K \subseteq G$ be the corresponding cycle obtained by replacing w_1 by $v_2v_1v_5x_5$ or w_2 by $x_4v_4v_3x_3$ or both. If we replaced both, then K has length at most 10 and it has two chords v_2v_3 and v_4v_5. Thus one of them must belong to T, contradicting the assumption that f does not share an edge with T. Therefore, we expanded only one vertex in K', and hence $6 \leq |V(K)| \leq 7$. By (16), K does not bound a face. By (4) T is a subgraph of $\mathrm{ins}(K)$, and hence G'' satisfies $(*)$, as claimed.

Since G'' has fewer vertices than G, we conclude that G'' is isomorphic to one of the graphs from Figure 1. Let K' be the unique triangle of G''. Using (13), we conclude that $K' \neq T$. Let K be the corresponding cycle of length six in G. By (16) the cycle K does not bound a face. By (5) $\mathrm{ins}(K)$ is isomorphic to one of the graphs drawn in Figure 1. By (13) we conclude that G'' cannot be the graph in Figure 1(a), and hence G'' is isomorphic to the graph in Figure 1(b). We may therefore assume that t_1, t_2, t_3 are adjacent in G'' to r_1, r_5, r_3, respectively, where t_1, t_2, t_3 are the vertices of K'. From the symmetry we may assume that the edge t_1t_3 of K' corresponds to the path t_1abct_3 in K. Now either $\{t_1, a, b, c\}$ or $\{a, b, c, t_3\}$ is equal to $\{v_2, v_1, v_5, x_5\}$ or $\{x_3, v_3, v_4, x_4\}$, and so from the symmetry we may assume that the ordered quadruple (t_1, a, b, c) is equal to one of (v_2, v_1, v_5, x_5), (x_5, v_5, v_1, v_2), (x_3, v_3, v_4, x_4), and (x_4, v_4, v_3, x_3) (see Figure 3).

We claim that t_1, r_1 and t_3, r_3 are adjacent in G. Indeed, they are adjacent in G'', and so it remains to rule out the case that the edge t_1r_1 or t_3r_3 was created during the identifications that produced G'' from G. This is done by examining the four cases listed above. Let first $(t_1, a, b, c) = (v_2, v_1, v_5, x_5)$. Since $\mathrm{ins}(K)$ is isomorphic to one of the graphs in Figure 1 and T shares no edge with f, we deduce that the vertices v_3, v_4 do not belong to $\mathrm{ins}(K)$. Let $i \in \{1, 3\}$, and assume for a contradiction that r_i is not adjacent to t_i in G. But they are adjacent in G'', and so either one of x_3, x_4 is equal to t_i and the other is adjacent to r_i, or one of x_3, x_4 is equal to r_i and the other is adjacent to t_i. Since G is planar, it follows that if $i = 1$, then x_3 is adjacent or equal to t_i, and if $i = 3$, then the vertex x_4 is adjacent or equal to t_i. Thus in the former case the set $\{v_2, v_3, x_3\}$ induces a triangle distinct from T (because T is a subgraph of $\mathrm{ins}(K)$). In the latter case we deduce that x_4 is adjacent to t_3, for otherwise the set $\{t_3, x_4, v_4, v_5, x_5\}$ induces a cycle of length at most four, again a contradiction. Since x_4 is adjacent to

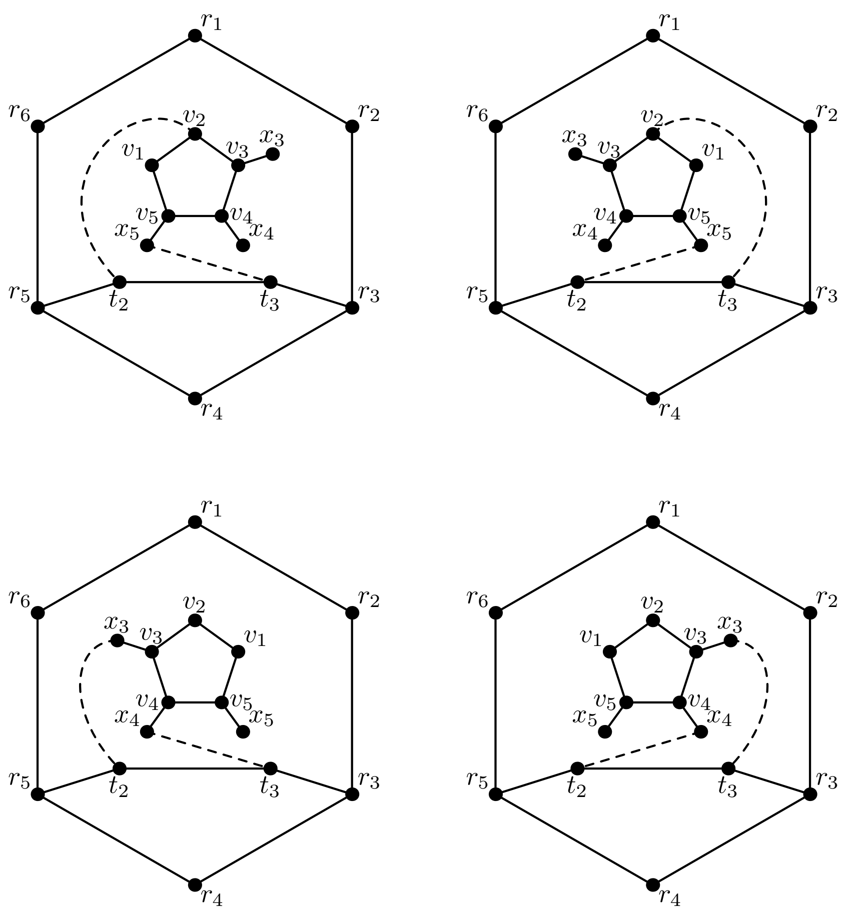


Figure 3: Possible configurations in the proof of (17).

t_3, we have $x_3 = r_3$. From (6) applied to the cycle $t_3t_2r_5r_4r_3v_3v_4v_5x_5$ we deduce that $t_3x_5v_5v_4x_4$ is a cycle of internal degree three vertices, contrary to (12). This completes the first of the four cases. The second case is handled similarly. In the last two cases we deduce, using the same argument as above, that v_1, v_2, v_5 do not belong to $\mathrm{ins}(K)$. Thus $\mathrm{ins}(K)$ has two adjacent vertices v_3 and v_4 of degree two, and hence is isomorphic to the graph in Figure 1(a). It follows that $t_2 \in V(T)$, contrary to (13). We have thus shown that t_1, r_1 and t_3, r_3 are adjacent in G.

Let D' denote the cycle $r_1t_1abct_3r_3r_2$ of length eight. Let us recall that T is a subgraph of $\mathrm{ins}(K)$. By (6) it follows that $\mathrm{ins}(D')\backslash E(D')$ includes at most one edge. However, f does not share an edge with T, thus (5) applied to K implies that $v_1, v_2, \ldots, v_5$ all belong to $\mathrm{ins}(D')$, a contradiction. This proves (17).

(18) *The cycle R has no subpath $z_1z_2z_3$ with* $\deg(z_2) = 3$ *and* $\deg(z_1) = \deg(z_3) = 2$.

To prove (18) suppose for a contradiction that say r_2 and r_4 have degree two and r_3 has degree three. By (14) the vertices r_1 and r_5 have degree at least three. By (16) the face incident with r_2 distinct from the outer face is bounded by a 5-cycle, say $r_1r_2r_3yx$. Similarly, there is a face bounded by a 5-cycle $r_3r_4r_5zy$, where $x \neq z$ by (1). Let K be the 6-cycle $r_1xyzr_5r_6$. By (1) and (5) the graph $\mathrm{ins}(K)$ is isomorphic to one of the graphs in Figure 1, contrary to (13). This proves (18).

(19) *If R has at least two vertices of degree two, then it has at least one vertex of degree at least four.*

To prove (19) suppose for a contradiction that R has at least two vertices of degree two and the remaining vertices of degree at most three. By (14) and (18) G has exactly two vertices of degree two, and the distance in R between them is three. We may therefore assume that r_1 and r_4 have degree two, and r_2, r_3, r_5, r_6 have degree three. By (16), G has a 6-cycle $C = x_1x_2x_3x_4x_5x_6$ such that $x_1r_2, x_3r_3, x_4r_5, x_6r_6 \in E(G)$. By (5) the graph $\mathrm{ins}(C)$ is isomorphic to one of the graphs in Figure 1. It follows that either x_2 or x_5 has degree two, contrary to (1). This proves (19).

We are now ready to complete the proof of Theorem 7 using the so-called discharging argument. Let us assign charges to the vertices and faces of G

in the following way: Each face f of length $|f|$ not bounded by R or T gets a charge of $1 = |f|-4$, the face bounded by T gets charge $2 = (|V(T)|-4)+3$, and the face bounded by R gets charge $0 = (|V(R)| - 4) - 2$. A vertex $v \in V(R)$ of degree two gets charge $-1/3 = (\deg(v) - 4) + 5/3$, a vertex $v \in V(R)$ of degree three gets charge $0 = (\deg(v) - 4) + 1$, and all other vertices v get charge $\deg(v) - 4$.

(20) *The total sum of the charges is at most* $-1/3$.

To prove (20) we deduce from Euler's formula the sum of the charges is at most $\sum_{f\in F(G)}(|f| - 4) + \sum_{v\in V(G)}(\deg(v) - 4) + n_3 + 5n_2/3 + 1 = n_3 + 5n_2/3 - 7$, where n_2 is the number of vertices of degree two and n_3 is the number of vertices of R of degree three in G. By (14) $n_2 \leq 3$. By (18), if $n_2 = 3$ then $n_3 = 0$. By (19), if $n_2 = 2$, then $n_3 \leq 3$. It follows that $n_3 + 5n_2/3 \leq 20/3$, and hence the sum of the charges is at most $-1/3$, as desired. This proves (20).

Let us now redistribute the charge according to the following rules: every face distinct from R sends $1/3$ to each incident vertex of degree two and each incident internal vertex of degree three. The face T sends $1/3$ to each face that shares an edge with it. The final charge of each vertex and of the faces R and T is clearly non-negative. Since the sum of the final charges is equal to the sum of the initial charges, it follows from (20) that G has a face f of strictly negative final charge. The face f has length five; let v_1, v_2, v_3, v_4, v_5 be the incident vertices in order.

If say v_2 were a vertex of degree two, then by (14), v_1 and v_3 would be vertices of R of degree at least three, and hence f would send no charge to them, contrary to the fact that the final charge of f is strictly negative. It follows that all vertices of f have degree at least three, and since the final charge of f is negative, f sends charge to at least four of them. Therefore, at least four of the vertices incident with f are internal and have degree three. The fifth vertex has degree at least four by (12): this is clear if it is internal, and otherwise it has two neighbors on R and two neighbors on f. By (17) f shares an edge with T. However, f sends $1/3$ to each of its incident vertices of degree three and nothing to the fifth vertex, and receives $1/3$ from T; hence the final charge of f is non-negative, a contradiction. $\square$

We are now ready to prove Theorem 7.

Proof of Theorem 7. Let G, T and ϕ be as in Theorem 7. We may assume that R bounds the outer face. Let G' be a minimal subgraph of G such that R is a subgraph of G and ϕ does not extend to a 3-coloring of G'. It follows that G' is R-critical. If T is not a subgraph of G', then we let f_0 be any face of G'; otherwise we let f_0 denote any face of G' that is contained in the closed disk bounded by T. Then G' satisfies hypothesis $(*)$ of Theorem 12. By Theorem 12 the graph G' is isomorphic to one of the graphs depicted in Figure 1. If neither of the two outcomes of Theorem 7 holds, then ϕ extends to a 3-coloring of G', a contradiction. $\square$

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