Packing six T-joins in plane graphs

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Abstract

Let G be a plane graph and T an even subset of its vertices. It has been conjectured that if all T-cuts of G have the same parity and the size of every T-cut is at least k, then G contains k edge-disjoint T-joins. The case $k = 3$ is equivalent to the Four Color Theorem, and the cases $k = 4$, which was conjectured by Seymour, and $k = 5$ were proved by Guenin. We settle the next open case $k = 6$.

1 Introduction

We study packings of T-joins in plane graphs. Let G be a graph and T an even-size subset T of its vertices. A *T-join* is a subgraph H of G such that the odd-degree vertices of H are precisely those in T. A *cut* is a partition of the vertex set of a graph G into two sets A and B, which we refer to as *sides*; the *size* of the cut is the number of edges with one end-vertex in A and the other end-vertex in B. A cut is *trivial* if one its sides consists of a

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single vertex and a cut is *odd* if the size of A is odd. Finally, a *T-cut* is a cut such that $|T \cap A|$ is odd.

Clearly, if G has a T-cut of size k, it cannot have more than k edge-disjoint T-joins. We are interested when the converse is also true. Seymour [7] (also see Problem 12.18 in [3]) conjectured the following for $k = 4$.

Conjecture 1. *Let G be a plane graph and T an even-size subset of its vertices. If the sizes of all T-cuts in G have the same parity and the size of every T-cut is at least k, then G contains k edge-disjoint T-joins.*

The case $k = 3$ is equivalent to the Four Color Theorem. The cases $k = 4$ and $k = 5$ were proved by Guenin [4]. We remark that the case $k = 4$ implies the Four Color Theorem, as pointed out by Seymour [7]. Here, we prove the next open case. Our main result is the following.

Theorem 1. *Let G be a plane multigraph and T an even subset of its vertices. If every T-cut of G has the same parity and the size of every T-cut is at least six, then G contains six edge-disjoint T-joins.*

Guenin [4] argued that it suffices to prove Conjecture 1 for plane graphs G with $V(G) = T$ that are k-regular, i.e., every vertex has degree k. In such case, the existence of k edge-disjoint T-joins is equivalent to the existence of a k-edge-coloring; a *k-edge-coloring* is an assignment of colors to the edges such that no vertex is incident with two edges of the same color. Hence, Theorem 1 for $k = 6$ is equivalent to the next theorem which we prove in the following sections of the paper.

Theorem 2. *Let G be a 6-regular plane multigraph. If G has no odd cut of size less than 6, then G has a 6-edge-coloring.*

Note that the condition that G has no odd cut of size less than six implies that the number of vertices of G is even (otherwise, consider a cut with one of the sides empty). Let us remark that Conjecture 1 would be implied by the following more general conjecture of Seymour (replacing the condition of not containing Petersen by a stronger condition of being planar yields a statement equivalent to Conjecture 1).

Conjecture 2. *Let G be a k-regular graph with no Petersen minor. The graph G is k-edge-colorable if and only if every odd cut of G has size at least k.*

The case $k = 3$ is the well-known Tutte's three edge-coloring conjecture, whose solution is announced by Robertson, Sanders, Seymour and Thomas (see [5]). Indeed, the case $k = 3$ is a special case of another well known conjecture by Tutte, which is known as the Tutte's four flow conjecture.

Conjecture 2 would also imply the following conjecture of Conforti and Johnson [2], also see [1].

Conjecture 3. *Let G be a graph with no Petersen minor and T a set of its odd-degree vertices. Then, the maximum number of edge-disjoint T-joins is equal to the size of the smallest T-cut.*

Conjectures 1, 2 and 3 are paid attention by many researchers, because they are connected not only to T-joins, T-cuts and edge-coloring but also to cycle covers and flows. For more details, we refer the reader to the books by Cornuéjols [1] and by Schrijber [6], respectively.

2 Notation

We now introduce notation used throughout the paper. A vertex of degree d is called a *d-vertex*. An $\geq d$*-vertex* is a vertex of degree at least d and an $\leq d$*-vertex* is a vertex of degree at most d. In a 2-connected plane graph, a d-face is a face incident with exactly d edges. Analogously to vertices, we use an $\leq d$-face and an $\geq d$-face.

Two faces in a plane graph are *adjacent* if they share a common edge. We say that a face is *k-big* if it is adjacent to $k \geq 4$-faces. A *bigon* is a 2-face and a series of consecutively adjacent bigons is called *multigon*. The *order* of a multigon is the number of edges forming it, i.e., the number of bigons forming it increased by one. Multigons of order three are called *trigons* and those of order four *quadragons*. Two multigons are *incident* if they share a vertex. If f is a face, then two multigons are *f-incident* if they contain edges consecutive on the boundary of f. We extend this notion to a multigon and a face and to two faces in the natural way.

A 5-face is *dangerous* if either it is adjacent to two trigons and at least one bigon or it is adjacent to a trigon and at least three bigons. A 7-face is *dangerous* if it is adjacent to three trigons and three bigons. Finally, a multigon t is *dangerous* if

- t is a quadragon,
- t is a trigon adjacent to a dangerous 5-face f such that t is f-incident with a multigon, or

- t is a trigon adjacent to a dangerous 7-face f such that t is f-incident with two multigons.

The rest of the paper is devoted to proving Theorem 2. With respect to this proof, a plane graph G is said to be a *minimal counterexample* if G satisfies the assumptions of Theorem 2, i.e.,

- G is 6-regular,
- every odd cut of G has size at least six, and
- G has no 6-edge-coloring, and

it also holds that

- subject to the previous conditions, G has the smallest order,
- subject to the previous conditions, G has as many quadragons as possible,
- subject to the previous conditions, G has as many trigons as possible, and
- subject to the previous conditions, G has as many bigons as possible.

By Lemma 2.2 from [4], in order to prove Theorem 2, it is enough to exclude the existence of a minimal counterexample.

In our arguments, we will often need to transform an edge-coloring to another one. To simplify our arguments, we will use the letters α, β, γ, δ, ε and φ to denote the colors used on edges. If G is a graph with maximum degree d that is d-edge-colored, then an *$\alpha\beta$-chain*, where α and β are two colors used on the edges of G, is a cycle or a maximal path formed by edges with the colors α and β only. *Swapping* the colors the of edges on an $\alpha\beta$-chain means recoloring α-colored edges of the chain with β and β-colored edges with α.

3 Structure of a minimal counterexample

In this section, we analyze structure of a minimal counterexample; in the next section, we then prove that there exists no minimal counterexample using the discharging method.

3.1 Odd cuts

We start with analyzing sizes and structure of odd cuts in a minimal counterexample. As the first step, we prove the following simple observation.

Lemma 3. *Every non-trivial odd cut in a minimal counterexample G has size at least eight.*

Proof. Since G is 6-regular, every cut in G has even size. Hence, if G has a non-trivial odd cut of size less than eight, its size must be six. Let A and B be the sides of such a non-trivial odd cut.

Let G_A be the (plane) graph obtained from G by replacing A with a single vertex incident with the six edges of the cut (A, B). Similarly, G_B is the graph obtained from G by replacing B with a single vertex incident with the six edges of the cut (A, B). By the minimality of G, both G_A and G_B have 6-edge-colorings. These edge-colorings combine to a 6-edge-coloring of G (the edges of the cut receive six distinct colors) which contradicts that G is a minimal counterexample. □

Using Lemma 3, we prove the following simple observation on the structure of multigons in a minimal counterexample.

Lemma 4. *In a minimal counterexample G, the order of every multigon is a most four and the sum of the orders of any two incident multigons is at most five.*

Proof. If G contains a multigon of order five or two incident multigons with orders summing to six, then there is a vertex v that has only two neighbors, say v' and v''. Unless G has exactly four vertices, the set $\{v, v', v''\}$ forms a side of a non-trivial odd cut of size six which is impossible by Lemma 3. Hence, G has exactly four vertices and it is straightforward to show that it can be 6-edge-colored. □

One of possible ways to to obtain an edge-coloring of minimal counterexample, which is assumed not to exist, is reducing a minimal counterexample to another 6-regular graph of the same order but with multigons of larger order. Such a reduction can only be possible if the resulting graph has no odd cuts of size less than six. One operation that will observe this property is the swapping operation that we now introduce.

If G is a plane graph such that it is possible to draw a closed curve in the plane that intersects G only at vertices $v_1, \ldots, v_k$ for k even and G contains edges v_2v_3, v_4v_5, …, $v_{k-2,k-1}$ and v_kv_1 (such a k-tuple of vertices $v_1, \ldots, v_k$ is called *eligible*), then the graph obtained from G by the $v_1 \ldots v_k$-*swap* is the plane graph obtained by removing the edges v_2v_3, v_4v_5, …, $v_{k-2,k-1}$ and v_kv_1 and inserting the edges $v_{2i-1}v_{2i}$ for $i = 1, \ldots, k/2$.

A crucial property of this operation is that the size of odd cuts can decrease by at most two if k is four or six. We prove this in the next two lemmas.

Lemma 5. *Let G be a minimal counterexample and let k be either four or six. Any graph G' obtained from G by a $v_1v_2v_3v_k$-swap for eligible vertices $v_1, \ldots, v_k$ is 6-regular and has no odd cut of size less than six.*

Proof. Clearly, the graph G' is 6-regular. Consider an odd cut with sides A and B of G'. Observe that this cut is also odd in G. By symmetry, we can assume that $|A \cap \{v_1, \ldots, v_k\}| \leq 3$. Let $A' \subseteq A$ be those vertices v_i of A such that v_{i-1} or v_{i+1} is in A. Since $|A \cap \{v_1, \ldots, v_k\}| \leq 3$, the set A' is either empty or formed by two or three consecutive vertices. If A' is empty or formed by three consecutive vertices, the number of edges leaving A' to B is the same in G and G'. If A is formed by two consecutive vertices, then the number of such edges leaving A' to B is either increased or decreased by two. Since the number of edges between the vertices of $A \setminus A'$ and the vertices of B is preserved, every odd cut of G' has size at least six. □

3.2 Existence of e-colorings

A crucial property of a minimal counterexample is the existence of an e-coloring. Let us define this notion formally. If G is a 6-regular graph and e an edge of G, then an *e-coloring* of G is a coloring of edges of G with six colors such that every edge except for e is assigned one color, e is assigned three or more colors, and for every color, each vertex is incident with an odd number of edges assigned that color. Observe that in an e-coloring, every vertex except the end-vertices of e must be incident with edges of mutually different colors, i.e., an e-coloring is proper at all vertices except for the end vertices of e. Also observe that the number of colors assigned to e in an e-coloring must always be odd.

The following lemma appears as Lemma 2.5 in [4].

Lemma 6. *Let G be a minimal counterexample. For every edge e, there exists an e-coloring.*

We now strengthen Lemma 6 for the case when e is contained in a multigon of order at least three.

Lemma 7. *Let G be a minimal counterexample and $e = vv'$ an edge of G contained in a multigon of order three or more. There exists an e-coloring such that e is assigned precisely three colors, say α, β and γ, and one of these colors, say α, is assigned to other two edges incident with v as well as other two edges incident with v'.*

Proof. By Lemma 6, there exists an e-coloring of G. Recall that e must be assigned an odd number of colors. Assume that e is assigned exactly three colors, say α, β and γ. If an edge contained in the multigon with e is assigned one of these three colors, say γ, then we can obtain a 6-edge-coloring of G by assigning to this edge the color α and assigning the edge e the color β only. Hence, either e is contained in a trigon and the other two edges of the trigon are assigned the colors δ and ε (by symmetry) or e is contained in a quadragon and the other three edges are assigned the colors δ, ε and φ.

By permuting the colors of the edges of the multigon and exchanging the roles of v and v', we can assume that one of the following cases apply:

- **The vertex v is incident with two edges colored with α in addition to e and the vertex v' is also incident with two edges colored with α in addition to e.** In this case, the edge-coloring is of the type described in the statement of the lemma.

- **The vertex v is incident with two edges colored with α in addition to e and the vertex v' is also incident with two edges colored with β in addition to e.** Let e_1 be one of the edges incident with v' assigned the color β. Let e_2 be the end edge of the $\alpha\beta$-chain of $G \setminus e$ starting at e_1. The edge e_2 is either an edge colored with α incident with v or the other edge colored with β incident with v'. If e_2 is incident with v, we can obtain a proper 6-edge-coloring of G by swapping the colors on the chain and assigning e the color γ only. Hence, e_2 is incident with v'. In this case, we swap the colors on the chain and obtain an e-coloring of the type described in the statement of the lemma.

- **The vertex v is incident with two edges colored with α in addition to e and the vertex v' is incident with three edges**

colored with φ. Observe that the order of the multigon must be three in this case.

Consider the $\alpha\varphi$-chains of $G \setminus e$ starting at edges incident with v'. Since v' is incident with three edges colored with φ and no edge colored with α in $G \setminus e$, at least one of these chains ends at an edge incident with v. Let e_v be this edge.

If e_v is colored with α, swap the colors α and φ on the $\alpha\varphi$-chain and color e with the colors β, γ and φ. An e-coloring of the desired type can now be obtained by exchanging the colors α and φ in the whole graph.

Hence, we can assume that there is no $\alpha\varphi$-chain starting an edge incident with v' and ending at an edge incident with v with the color α. Consequently, there exists an $\alpha\varphi$-chain starting and ending at an edge incident with v'. By swapping the colors α and φ on this chain, we obtain an e-coloring of the type described in the statement of the lemma.

- **Each of the vertices v and v' is incident with three edges colored with φ.** Observe that the order of the multigon must again be three.

  Since v is incident with three edges colored with φ and no edge colored with α in $G \setminus e$, there exists a $\alpha\varphi$-chain in $G \setminus e$ that starts at an edge incident with v and ends at an edge incident with v'. Swap now the colors α and φ on this chain and color e with the colors β, γ and φ. The desired edge coloring of G can now be obtained by exchanging the colors α and φ in the whole graph.

Assume now that e is assigned five colors, say α, β, γ, δ and ε. If one of these colors, say ε, is assigned to another of edge of the multigon, then assign e the colors α, β and γ only and assign δ to the edge colored ε. On the other hand, if other two edges in the multigon are assigned the color φ, then assign to e the colors α, β and γ only and recolor the other two edges with δ and ε. In either of the two cases, we have found an e-coloring of G assigning e exactly three colors. Since we have already analyzed such e-colorings, the proof is now completed. □

Hereafter, we will sometimes refer to the assumption of Lemma 7. In particular, the color α will be the color assigned to other two edges incident with v as well as other two edges incident with v'.

The existence of e-colorings is related to the notion of mates which also appeared in [4]. We use a slightly different but equivalent terminology to that in [4]. If G is a minimal counterexample, e an edge of G and c one of the colors used in an e-coloring, then a *c-mate* M_c is a set of edges of G that form an odd cut containing e such that, for every color $c' \neq c$, M_c contains exactly one edge that is assigned the color c'. A c-mate is *non-trivial* if it forms a non-trivial cut of G. Lemma 2.6 in [4] asserts the existence of mates in a minimal counterexample.

Lemma 8. *Let G be a minimal counterexample and e an edge of G. For every e-coloring and every color c, there exists a c-mate.*

The following observation on the structure of mates is often used in our arguments. We state it as a proposition for future references.

Proposition 9. *Let G be a minimal counterexample and e an edge of G. In each e-coloring, every non-trivial c-mate M_c contains at least five edges (possibly including e) assigned the color c.*

Proof. By Lemma 3, the mate M_c contains at least eight edges. Since the edge e is assigned at least three colors and M_c contains exactly one edge assigned each of the colors $c' \neq c$, M_c must include at least five edges assigned the color c. □

We use the existence of a c-mate M_c in the following way: if M_c contains an edge adjacent to a face f or an edge in a multigon adjacent to f, then f is ≥ 4-face unless f is incident with an end-vertex of e, or f is adjacent to an edge of M_c colored with a color different from c or f is adjacent to a multigon containing edges with colors different from c only.

Let us demonstrate the use of this approach in the following lemma.

Lemma 10. *Every face f adjacent to a quadragon is ≥ 5-big.*

Proof. Let e be the edge of the quadragon incident with f and consider an e-coloring as described in Lemma 7. Let M_c be a c-mate for $c \neq \alpha$ and e_c another edge of f contained in M_c. Let f_c be the other face containing the edge e_c. Since the edges of the quadragon are assigned all six colors and the mate M_c contains at least five edges with the color c by Proposition 9, the face f_c contains another edge with the color c. Hence, f_c is ≥ 4-face. Since the faces f_c differ for different choices of c, we conclude that f is ≥ 5-big. □

Another application of the mates is the following.

Lemma 11. *Every face f adjacent to a trigon is adjacent to at least one edge not contained in a multigon.*

Proof. Let e be an edge of the trigon and consider an e-coloring as described in Lemma 7. Let M_φ be a φ-mate. Since the edges of the trigon have all the five colors different from φ, all the other edges contained in M_φ have the color φ. Since the only vertices incident with two edges of the same color are the end vertices of e, no edge of M_φ except for those contained in the trigon is contained in a multigon. In particular, the edge of f contained in M_φ and not in the trigon is not contained in a multigon. □

In the rest of this section, we will focus in more detail on multigons adjacent to faces of various sizes.

3.3 Structure of 3-faces

In this subsection, we focus on 3-faces. We start with 3-faces adjacent to a trigon.

Lemma 12. *If a 3-face f in a minimal counterexample is adjacent to a trigon, then f is adjacent to no other multigon, both the faces adjacent to f are ≥ 5-big and the other face adjacent to the trigon is also ≥ 5-big. In particular, it is ≥ 6-face.*

Proof. Let $e = v_1v_2$ be an edge of the trigon adjacent to f and consider an e-coloring as in Lemma 7. Since the edges incident with v_1 and v_2 not contained in the trigon have the colors α and φ only and no vertex except for v_1 and v_2 is incident with two edges of the same color, the face f cannot be adjacent to another multigon and the other two edges adjacent to f have the colors α and φ.

Let v_3 be the remaining vertex of the face f, f' the other face adjacent to the trigon, f'' the face incident with the edge v_1v_3 and v_4 a neighbor of v_3 on the face f'' (see Figure 1). The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a 6-coloring by the minimality of G (there is a quadragon between v_1 and v_2 after the swap) and Lemma 5. By symmetry, we can assume that the colors of the edges of the quadragon between v_1 and v_2 are β, γ, δ and ε, and the color of the edge v_1v_3 is φ. If the edge v_3v_4 has a color different from α, then we can obtain a proper 6-edge-coloring of G by assigning the edges v_2v_3 and v_4v_1 the color of the edge v_3v_4. Hence, the color of v_3v_4 is

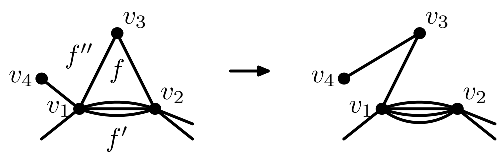

Figure 1: Notation used in the proof of Lemma 12 and the swap.

α. Consider now the following e-coloring: the edge e is colored with α, β, γ, the other edges of the trigon with δ and ε, the edge v_2v_3 is colored with α, the edge v_1v_4 is also colored with α and the remaining edges of G have the same colors as in G'.

Let M_c be a c-mate for $c \neq \alpha$. Observe that M_c contains the three edges of the trigon and the edge v_1v_3 which is colored with φ (it cannot contain the edge v_2v_3 because its color is α). We show that both f' and f'' are $\geq$ 5-big. Since G is 2-connected (it is 6-regular and 6-edge-connected), the faces f' and f'' are different. We now argue that f' is $\geq$ 5-big. Let e_c be the edge incident with f' that is contained in M_c and that is not contained in the trigon. Clearly, the color of e_c is c. Let f_c be the face adjacent to e_c that is distinct from f'. Since the mate M_c contains at least five edges with the color c by Proposition 9, f_c is also distinct from f''. Since f_c contains another edge colored with c by Proposition 9, f_c must be $\geq$ 4-face. Since the faces f_c are different for different choices of $c \neq \alpha$, the face f' is $\geq$ 5-big. The argument that f'' is $\geq$ 5-big follows the same lines.

Switching the roles of v_1 and v_2 yields that the face adjacent to f distinct from f'' is also $\geq$ 5-big. □

We now focus on 3-faces adjacent to bigons.

Lemma 13. *If a* 3*-face of a minimal counterexample is adjacent to at least two bigons, then it is adjacent exactly to two bigons and the other faces adjacent to these bigons are* $\geq$ 5*-big.*

Proof. Let v_1, v_2 and v_3 be the vertices of f in such an order that there is a bigon between v_1 and v_2 and between v_1 and v_3. Let f' be the other face adjacent to the bigon between v_1 and v_2 and f'' the other face adjacent to the bigon between v_1 and v_3. Finally, let v_4 be the neighbor of v_1 on f'' different from v_3. Also see Figure 2.

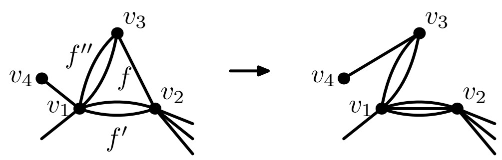

Figure 2: Notation used in the proof of Lemma 13 and the swap.

By the minimality of G and Lemma 5, the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a 6-edge-coloring. Let β, γ and δ be the colors assigned to the edges between v_1 and v_2 and ε and φ the colors of the edges between v_1 and v_3. Since the edge v_3v_4 cannot have the color ε or φ, we can assume that its color is α or β by symmetry. In the latter case, coloring the edges of the bigon between v_1 and v_2 with the colors γ and δ, the edges v_1v_4 and v_2v_3 with β and the other edges with their colors in G' yields a 6-edge-coloring of G. Hence, the color of v_3v_4 is α. Observe that this implies that the edge v_2v_3 is not contained in a multigon.

Consider now the following e-coloring where e is one of the edges of the bigon between v_1 and v_2: the edge e is assigned the colors α, β and γ, the other edge of the bigon the color δ, the edges v_1v_4 and v_2v_3 the color α and the remaining edges preserve their colors. Consider a c-mate M_c for $c \neq \alpha$. This mate must contain all the edges of the bigons between v_1 and v_2 and between v_1 and v_3. Let e_c be the edge of f' contained in M_c. The color of e_c must be c and the other face containing e_c must contain another edge with the color c. Consequently, it is a ≥ 4-face. We conclude (by considering all choices of c) that f' is ≥ 5-big face. A symmetric argument applies to the face adjacent to the bigon between v_1 and v_3. ☐

Before considering 3-faces adjacent to a single bigon, we have to prove the following lemma:

Lemma 14. *In a minimal counterexample, any trigon adjacent to an ≤ 2-big face is also adjacent to an ≥ 4-big face.*

Proof. Let e be an edge of the trigon, f the ≤ 2-big face adjacent to the trigon and f' the other face adjacent to the trigon. Consider an e-coloring described in Lemma 7 and let M_c be a c-mate for $c \neq \alpha$.

The mate M_φ contains at least five edges colored with φ by Proposition 9 and no edges of other colors except for those contained in the trigon. Hence,

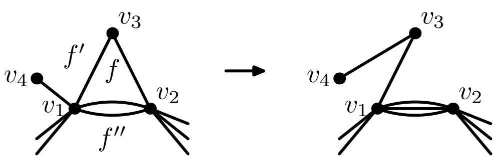

Figure 3: Notation used in the proof of Lemma 15 and the swap.

one of the (at most) two $\geq$ 4-faces adjacent to f shares with f an edge colored φ. By symmetry with respect to the colors β, γ, δ and ε, we can assume that the other $\geq$ 4-face adjacent to f (if it exists) shares with f an edge with color different from β, γ and δ.

Consider now a mate M_c, $c \in \{\beta, \gamma, \delta, \varphi\}$. On the face f, the mate M_c either contains an edge colored with φ or (if $c \neq \varphi$) an edge colored with the color c that lies on a $\leq$ 3-face (which forces M_c to contain an edge incident with this face that is colored with φ). In both cases, since the mate M_c contains at least five edges colored with c by Proposition 9, one of these edges (which are colored with c) must lie on the face f' and the face containing this edge different from f' is $\geq$ 4-face. Since these faces are different for different values of c in $\{\beta, \gamma, \delta, \varphi\}$, the face f' is $\geq$ 4-big. $\square$

We are now ready to consider 3-faces adjacent to a single bigon.

Lemma 15. *If a 3-face f in a minimal counterexample is adjacent to a single bigon and the the other face adjacent to this bigon $\leq$ 2-big, then the other two faces adjacent to f are $\geq$ 3-big.*

Proof. Let v_1, v_2 and v_3 be the vertices of f in such an order that the bigon is between v_1 and v_2. Let f' be the face incident with the edge v_1v_3, f'' the other face adjacent to the bigon and let v_4 be the neighbor of v_1 on f' different from v_3. Also see Figure 3.

If G contains a trigon between the vertices v_1 and v_4, the claim follows from Lemma 14. Otherwise, the minimality of G and Lemma 5 implies that the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a 6-edge-coloring. By symmetry, we can assume that the colors assigned to the edges between v_1 and v_2 are β, γ and δ, and ε is the color of the edge v_1v_3. If the color of the edge v_3v_4 is one of the colors β, γ and δ, we can obtain a 6-coloring of G as in the proof of Lemma 12. Hence, we can assume that the color of v_3v_4 is α.

Let e be one of the edges of the bigon and consider the following e-coloring: the edge e is assigned the colors α, β and γ, the other edge of the bigon the colors δ, the edges v_1v_4 and v_2v_3 are assigned the color α and the remaining edges preserve their colors. Let M_c be a c-mate for $c \neq \alpha$. Observe that each mate M_c, $c \neq \alpha$, contains the two edges of the bigon as well as the edge v_1v_3.

By Proposition 9, the mate M_φ contains at least five edges colored with φ. The construction of the e-coloring yields that all these edges must lie on ≥ 4-faces (since their end-vertices are distinct). Hence, one of the (at most) two ≥ 4-faces adjacent to f'' shares an edge with color φ with f''.

The argument that we now present is symmetric with respect to the colors β, γ, δ and ε. Hence, we can assume that the other ≥ 4-face adjacent to f'' (if it exists) shares with f'' an edge with color different from ε. On the face f'', each of the mates M_c, $c \in \{\beta, \gamma, \delta\}$, either contains an edge colored with φ or an edge colored with the color c that lies in a ≤ 3-face. In the latter case, the other edge of that face contained in M_c must have the color φ. Hence, M_c contains an edge φ in both cases. Since the mate M_c contains at least five edges colored with c by Proposition 9, it contains at least three additional edges colored with c. One of these edges must lie on the face f' and the face containing this edge different from f' must be a ≥ 4-face. Since these faces are different for different values of $c \in \{\beta, \gamma, \delta\}$, the face f' is ≥ 3-big. □

We finish this subsection with an observation on faces around 3-faces adjacent to trigons.

Lemma 16. *A minimal counterexample G does not contain a vertex v_1 incident with mutually adjacent 3-face f_3 and dangerous a 5-face f_5, a bigon contained in f_3 but not in f_5, and a trigon t contained in f_5 such that t is f_5-incident to a bigon.*

Proof. Let v_2 be the other vertex contained in t, let v_3, v_4 and v_5 be the other vertices of f_5 (in this order) and let v_6 be the remaining vertex of f_3. Also see Figure 4.

Assume first that f_5 is adjacent to a trigon between the vertices v_3 and v_4. Let G' be the graph obtained from G by the $v_1v_2v_3v_4v_5v_6$-swap. By the minimality of G (none of the affected multigons has order four in G by Lemma 10) and Lemma 5, the graph G' has a 6-edge-coloring.

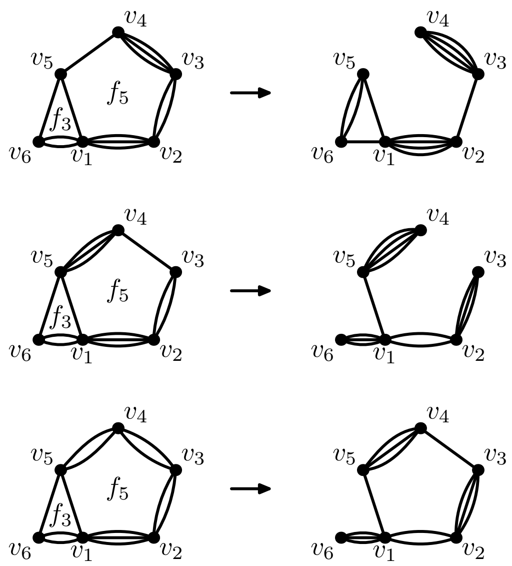

Figure 4: Notation used in the proof of Lemma 16 and possible swaps.

Let C be the set of colors assigned to the four edges between v_1 and v_2 in G'. Both the colors of the edges between v_5 and v_6 are in C: the two colors not contained in C are assigned to the edges v_1v_6 and v_1v_5 and thus the two edges between v_5 and v_6 cannot have either of these two colors.

At least three of the colors from C are used on the edges of the quadragon between v_3 and v_4 in G'. Hence, there is a color c assigned to an edge between v_1 and v_2, an edge between v_3 and v_4, and an edge between v_5 and v_6. Remove the three edges colored with c between these pairs of vertices and insert the edges of G missing in G'. Coloring the new edges with c yields a 6-edge-coloring of G.

Assume next that G contains a trigon between v_4 and v_5. Let G'' be the graph obtained by the $v_2v_3v_4v_5v_6v_1$-swap and consider its 6-edge-coloring which exists by the minimality of G and Lemma 5. Let C_{16} be the colors of the edges in the trigon between v_1 and v_6 in G'', C_{23} the colors of the edges in the trigon between v_2 and v_3, and C_{45} the colors of the edges in the quadragon between v_4 and v_5. Since G'' contains a bigon between v_1 and v_2, it holds that $|C_{16} \cap C_{23}| \geq 2$. Similarly, because the vertices v_1 and v_5 are joined by an edge, it holds that $|C_{16} \cap C_{45}| \geq 2$. Consequently, there exists a color $c \in C_{16} \cap C_{23} \cap C_{45}$. Removing edges of this color from the quadragon and the two trigons and coloring the edges of G missing in G'' with c yields a 6-edge-coloring of G.

The final case to consider is that f_5 is adjacent to three bigons but the only trigon adjacent to f_5 is the one formed by the edges between v_1 and v_2. Let G'' be again the graph obtained by the $v_2v_3v_4v_5v_6v_1$-swap and consider its 6-edge-coloring which exists by the minimality of G and Lemma 5. Let C_{16} be the colors of the edges in the trigon between v_1 and v_6 in G'', C_{23} the colors of the edges in the trigon between v_2 and v_3, and C_{45} the colors of the edges in the quadragon between v_4 and v_5. Since G'' contains a bigon between v_1 and v_2, it holds that $|C_{16} \cap C_{23}| \geq 2$. If C_{45} does not contain a color from $C_{16} \cap C_{23}$, then the edges v_1v_5 and v_3v_4 have the same color that is the unique color not contained in $C_{45} \cup (C_{16} \cap C_{23})$. This however implies that the sets C_{16} and C_{23} must be the same: they do not contain the colors of the two edges between v_1 and v_2 and the common color of the edges v_1v_5 and v_3v_4. Since the sets C_{23} and C_{45} have a common color, we conclude that there exists a color c contained in all the sets C_{16}, C_{23} and C_{45}. Removing the edges colored with c from the three trigons and coloring the edges of G missing in G'' with c yields a 6-edge-coloring of G. $\square$

3.4 Structure of 4-faces

In this subsection, we just prove two simple lemmas on 4-faces.

Lemma 17. *If a trigon in a minimal counterexample G is adjacent to a 4-face f, then f is adjacent to no other multigon.*

Proof. Let $v_1, \dots, v_4$ be the vertices of f in such an order that the trigon adjacent to f is between v_1 and v_2. Apply the $v_1v_2v_3v_4$-swap. Since the resulting graph G' contains a new quadragon (and f is not adjacent to a quadragon in G by Lemma 10), G' has a 6-edge-coloring by the minimality of G and Lemma 5. Let β, γ, δ and ε be the colors of the edges of the quadragon in G'. If one of these colors appears on the edges between v_3 and v_4, we can use this color for the edges v_1v_4 and v_2v_3 and obtain a proper coloring of G. Hence, G' contains a bigon between the vertices v_3 and v_4, the colors of its two edges are α and φ, and G' contains neither an edge v_1v_4 nor v_2v_3. In particular, in G, f is adjacent to a single multigon which is the considered trigon. $\square$

Lemma 18. *No 4-face in a minimal counterexample is adjacent to three or four bigons.*

Proof. Let $v_1, \dots, v_4$ be the vertices of f in such an order that the bigons are (at least) between the pairs v_1 and v_2, v_2 and v_3, and v_3 and v_4. By the minimality of G and Lemma 5, the graph G' obtained by the $v_1v_2v_3v_4$-swap has a 6-edge-coloring. Since G' contains an edge v_2v_3, the trigons between the pairs v_1 and v_2, and v_3 and v_4 must have two edges with the same color. Remove edges of this color from the trigons and insert them as edges between v_2 and v_3, and v_1 and v_4. This yields a 6-edge-coloring of G. $\square$

3.5 Structure of ≥ 5-faces

We start this subsection with a lemma on 5-faces.

Lemma 19. *In a minimal counterexample, if a 5-face f is adjacent to 5 multigons, then all these multigons are bigons and each such bigon b is adjacent to a ≥ 4-big face.*

Proof. By Lemmas 10 and 11, f is adjacent to five bigons. Let e be an edge of the bigon b and f' the other face adjacent to it. Consider an e-coloring and c-mates M_c for different colors c. If e is assigned five colors, then each mate M_c includes all six colors on the edges of b. Since a mate M_c cannot include additional edges with a color different from c, such a mate cannot include any other bigons which is impossible. Hence, we can assume that e is assigned three colors, say α, β and γ. By symmetry, we can assign that the other edge of b is colored with δ.

The mate M_ε includes in addition to the bigon b another bigon adjacent to f. The two edges of this bigon must be colored with the colors ε and φ (they cannot have the same color since they are incident with a vertex that is not an end-vertex of e). Let b' be this bigon. Similarly, the mate M_φ must include a bigon with one edge colored ε and the other φ.

We claim that the bigons b and b' are f-incident. If they were not, there would a bigon b'' f-incident with both b and b'. Let v be the vertex shared by b'' and b. Since v is incident with three edges assigned one of the colors and the remaining five colors are assigned to a single edge each, one of the edges b' has the color ε or the color φ. This is however not possible, since the edges of b'' have the colors ε and φ.

Observe that the face f is adjacent to at most two bigons containing an edge colored ε, one of these bigons being b'. Let b_ε be such a bigon different from b' if it exists. Similarly, let b_φ be the bigon different from b' containing an edge colored φ (if it exists). Let c_ε and c_φ be the colors of the other edges contained in b_ε and b_φ (if the bigons do not exist, choose c_ε and c_φ arbitrarily).

Let c be a color different from c_ε and c_φ. The mate M_c includes the two edges assigned colors α, β, γ and δ which appear on the bigon b. Since it contains exactly one edge of each color different from c, M_c must include a bigon with edges colored ε and φ (which can be b' or a different bigon of this type). Consequently, M_c includes an edge colored with c on a face f' and this edge must be contained in a ≥ 4-face since M_c includes more edges colored with c (see Proposition 9). Since there are four choices of c different from c_ε and c_φ, the face f' is ≥ 4-big. □

In the next two lemmas, we apply the six-vertex swap operation to analyze structure of 6-faces.

Lemma 20. *In a minimal counterexample G, no 6-face f is adjacent to one or two bigons and three trigons.*

Proof. By Lemma 4, the trigons adjacent to f cannot be f-incident. Hence, we can assume that the trigons adjacent to f are between the pairs of the vertices v_1 and v_2, v_3 and v_4, and v_5 and v_6. By symmetry, there is a bigon between the vertices v_2 and v_3. Consider the graph G' obtained from G

by the $v_1v_2v_3v_4v_5v_6$-swap. By the minimality of G and Lemma 5, G' has a 6-edge-coloring. Let C_1, C_2 and C_3 be the sets of colors of the three quadragons.

Since the vertices v_2 and v_3 are joined by an edge in G', we have $|C_1 \cap C_2| \geq 3$. Since $|C_3| = 4$, there exists a color $c \in C_1 \cap C_2 \cap C_3$. Removing the edges of the three quadragons colored with c and inserting the edges v_2v_3, v_4v_5 and v_6v_1 colored with c yields a 6-edge-coloring of G. This contradicts the fact that G is a minimal counterexample. □

Lemma 21. *In a minimal counterexample G, no 6-face f is adjacent to three bigons and two trigons.*

Proof. Let $v_1 \cdots v_6$ be the vertices of the face f. By Lemma 11, f is incident with an edge not contained in a multigon; let v_1v_6 be such an edge. Consider the graph G' obtained from G by the $v_1v_2v_3v_4v_5v_6$-swap. Let C_i, $i = 1, 2, 3$, be the set of colors assigned to the edges of the multigon b_i between v_{2i-1} and v_{2i}. Since the order of each multigon between v_{2i-1} and v_{2i}, $i = 1, 2, 3$, is at least three, the size of each C_i is at least three.

Assume first that b_2 is a trigon, i.e., $|C_2| = 3$. If b_1 is a quadragon, then $|C_1 \cap C_2| \geq 2$ since the multigons b_1 and b_2 are joined by an edge. If b_1 is a trigon, then it also holds $|C_1 \cap C_2| \geq 2$ since the multigons b_1 and b_2 are joined by two edges (there must be a trigon between v_2 and v_3 in G). A symmetric argument yields that $|C_2 \cap C_3| \geq 2$. Consequently, there exists a color $c \in C_1 \cap C_2 \cap C_3$. Removing the edges with the color c from the multigons b_1, b_2 and b_3, and coloring the edges of G not present in G' with c yields a 6-edge-coloring of G.

We now consider the case that b_2 is a quadragon, i.e., $|C_2| = 4$. Since no trigons in G can share a vertex, we can assume by symmetry that that b_1 is a quadragon and b_3 is a trigon, i.e., $|C_1| = 4$ and $|C_3| = 3$. Since the quadragons b_1 and b_2 are joined by an edge in G', it follows $|C_1 \cap C_2| \geq 3$. Similarly, b_2 and b_3 are joined by an edge and thus $|C_2 \cap C_3| \geq 2$. Since the size of C_2 is four, it follows that there exists a color $c \in C_1 \cap C_2 \cap C_3$. We can now obtain a 6-edge-coloring of G as in the case that b_2 is a trigon. □

We now prove a lemma on bigons and trigons adjacent to 7-faces.

Lemma 22. *In a minimal counterexample, every trigon t adjacent to a dangerous 7-face f is adjacent to ≥ 5-big face (which is different from f).*

Proof. Let e be an edge of the trigon and consider an e-coloring as described in Lemma 7. Let M_c be a c-mate. The mate M_φ cannot contain any edge with color $c \neq \varphi$ except for those contained in the trigon. Hence, M_φ must include the only edge of f not contained in a multigon and this edge must be colored with φ. Consequently, every mate M_c, $c \neq \varphi$, includes either this single edge, which has the color φ, or a multigon which has one edge of color c and the other of color φ. We conclude that every mate M_c contains the edge with the color φ either incident to f or contained in a multigon adjacent to f.

Let f' be the face adjacent to t different from f. The mate M_c, $c \neq \alpha$, can only contain an edge colored with c on f' and the face f_c containing this edge must contain another edge colored with c by Proposition 9. Since no vertex is incident with two edges with the color c, the face f_c is ≥ 4-face. We conclude that f' is ≥ 5-big. □

In the next lemma, we consider 8-faces.

Lemma 23. *In a minimal counterexample, no* 8*-face* f *is adjacent to three bigons and four trigons.*

Proof. Let $v_1, \ldots, v_8$ be the vertices of f in the cyclic order around f. By Lemma 4 and the symmetry, we can assume that the trigons are between the vertices v_i and v_{i+1} for $i = 1, 3, 5, 7$ and the bigons are between the vertices v_i and v_{i+1} for $i = 2, 4, 6$. Hence, there is either a single edge or a bigon between the vertices v_1 and v_8.

Consider now the graph G' obtained from G by the $v_1 \ldots v_8$-swap. We claim that G' has no odd cut of size less than six. Consider an odd cut with sides A and B that has the smallest size in G'. By symmetry, we can assume that $|A \cap \{v_1, \ldots, v_8\}| \leq 4$. Let $A' \subseteq A$ be those vertices v_i of A such that v_{i-1} or v_{i+1} is contained in A. Observe that $|A'| \leq 4$.

If A' is empty or A' is formed by a single sequence of an odd number of vertices, then the size of the cut with sides A and B is the same in G and G'. If A' is formed by a single sequence of an even number of vertices, then the size is either increased or decreased by two, and thus the size of the cut is at least six by Lemma 3. Otherwise, A' is formed by at least two sequences. Since $|A'| \leq 4$, A' is formed by two sequences of consecutive vertices, each of length two.

Suppose that the size of the cut with the sides A and B is less than six in G'. Since its size in G is at least eight and the size of the cut cannot decrease by more than two unless $A' = \{v_1, v_2, v_5, v_6\}$ or $A' = \{v_3, v_4, v_7, v_8\}$, A' must be equal to one of these two sets. By symmetry, $A = \{v_1, v_2, v_5, v_6\}$ and thus $B \cap \{v_1, \ldots, v_8\} = \{v_3, v_4, v_7, v_8\}$. Note that the roles of A and B are now completely symmetric. Since the sizes of the cut with the sides A and B in G and G' can differ by at most four, it follows that its size in G is eight and in G' is four.

Since G' is a plane graph, then v_1 and v_5 are not joined by a path formed by vertices of A only or v_3 and v_7 are not joined by a path formed by vertices of B only. By symmetry, we can assume the former to be the case: let A_1 be the component of the side A containing the vertices v_1 and v_3, and set $A_2 = A \setminus A_1$. Since the size of the cut with sides A and B in G' is four, the size of the cut with sides A_1 and $B \cup A_2$ is at most two or the size of the cut with sides A_2 and $B \cup A_1$ is at most two. This contradicts the choice of A and B.

We have shown that G' has no odd cuts of size less than six which implies that G' has a 6-edge-coloring by the minimality of G. Let C_i be the set of the four colors assigned to the edges between v_{2i-1} and v_{2i}, $i = 1, 2, 3, 4$, in G'. Since the vertices v_{2i} and v_{2i+1} for $i = 1, 2, 3$ are joined by an edge, it holds that $|C_i \cap C_{i+1}| \geq 3$ for $i = 1, 2, 3$. Hence, $|C_1 \cap C_2 \cap C_3| \geq 2$. This combines with $|C_3 \cap C_4| \geq 3$ to the fact that $|C_1 \cap C_2 \cap C_3 \cap C_4| \geq 1$. Let c be the color contained in all the sets C_i, $i = 1, 2, 3, 4$. We can now obtain a 6-edge-coloring of G by removing the edges in the quadragons of G' colored with c and coloring the edges of G not contained in G' with c. $\square$

We finish this section with two lemmas on dangerous trigons which we will apply in the proof of Lemma 32.

Lemma 24. *Let G be a minimal counterexample and f a ≥ 5-big face of G. If f is adjacent to a dangerous multigon t, then f is adjacent to at least four ≥ 4-faces that are not f-incident with t.*

Proof. Consider an e-coloring for an edge e contained in t of the type described in Lemma 7 and let M_c be a c-mate.

If t is a quadragon, then M_c contains only edges colored with c except for the edges contained in t. Hence, each M_c, $c \neq \alpha$, contains an edge on f colored with c contained in ≥ 4-face. Since the edges f-incident with t are colored with α, we conclude that f is adjacent to at least five ≥ 4-faces that are not f-incident with t.

In the rest, we assume that t is a trigon. Let f' be the dangerous face adjacent to t. If f' is a 7-face, then t is f-incident with two edges not contained in a bigon. The colors of these edges are α. The mate M_φ must contain an edge of f' not contained in a multigon and thus the only such edge incident with f' has the color φ. Every mate M_c, $c \neq \alpha$, either contains the edge of f' not contained in a multigon, which is colored with φ, or a multigon adjacent to f'. In either of the two cases, it includes an edge colored with φ. Consequently, it must include an edge of f that is colored with c (and thus not f-incident with t) and that is contained in $\geq$ 4-face. We conclude that f is adjacent to at least five $\geq$ 4-faces not f-incident with t.

It remains to consider the case that f' is a 5-face. If t is f'-incident with two multigons, the same argument as in the case of a 7-face applies. Hence, we can assume that t is f'-incident with a single multigon and let e' be the edge incident with f' that is f'-incident to t and not contained in a multigon. If the color of e' is α, then f' must contain an edge with the color φ not contained in a multigon. Each mate M_c, $c \in \{\beta, \gamma, \delta, \varepsilon\}$, either contains the edge colored with φ or a multigon adjacent to f (which has one of its edges colored with φ). Hence, M_c must contain an edge of f colored with c that is contained in a $\geq$ 4-face. Such a $\geq$ 4-face cannot be f-incident with t and the claim follows.

Assume now that the color of e' is φ. Hence, the two edges f-incident with t have the color α. Let c_0 be the color of the other edge of f' not contained in a multigon (if it exists). Each mate M_c, $c \notin \{\alpha, c_0\}$, contains an edge colored with φ contained in a multigon adjacent to f' and thus it contains an edge colored with c incident with f that is contained in a $\geq$ 4-face different from f. Again, f is adjacent to at least $\geq$ 4-faces that are not f-incident with t. □

Lemma 25. *Let G be a minimal counterexample and f a ≥ 5-big face of G. If f is a adjacent to a dangerous multigon t that is f-incident with a bigon, then f is adjacent to at least five ≥ 4-faces that are not f-incident with t.*

Proof. Lemma 4 and the fact that G is 6-regular implies that the face f' adjacent to t that is different from f is a dangerous 5-face. Let e be an edge of t and consider an e-coloring as described in Lemma 7. Let e' be the edge of f' that is f'-incident with t and not contained in the bigon f'-incident with t. Observe that the color of e' must be α (the bigon f-incident with t must have edges with the colors α and φ). Let M_c be a c-mate for $c \neq \alpha$. Since the bigon f-incident with t contains an edge colored with α and the edge f-incident with t (which is on the other side of t than the bigon) is colored with α, the mate M_c, $c \neq \alpha$ contains neither this bigon nor this edge.

Let us focus on the mate M_φ. Since the trigon t contains edges of all the colors different from φ, the mate M_φ cannot include any multigon adjacent to f'. Consequently, f' is adjacent to an edge e'' different from e' that is not contained in a multigon and the color of this edge is φ. Since f' is dangerous, we conclude that f' is adjacent to the trigon t, another trigon, a bigon and the edges e' and e''.

The mate M_φ contains the trigon t, the edge e'' and some edges colored with φ. Since no two edges colored with φ share the same vertex, we conclude that f is incident with an edge colored with φ that is not f-incident with t and that is contained in a ≥ 4-face different from f.

The mate M_c, $c \in \{\beta, \gamma, \delta, \varepsilon\}$, contains the trigon t and the edge e'' (note that the edges of the bigon f'-incident with t are α and φ). Hence, M_c must contain an edge of f colored with c that is not f-incident with t and that is contained in a ≥ 4-face different from f. It follows that f is adjacent to at least five ≥ 4-faces that are not f-incident with t. $\square$

4 Discharging phase

4.1 Discharging rules

We consider a minimal counterexample and assign every d-face, $d \geq 3$, $d-3$ units of charge, every bigon -1 unit of charge, every trigon -2 units of charge and every quadragon -3 units of charge. Note that the minimal counterexample can contain multigons of order at most four (see Lemma 4). Vertices are assigned no charge. Since the minimal counterexample is 6-regular, the Euler formula implies that the total sum of charge assigned to faces is negative.

Next, charge gets redistributed among ≥ 3-faces and multigons using the following rules (also see Figure 5). We attempt to name the rules mnemotechnically: the names start with R, followed by a character B, T, M and 3 to denote the type of faces it involves (bigons, trigons, multigons and 3-faces) and sometimes by another character to distinguish the rules further (e.g., "b" stands for big, "d" for dangerous, "t" for a trigon, etc.).

Rule RMb Every dangerous multigon t adjacent to a ≥ 4-big face f receives 1.5 units of charge from f.

Rule RMd Every dangerous multigon t adjacent to a dangerous face f receives 0.5 unit of charge from f.

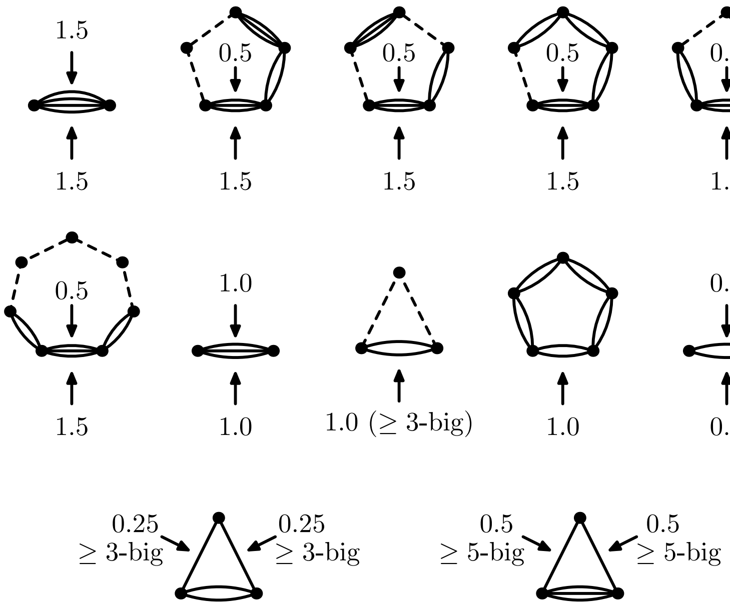

Figure 5: Illustration of discharging rules (dashed edges denote edges that can be single or contained in multigons).

Rule RT Every trigon such that neither Rule RMb nor Rule RMd applies to it receives 1 unit of charge from each adjacent face.

Rule RB3 A bigon adjacent to a 3-face and a $\geq$ 3-big face f receives 1 unit of charge from f.

Rule RB5 A bigon adjacent to a $\geq$ 4-big face f and to a 5-face adjacent to 5-multigons receives 1 unit of charge from f.

Rule RB A bigon such that neither Rule RB3 nor Rule RB5 applies to it receives 0.5 unit of charge from each adjacent face.

Rule R3 A 3-face f that is adjacent to a bigon and two $\geq$ 3-big faces receives 0.25 unit of charge from each adjacent $\geq$ 3-big face.

Rule R3t A 3-face f that is adjacent to a trigon and a $\geq$ 5-big face f' receives 0.5 unit of charge from f'.

Charge of $\geq$ 3-faces and multigons after these rules are applied is referred to as *final* charge. In the remainder of this section, we show that final charge of every face and every multigon is non-negative.

4.2 Final charge of multigons

We first analyze final charge of multigons.

Lemma 26. *The final amount of charge of every multigon in a minimal counterexample G is non-negative.*

Proof. Recall that the initial amount of charge of a multigon of order $k+1$ is $-k$. By Lemma 4, G can contain only bigons, trigons and quadragons. Every bigon receives either 1 unit of charge by Rule RB3 or RB5 from an adjacent face or 0.5 unit of charge by Rule RB from each adjacent face. Hence, every bigon receives at least 1 unit of charge in total. Let us consider trigons. If a trigon is not dangerous, then Rule RT applies twice. If a trigon is dangerous, then one of the faces adjacent to it is $\geq$ 4-big by Lemma 14 (note that a dangerous face must be $\leq$ 2-big) and Rules RMb and RMd apply (each once). Hence, every trigon also receives two units of charge in total. Finally, every quadragon receives 1.5 units of charge by Rule RMb from each adjacent face which is $\geq$ 4-big by Lemma 10. $\square$

4.3 Final charge of 3-faces and 4-faces

In this subsection, we analyze final charge of 3-faces and 4-faces. Let us start with 3-faces.

Lemma 27. *The final amount of charge of every* 3*-face* f *in a minimal counterexample* G *is non-negative.*

Proof. Faces of a minimal counterexample send charge to adjacent multigons and 3-faces only. Hence, if f sends out any charge, it is $\leq$ 2-big. In particular, f can send out some charge by Rules RT and RB only. Consequently, if f is adjacent to no multigon, f sends out no charge and its final charge is non-negative.

If f is not adjacent to multigons of order three or more, then it is adjacent to at most two bigons by Lemma 13. If it is adjacent to two bigons, each of these bigons is adjacent to a $\geq$ 5-big face by Lemma 13, and thus Rule RB3 applies to it. Consequently, Rule RB does not apply to f. If f is adjacent to a single bigon and the other face adjacent to the bigon is $\geq$ 3-big, again, Rule RB3 applies and f sends out no charge. If the other face adjacent to the bigon is $\leq$ 2-big, then f is adjacent to two $\geq$ 3-faces by Lemma 15. In this case, f sends a half of unit of charge to the bigon by Rule RB and receives twice a quarter of unit of charge by Rule R3.

If f is adjacent to a trigon, then Lemma 12 yields that f is adjacent to no other multigon and the other two faces adjacent to f are $\geq$ 5-big. Hence, f sends to the trigon 1 unit of charge by Rule RT and receives 0.5 from each of the adjacent $\geq$ 5-big face by Rule R3t. The final charge is again non-negative.

Since f cannot be adjacent to a quadragon by Lemma 10, the proof is completed. $\square$

Let us analyze final charge of 4-faces.

Lemma 28. *The final amount of charge of every* 4*-face* f *in a minimal counterexample* G *is non-negative.*

Proof. If f is $\geq$ 3-big, only Rules RT, RB3, RB or R3 can apply and at most one of them applies. Hence, f sends out at most one unit of charge and its final charge is non-negative.

In the rest of the proof, we assume that f is $\leq$ 2-big which implies that only Rules RT and RB can apply. If f is adjacent to a trigon, then f is adjacent to no other multigon by Lemma 17. Hence, Rule RT applies once and no other rule can apply to f. Consequently, f sends out one unit of charge and its final charge is zero.

If f is adjacent to no multigons of order three or more, then, by Lemma 18, f is adjacent to at most two bigons. Hence, Rule RB can apply at most twice to f and thus the final amount of charge of f is non-negative. □

4.4 Final charge of ≥ 5-faces

In this subsection, we analyze the amount of final charge of ≥ 5-faces. The case of 5-faces needs to be treated separately. So, we start with them.

Lemma 29. *The final amount of charge of every* 5*-face* f *in a minimal counterexample* G *is non-negative.*

Proof. If f is $\geq$ 4-big, then at most one of the rule applies to f. Consequently, f sends out at most 1.5 units of charge and its final charge is positive.

If f is 3-big, then either at most two rules apply to f each once or the same rule applies to f twice. Since f cannot send out charge by Rules RMb, f sends out at most two units of charge in total and its final charge is non-negative.

If f is $\leq$ 2-big, then only Rules RMd, RT and RB can apply to f. Since no two trigons are incident by Lemma 4, f is adjacent to at most two trigons. If f is adjacent to two trigons and no other multigons, Rule RT applies twice and the final charge of f is zero.

If f is adjacent to two trigons and one bigon, then at least one of the trigons is f-incident with the bigon. Observe that f is dangerous in this case. Hence, Rule RMd applies once (with respect to the trigon f-incident with the bigon), Rule RMd or RT applies once and Rule RB applies. We conclude that the final amount of charge of f is non-negative.

If f is adjacent to two trigons and two bigons, then both trigons are f-adjacent to a bigon. Again, f is dangerous. Hence, Rule RMd applies twice and Rule RB also applies twice. Consequently, final charge of f is zero.

Finally, f cannot be adjacent to two trigons and three bigons by Lemma 19.

If f is adjacent to a single trigon and at most two bigons, it sends out at most two units of charge (Rule RT applies once, Rule RB at most twice). If f is adjacent to a single trigon and three bigons, then Rule RMd applies once (note that f is dangerous) and Rule RB three times. Again, f sends out at most two units of charge. Finally, f cannot be adjacent to a single trigon and four bigons by Lemma 19. In all these case, f sends out at most two units of charge and its final amount of charge is non-negative.

If f is adjacent to no trigon and at most four bigons, f sends out at most two units of charge in total by Rule RB. On the other hand, if f is adjacent to five bigons, Lemma 19 yields that Rule RB5 will apply to each of the adjacent bigons and f sends out no charge. Again, the final amount of charge of f is non-negative. □

Before proving the final lemma (Lemma 32) of this section which deals with ≥ 6-faces, we have to state two auxiliary lemmas that will be useful in analyzing final charge of 4-big and 5-big faces. The next two lemmas use the same notation which we will later use in the proof of Lemma 32.

Lemma 30. *Let G be a minimal counterexample and f an ℓ-face, $\ell \geq 6$. Let $v_1, \ldots, v_\ell$ be the vertices incident with the face f in the cyclic order around f, and let $f_1, \ldots, f_\ell$ be the face or the multigon adjacent to f through the edge $v_i v_{i+1}$ (indices taken modulo ℓ). Finally, let s_i be the amount of charge sent by f to f_i, $i = 1, \ldots, \ell$. It holds that*

$$s_i + s_{i+1} \leq 2 \tag{1}$$

for every $i = 1, \ldots, \ell$ (indices again taken modulo ℓ).

Proof. Let w be the vertex shared by f_i and f_{i+1}. Assume that $s_i + s_{i+1} > 2$. Since the charge sent out by f following the discharging rules can only be equal to 0, 0.25, 0.5, 1 and 1.5, it follows that s_i or s_{i+1} is equal to 1.5. By symmetry, assume that $s_i = 1.5$. Consequently, Rule RMb applies to f_i and f_i is either a trigon or a quadragon. By Lemma 4, f_{i+1} cannot be a trigon or a quadragon. Consequently, $s_{i+1} = 1$ and one of Rules RB3 or RB5 must apply. Since G is 6-regular, Rule RB5 cannot apply. Consequently, f_{i+1} is a bigon adjacent to a 3-face and f_i is a trigon.

The trigon f_i cannot be adjacent to a dangerous 5-face f' and f'-incident with a bigon by Lemma 16. On the other, the trigon f_i cannot be adjacent to a dangerous 7-face f' and f'-incident with two bigons since G is 6-regular. Hence, Rule RMb cannot apply. The inequality (1) now follows. □

Lemma 31. *Let G be a minimal counterexample and f a ≥ 3-big ℓ-face, $\ell \geq 6$. Let $v_1, \ldots, v_\ell$ be the vertices incident with the face f in the cyclic order around f, and let $f_1, \ldots, f_\ell$ be the face or the multigon adjacent to*

f through the edge v_iv_{i+1} (indices taken modulo ℓ). Finally, let s_i be the amount of charge sent by f to f_i, $i = 1, \ldots, \ell$. It holds that

$$s_i + s_{i+1} + s_{i+2} \leq 3.5 \tag{2}$$

for every $i = 1, \ldots, \ell$ (indices again taken modulo ℓ) and the equality can hold if and only if $s_i = s_{i+2} = 1.5$, $s_{i+1} = 0.5$ and f_{i+1} is a bigon.

Proof. The values of s_i, s_{i+1} and s_{i+2} are among 0, 0.25, 0.5, 1 and 1.5. Hence, $s_i + s_{i+1} + s_{i+2} \leq 3 + s_{i+1}$. On the other hand, Lemma 30 yields that $s_i + s_{i+1} + s_{i+2} \leq 4 - s_{i+1}$. We conclude that (2) holds and the equality is attained only if $s_i = s_{i+2} = 1.5$ and $s_{i+1} = 0.5$.

Assume that $s_i = s_{i+2} = 1.5$ and $s_{i+1} = 0.5$. Observe that f_i and f_{i+2} are multigons of order three or four. Let w and w' be the vertices shared by f_i and f_{i+1} and f_{i+1} and f_{i+2}, respectively. If f_{i+1} is not a bigon and $s_{i+1} = 0.5$, then Rule R3t must apply. Hence, f_{i+1} is a 3-face adjacent to a trigon. Since both f_i and f_{i+2} are multigons of order three or more, this is impossible by Lemma 4. The lemma now follows. $\square$

We now analyze the amount of final charge of ≥ 6-faces.

Lemma 32. *In a minimal counterexample, the amount of final charge of every ≥ 6-faces f is non-negative.*

Proof. Let ℓ be the size of f. Assume first that f is ≤ 2-big. Hence, only Rules RMd, RT and RB can apply to f. If f is adjacent to no trigon, then f sends out at most $\ell/2$ units of charge. Since the initial amount of charge of f is $\ell - 3 \geq \ell/2$ (recall $\ell \geq 6$), the final charge of f is non-negative. If f is adjacent to a trigon, f is adjacent to at most $\ell - 1$ multigons by Lemma 11. Moreover, Lemma 4 implies that no two trigons are incident. Hence, Rule RT applies at most $\lfloor \ell/2 \rfloor$ times and Rules RMd, RT and RB together apply at most $\ell - 1$ times. Consequently, f sends out at most

$$\frac{1}{2}\left(\ell - 1 + \left\lfloor \frac{\ell}{2} \right\rfloor\right) \tag{3}$$

units of charge. The value of (3) is at most $\ell - 3$ unless $\ell \in \{6, 7, 8\}$. By considering the number of bigons and trigons adjacent to f, we derive that f sends out at most the amount of its initial charge unless one of the following cases applies (recall that one of the edges incident to f is not in a multigon and no two trigons can share a vertex):

1. **The face f is a 6-face and f is adjacent to two trigons and three bigons.** This case is excluded by Lemma 21.

2. **The face f is a 6-face and f is adjacent to three trigons and at least one bigon.** This case is excluded by Lemma 20.

3. **The face f is a 7-face and f is adjacent to three trigons and three bigons.** In this case, f is dangerous and at least one of the trigons is f-incident with two bigons. Hence, Rule RB applies three times and either Rule RMd applies twice and Rule RT once or Rule RMd applies once and Rule RT twice. In both cases, the face f sends out at most four units of charge and its final charge is non-negative.

4. **The face f is a 8-face and f is adjacent to four trigons and three bigons.** This case is excluded by Lemma 23.

Adopt now the notation from the statements of Lemmas 30 and 31. If f is 3-big, then Rule RMb never applies. Hence, it holds $s_i \leq 1$ for every $i = 1, \ldots, \ell$. Since f is 3-big, f sends no charge to at least three of $f_1, \ldots, f_\ell$, and thus f sends out at most $\ell - 3$ units of charge.

Let us assume that f is k-big. Further, let $i_1, \ldots, i_k$ be the indices i such that f_i is a ≥ 4-face and set $I_j = \{i_j + 1, \ldots, i_{j+1} - 1\}$ (indices modulo ℓ and k where appropriate). If $i_j + 1 = i_{j+1}$, then $I_j = \emptyset$. We now prove the following claim:

Claim 1. *It holds that*

$$\sum_{i \in I_j} s_i \leq |I_j| + 0.50 \tag{4}$$

for every $j = 1, \ldots, k$. Moreover, if $k = 4$ or $k = 5$, then it holds that

$$\sum_{i \in I_j} s_i \leq |I_j| + 0.25 \tag{5}$$

for every $j = 1, \ldots, k$.

If $|I_j| = 1$, the estimate (4) follows from the fact that no face or multigon receives more than 1.5 units of charge from f. If $|I_j| \geq 2$, the estimate (4) directly follows from Lemmas 30 and 31. Lemmas 30 and 31 also yield that the equality in (4) holds only if $|I_j|$ is odd, $s_{i_j+1} = s_{i_j+3} = \cdots = s_{i_{j+1}-1} = 1.5$ and $s_{i_j+2} = s_{i_j+4} = \cdots = s_{i_{j+1}-2} = 0.5$.

We now establish the inequality (5). If $|I_j| = 1$, then Rule RMb applies to f_{i_j+1} and the face f is adjacent to at least four ≥ 4-faces distinct from

f_{i_j} and $f_{i_{j+1}}$ by Lemma 24. This implies that f is $\geq$ 6-big, i.e., $k \geq 6$ which contradicts the assumptions.

Assume that $|I_j| \geq 3$. Since $s_{i_j+1} + s_{i_j+2} + s_{i_j+3} = 3.5$, f_{i_j+2} is a bigon by Lemma 31. Lemma 25 applied to f_{i_j+1} and the bigon f_{i_j+2} now implies that the face f is adjacent to at least five $\geq$ 4-faces besides f_{i_j} which again implies that $k \geq 6$. The proof of the claim is now finished.

By Claim 1, if f is 4-big or 5-big, then the sum $s_1 + \ldots + s_\ell$ is at most $|I_1| + \ldots + |I_k| + k/4 = \ell - 3k/4$. In particular, the charge sent out by the face f is at most $\ell - 3k/4 \leq \ell - 3$ and the final charge of f is non-negative. If f is $\geq$ 6-big, then Claim 1 yields that the sum $s_1 + \ldots + s_\ell$ is at most $|I_1| + \ldots + |I_k| + k/2 = \ell - k/2$. Again, the charge sent out by f is at most $\ell - 3$ and its final charge is non-negative. $\square$

4.5 Finale

In order to prove Theorem 2 which implies Theorem 1, we have to exclude the existence of a minimal counterexample. Assume that G is a minimal counterexample and assign charge to the multigons and $\geq$ 3-faces of G as described in Subsection 4.1 and apply the Rules as described. By Lemmas 26–32, the final amount of charge of every multigon and every face of G is non-negative. Since charge is preserved during the application of the rules and the sum of the amounts of initial charge is negative, a minimal counterexample cannot exist. This establishes Theorem 2.

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