Khayyam-Pascal Determinantal Arrays, Star of David Rule and Log-Concavity

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Abstract

In this paper we develop a new geometric method to answer the log-concavity questions related to a nice class of combinatorial sequences arising from the Khayyam-Pascal triangle.

1 Introduction

One of the important task in *enumerative combinatorics* is to determine *log-concavity* of a combinatorial sequence.

Definition 1.1. A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be *concave* if $\frac{a_{i-1}+a_{i+1}}{2} \leq a_i$ for all $1 \leq i \leq n-1$, and *logarithmically concave* (or log-concave for short) if $a_{i-1}a_{i+1} \leq a_i^2$ for all $1 \leq i \leq n-1$.

Definition 1.2. The sequence $a_0, a_1, \ldots, a_n$ is called *symmetric* if $a_i = a_{n-i}$ for $0 \leq i \leq n$.

Definition 1.3. We say that a polynomial $a_0+a_1q+\cdots+a_nq^n$ has a certain property (such as log-Concave or symmetric) if its sequence $a_0, a_1, \ldots, a_n$ of coefficients has the property.

There are many ways to prove the log-concavity of a combinatorial sequence. One of the classic method of proof is direct combinatorial approach, which is of significant interest for combinatorial people.

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Example 1.1. *The best-known log-concave sequence is the n-th row of Khayyam-Pascal's triangle:*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

Here, the log-concavity is easy to show directly because of the explicit formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. *Indeed,*

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1,$$

which is equivalent to $n > -1$ (or $n \geq 0$), as required.

Example 1.2. *For the sequence of the n-th diagonal of the Khayyam-Pascal triangle:*

$$\binom{n}{0}, \binom{n+1}{1}, \binom{n+2}{2}, \ldots, \binom{n+k}{k}, \ldots,$$

again, we have

$$\frac{\binom{n+i}{i}^2}{\binom{n+i-1}{i-1}\binom{n+i+1}{i+1}} = \frac{(n+i)(i+1)}{i(n+i+1)} > 1,$$

which is equivalent to $n > 0$.

In spite of the *geometric idea* behind the definition of the log-concavity of a sequence, there is no geometric approach to tackle this issue. In this paper, we develop a new geometric method to answer the log-concavity questions related to a nice class of combinatorial sequences arising from the Khayyam-Pascal triangle.

2 Khayyam-Pascal Array and Parallelepiped Determinantal Identities

Consider a 45° rotation of the Khayyam-Pascal triangle which we call it *Khayyam-Pascal squared array* [1]. Now, we construct a parallelepiped with two triangles as its bases which is shown with six entries of this array and the corresponding edges in Fig 1.

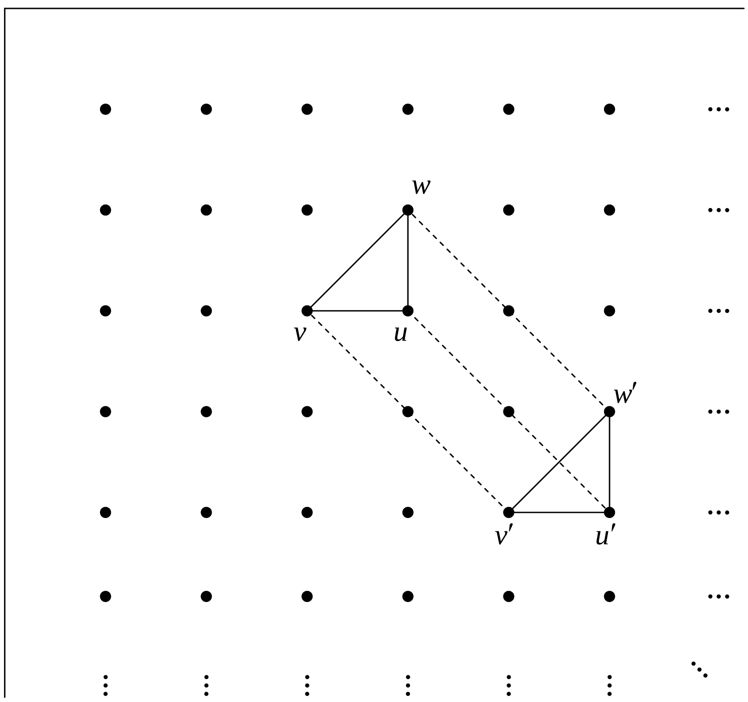

Fig 1. A Determinantal Parallelepiped

Then, we have the following determinantal identities which are the direct consequence of the recurrence relation for the Khayyam-Pascal array. For a generalization to higher dimensions and other possible proofs see the paper [2].

Proposition 2.1. *(Parallelepiped Determinantal Identities)*

$$i) \quad \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} w & v \\ w' & v' \end{vmatrix},$$

$$ii) \quad \begin{vmatrix} w & v \\ w' & v' \end{vmatrix} = \begin{vmatrix} w & u \\ w' & u' \end{vmatrix}.$$

In other words, the determinants formed by three faces of the parallelepiped $uvwu'v'w'$ in Fig 1 are equal.

Proof. By the rule of Khayyam-Pascal array, we have

$$u = v + w$$
$$u' = v' + w'.$$

Now, multiplying the above equalities by v and v', respectively, we get

$$uv' = vv' + wv'$$
$$u'v = vv' + w'v.$$

Subtracting the above equalities, we obtain

$$uv' - u'v = wv' - w'v,$$

or equivalently

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} w & v \\ w' & v' \end{vmatrix},$$

which is the first determinantal identity. The second one can be proved in a similar way and left to the reader as a simple exercise. □

Proposition 2.2. *Every diagonal of the Khayyam-Pascal triangle is log-concave.*

Proof. First of all note that the diagonals of the Khayyam-Pascal triangle correspond to the columns (rows) of the Khayyam-Pascal squared array. Now, we use the previous determinantal identities in their special cases to give a new geometric proof of the log-concavity of the diagonals of the Khayyam-Pascal triangle. To this end, consider three consecutive terms a_{k-1}, a_k, a_{k+1} in any arbitrary column of the Khayyam-Pascal squared array, as shown in Fig 2. We consider a parallelepiped in its special case where two antipodal vertices (u and w' in Fig 1) coincide. Here, those vertices correspond to two equal entries a_k. By Proposition 2.1, we have

$$\begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} = \begin{vmatrix} b_k & a_k \\ b_{k+1} & a_{k+1} \end{vmatrix}.$$

But, we already know that the 2-by-2 determinant in the right-hand side of the above identity is a Narayana number [4]. Therefore, we obtain

$$\begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} \geq 0,$$

and this completes the proof. $\square$

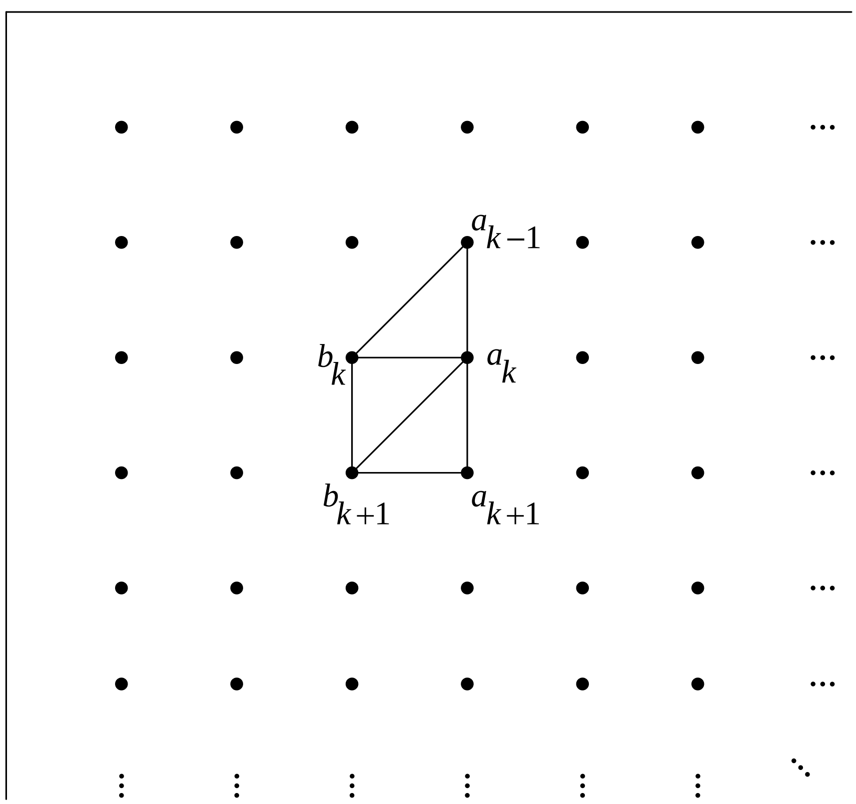

Fig 2. Log-Concavity of Diagonals of the Khayyam-Pascal Triangle Array.

Next we prove the log-concavity of the rows of the Khayyam-Pascal triangle, using the same technique.

Proposition 2.3. *Every row of the Khayyam-Pascal triangle is log-concave.*

Proof. We note that the rows of the Khayyam-Pascal triangle correspond to the diagonals of the the Khayyam-Pascal squared array. Consider an special parallelepiped $vuwv'u'v$, as shown in Fig 3.

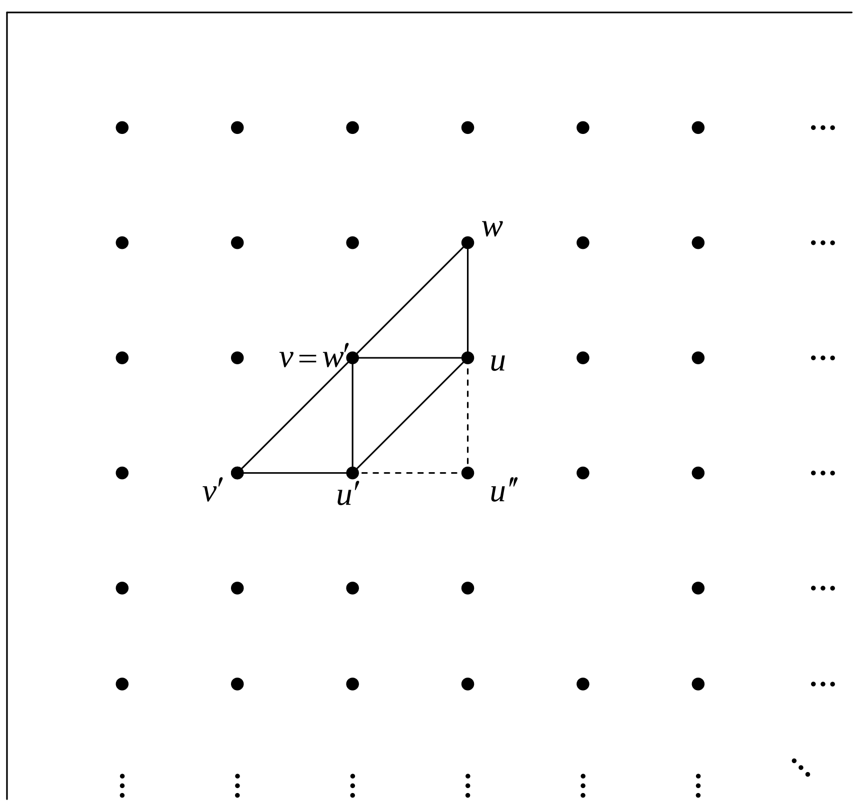

Fig 3. Log-Concavity of Rows of Khayyam-Pascal Triangle Array.

Then, we have

$$\left| \begin{array}{cc} v & w \\ v' & v \end{array} \right| = \left| \begin{array}{cc} w' & w \\ u' & u \end{array} \right|.$$

On the other hand, from the parallelepiped $w'uwu'u''u$ we get

$$\left| \begin{array}{cc} w' & w \\ u' & u \end{array} \right| = \left| \begin{array}{cc} w' & u \\ u' & u'' \end{array} \right|.$$

Therefore, we conclude that

$$v^2 - wv' = \left| \begin{array}{cc} w' & u \\ u' & u'' \end{array} \right|.$$

But, again the last determinant in the above equality is the Narayana number and a non-negative integer. This completes the proof. □

Definition 2.1. We call an array a *row log-concave* (diagonal log-concave) array, if every row (diagonal) of this array is log-concave.

As in the paper of Peter R.W. McNamara and Bruce E. Sagan [3] for every array $A = (a_{ij})_{i,j \geq 0}$, we will call the determinants $\left| \begin{array}{cc} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{array} \right|$, its *adjacent minors*. From the proofs of the two previous propositions, we get the following interesting result.

Corollary 2.4. *Every diagonal log-concave array with non-negative adjacent minors, is also a row log-concave array.*

3 Khayyam-Pascal Determinantal Arrays

In this section, we introduce an infinite class of arrays of numbers as a generalization of the standard Khayyam-Pascal squared array. We will denote the entries of the the Khayyam-Pascal squared array by $P_{i,j} = P_{i,j}^{(1)} = \binom{i+j}{i}$, $i, j \geq 0$. Our main goal here is to prove that the members of this new class of arrays are diagonal and row log-concave, again using geometric ideas.

Definition 3.1. A Khayyam-Pascal determinantal array of order k, $k \geq 1$, is an infinite array with entries $P_{i,j}^{k}$, $(i, j \geq 0)$, where $P_{i,j}^{k}$ is the determinant of a k-by-k subarray of the Khayyam-Pascal squared array. Namely,

$$P_{i,j}^{(k)} := \begin{vmatrix} P_{i,j} & \dots & P_{i,j+k-1} \\ \vdots & \ddots & \vdots \\ P_{i+k-1,j} & \dots & P_{i+k-1,j+k-1} \end{vmatrix}.$$

Example 3.1. *A Khayyam-Pascal determinantal array of order 2 has shown in Fig 4. This is a well-known array which is the squared-form of the so-called Narayana triangular array (see A001263 in* [4]*).*

1	1	1	1	1	1	1
1	3	6	10	15	21	28
1	6	20	50	105	196	336
1	10	50	175	490	1176	2520
1	15	105	490	1764	5292	13860
1	21	196	1176	5292	19404	60984
1	28	336	2520	13860	60984	226512

Fig 4. Khayyam-Pascal Determinantal Array of Order 2.

In [5], the authors have shown that if we define the weight of any arbitrary rectangle whose vertices are the entries of the Khayyam-Pascal determinantal array of order k as shown in Fig 5, by

$$W := \frac{P^k_{i+m,j+l} \cdot P^k_{i,j}}{P^k_{i+m,j} \cdot P^k_{i,j+l}},$$

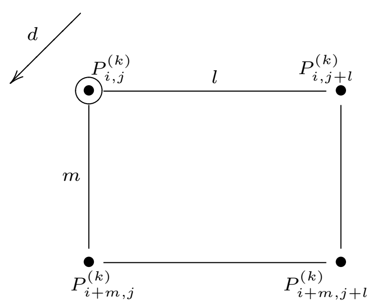

Fig 5. Weighted Version of Star of David.

then when we move the anchor, the circled-vertex, along the diagonal of the Khayyam-Pascal determinantal array (indicated by the arrow d in Fig 5), the weights remain unchanged. They called this property the *weighted-version of the Star of David Rule*. As they have shown in another paper [6], the weighted-version of the Star of David Rule can also be used to prove the following interesting property of this new class of arrays.

Proposition 3.1. *In any Khayyam-Pascal determinantal array, the ratio of any pair of r-by-r minors along any arbitrary diagonal $x + y = d$ of the array is the same as the ratio of the product of the entries appearing in their back diagonals parallel to d (see Fig 6). In other words, we have*

$$\frac{\begin{vmatrix} P^{(k)}_{i,j} & \cdots & P^{(k)}_{i,j+r-1} \\ \vdots & \ddots & \vdots \\ P^{(k)}_{i+r-1,j} & \cdots & P^{(k)}_{i+r-1,j+r-1} \end{vmatrix}}{\begin{vmatrix} P^{(k)}_{i',j'} & \cdots & P^{(k)}_{i',j'+r-1} \\ \vdots & \ddots & \vdots \\ P^{(k)}_{i'+r-1,j'} & \cdots & P^{(k)}_{i'+r-1,j'+r-1} \end{vmatrix}} = \frac{P^{(k)}_{i,j+r-1} \cdots P^{(k)}_{i,j+r-1}}{P^{(k)}_{i',j'+r-1} \cdots P^{(k)}_{i',j'+r-1}}$$

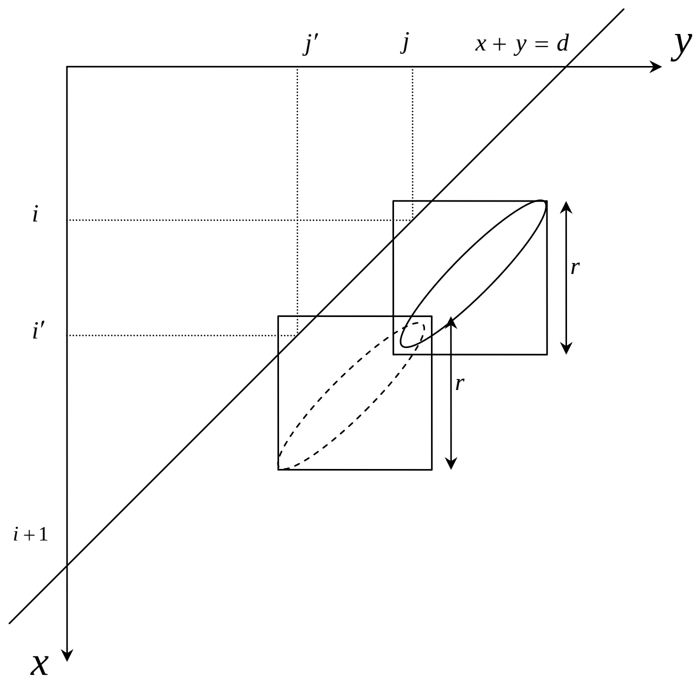

Fig 6. Ratio of Determinants in Khayyam-Pascal Determinantal array

The following lemma is the key in the proof of diagonal log-concavity of the Khayyam-Pascal determinantal Arrays.

Lemma 3.2. *For every integer $n \geq 1$, the log-concave sequence $\{a_i\}_{i\geq 1}$ satisfies the following inequality*

$$\frac{a_2 a_{n+1}}{a_1 a_{n+2}} \geq 1.$$

Proof. We use induction on n. The basis case, $n = 1$, is just the definition of the log-concavity of the sequence $\{a_i\}_{i\geq 1}$. Now, let us assume by induction hypothesis that the assertion is true for $n-1$. Hence, we have

$$1 \leq \frac{a_2 a_n}{a_1 a_{n+1}} = (\frac{a_2 a_n}{a_1 a_{n+1}})(\frac{a_{n+1} a_{n+2}}{a_{n+1} a_{n+2}}) = (\frac{a_2 a_{n+1}}{a_1 a_{n+2}})(\frac{a_n a_{n+2}}{a_{n+1}^2}).$$

Thus, we get

$$\frac{a_2 a_{n+1}}{a_1 a_{n+1}} \geq \frac{a_{n+1}^2}{a_n a_{n+2}} \geq 1.$$

The later inequality holds because of the definition of the log concavity of the sequence $\{a_i\}_{i\geq 1}$. This completes the proof by induction. $\square$

Now, we are at the position to state our main result of this section.

Theorem 3.3. *For every integer $k \geq 1$, the Khayyam-Pascal determinantal array of order k is diagonal log-concave.*

Proof. Assume that α, β, θ, γ are four entries of the Khayyam-Pascal determinantal array of order k such that β, θ, γ are three consecutive diagonal entries, as shown in Fig 7.

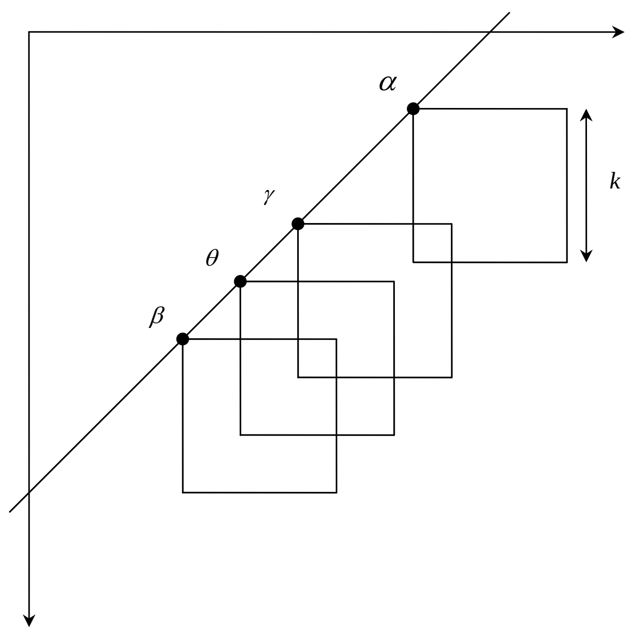

Fig7. Four Entries of A diagonal of Khayyam-Pascal Determinantal Array

Clearly the back diagonal entries of these four entries of the Khayyam-Pascal determinantal array of order k, as the four k-by-k minors of the Khayyam-Pascal squared array, lie in some diagonal of the Khayyam-Pascal squared array. For simplicity of arguments, we will show their entries from south-west to north-east by β_1, β_2, ..., β_k, θ_1, θ_2, ..., θ_k, γ_1, γ_2, ..., γ_k and α_1, α_2, ..., α_k, respectively. It is not hard to see that we have the following relations among their entries:

$$\beta_2 = \theta_1, \beta_3 = \theta_2, \ldots, \beta_k = \theta_{k-1},$$
$$\theta_1 = \gamma_1, \theta_3 = \gamma_2, \ldots, \theta_k = \gamma_{k-1}.$$

To prove the log-concavity, it suffices to show that $\theta^2 - \beta\gamma \geq 0$. But, using the determinants ratio Proposition 3.1 and the above relations, we have

$$\begin{aligned}
&\theta^2 - \beta\gamma \\
&= \left(\frac{\alpha}{\alpha_1 \cdots \alpha_{k-1}\alpha_k}\right)^2 \left[(\theta_1 \cdots \theta_{k-1}\theta_k)^2 - (\beta_1 \cdots \beta_{k-1}\beta_k)(\gamma_1 \cdots \gamma_{k-1}\gamma_k)\right], \\
&= \left(\frac{\alpha}{\alpha_1 \cdots \alpha_{k-1}\alpha_k}\right)^2 \left[(\beta_2\beta_3^2 \cdots \beta_k^2\gamma_{k-1})(\beta_2\gamma_{k-1} - \beta_1\gamma_k\right].
\end{aligned}$$

Therefore we need to prove that $\frac{\beta_2\gamma_{k-1}}{\beta_1\gamma_k} \geq 1$, which is nothing more than the inequality of the key lemma, Lemma 3.2, by renaming technique. □

Next, we prove the row log-concavity of the Khayyam-Pascal determinantal array.

Theorem 3.4. *For every integer $k \geq 1$, the Khayyam Pascal determinantal array of order k is a row log-concave array.*

Proof. Using Corollary 2.4, it is only suffices to prove that every adjacent minor of the Khayyam-Pascal determinantal array of order k is nonnegative. Now by the Proposition 3.1 about the ratio of determinants along the diagonal $x + y = d$, we get

$$\frac{\begin{vmatrix} P_{i,j}^{(k)} & P_{i,j+1}^{(k)} \\ P_{i+1,j}^{(k)} & P_{i+1,j+1}^{(k)} \end{vmatrix}}{\begin{vmatrix} 1 & P_{i+j,1}^{(k)} \\ 1 & P_{i+j+1,1}^{(k)} \end{vmatrix}} = \frac{P_{i+1,j}^{(k)} P_{i,j+1}^{(k)}}{P_{i+j,1}^{(k)}},$$

which is clearly a positive integer. Thus, to prove that the adjacent minor $\begin{vmatrix} P^{(k)}_{i,j} & P^{(k)}_{i,j+1} \\ P^{(k)}_{i+1,j} & P^{(k)}_{i+1,j+1} \end{vmatrix}$ is a nonnegative integer, we only need to show that $\begin{vmatrix} 1 & P^{(k)}_{i+j,1} \\ 1 & P^{(k)}_{i+j+1,1} \end{vmatrix}$ is positive for every $i, j \geq 0$, which is equivalent to show that the first column, starting form 0, of the Khayyam-Pascal determinantal array of order k is an increasing sequence. It is not hard to see that this first column is indeed the kth column of the Khayyam-Pascal squared array [1]. Finally we need to show that for every $l \geq 0$, we have

$$\frac{\binom{l+k}{k}}{\binom{(l-1)+k}{k}} > 1,$$

which is equivalent to inequality $k > 0$ or $k \geq 1$, as required. $\square$

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