Determinants, Lyndon Covers and The Coin Arrangements Lemma

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Abstract

In this paper we first give a multiset version of the well-known graph theoretical interpretation of determinant of a matrix as a signed weighted sum over the cycle covers of the associated digraph of the matrix. Then, as a direct consequence of this new result, we give a multivariate generalization of the coin arrangements lemma.

1 Introduction

The cyclic decomposition of the permutation of a finite set is a fundamental result in algebraic combinatorics. It simply leads to the definition of an important combinatorial number, i.e., the Stirling cycle number $S_1(n,k)$ which is the number of the permutations of an n-element set with exactly k cycles. In addition, the canonical cyclic representation of a permutation, i.e., starting each cycle with the largest element and ordering cycles in increasing order of their first elements gives us an important result that $S_1(n,k)$ is also equal to the number of permutations of an n-element set with exactly k left-to-right maxima [1]. But this is not the only place that the cyclic decomposition plays a crucial role. The classical notion of a determinant of a matrix A with row-index set (or column-index set) $\{1,\ldots,n\}$ is yet another important example that the cyclic decomposition has the key role. The combinatorial definition of the determinant is nothing

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more than a signed weighted sum over the set of all permutations of the set $\{1, \ldots, n\}$. Hence, the cyclic decomposition gives a nice graph theoretical interpretation of a determinant as a signed weighted sum over the cycle covers in the associated digraph of the matrix A.
A multiset version of the notion of the left-to-right maxima has already been given by Foata as the first fundamental transformation, using the idea of flow and circuit monoids [2]. In this paper, we attempt to give a multiset version of the graph-theoretical interpretation of a determinant. Therefore, the next section provides a review of basic concepts of multiset theory and we will explicitly define what we mean mathematically by the multiset version of a combinatorial identity about permutations. In section 3, we also briefly review the combinatorics of Lyndon words and its connection with the coin arrangements lemma. In section 4, we review the folkloric result of the graph-theoretical interpretation of the classic determinant. Then, in the last chapter, we will present its multiset version, using the natural generalization of all key notions from the set to the multiset. As a result of this new theory, we give a multivariate generalization of the coin arrangements lemma [3].

2 Mathematics of Multisets

Here, we review some basics of the *multiset theory* that will be used throughout the paper which are mainly based on the references [4, 5]. For more thorough discussions, we refer reader to [6].
Ordinary sets are composed of pairwise different elements, i.e., no two elements are the same. If we relax this condition, i.e., if we allow multiple but finite occurrences of any element, we get a generalization of the notion of a set which is called a multiset. Here is the formal definition. We will denote the set of nonnegative integers by $\mathbb{N}_0$.

Definition 2.1. (see [4]) Let A be a set. A multiset over A is just a pair $\langle A, f\rangle$, where A is a set and $f : A \mapsto \mathbb{N}_0$ is a function. Value $f(a)$ gives multiplicity of symbol a.

Example 2.1. *For the multiset $M = \{a, a, a, b, b, c, d, d, d, d\}$, we have $A = \{a, b, c, d\}$ and $f : A \mapsto \mathbb{N}_0$ with $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 4$.*

Definition 2.2. For every $a \in A$, $f(a)$ is called the multiplicity of the element a of the multiset.

There are many ways to represent a multiset by listing its elements. For example, consider the multiset

$$M = \{a, a, a, b, b, c, d, d, d, d\}.$$

We can show M in the following way by indicating the multiplicities of its elements

$$M = \{3.a, 2.b, 1.c, 4.d\},$$

or equivalently

$$M = \{a^3, b^2, c^1, d^4\}.$$

Throughout this paper, we will use the later representation for presenting a multiset.

Definition 2.3. For a multiset $M = \{1^{m_1}, 2^{m_2} \cdots n^{m_n}\}$, we will call the vector $\overrightarrow{m} = (m_1, m_2, \cdots, m_n)$ the *multiplicity vector* of the multiset M.

Next, we proceed with the definitions of the notion of the subset a of multiset and the operations between multisets.

Definition 2.4. Assume that $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$ are two multisets. Then we say that $\mathcal{A}$ is a *submultiset* of $\mathcal{B}$ denoted by $\mathcal{A} \subseteq \mathcal{B}$ if for all $a \in A$

$$f(a) \leq g(a).$$

Definition 2.5. Let $\mathcal{A} = \langle A, f\rangle$, be a multiset. Then $\mathcal{A}$ is the *empty multiset* if for all $a \in A$, $f(a) = 0$.

Definition 2.6. Suppose that $\mathcal{A} = \langle A, f\rangle$ is a multiset, then its *cardinality*, denoted by $card(\mathcal{A})$, is defined as

$$card(\mathcal{A}) = \sum_{a \in A} f(a).$$

Now we define the operations sum and removal of two multisets.

Definition 2.7. Suppose that $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$ are two multisets. Then their *sum*, denoted by $\mathcal{A} \biguplus \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h\rangle$ where for all $a \in A$:

$$h(a) = f(a) + g(a).$$

Definition 2.8. Suppose that $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$ are two multisets. Then the *removal* of multiset $\mathcal{B}$ from $\mathcal{A}$, denoted by $\mathcal{A} \ominus \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h\rangle$ where for all $a \in A$:

$$h(a) = max\left(f(a) - g(a), 0\right).$$

Next, we give a generalization of the definition of the permutation for the case of the multiset.

Definition 2.9. Let M be a multiset and let r be a positive integer satisfying $r \leq card(M)$. We define an *r-permutation* of M as an *ordered arrangement* of r objects of M. In particular, if $N = card(M)$, an N-permutation of M will also be called a permutation of M.

Example 2.2. *If* $M = \{a^3, b^2, c^1, d^4\}$*, then*

$$cabddabdad,$$

is one-line notation of a permutation of the multiset M*.*

It is a classic result in combinatorics [5] that the number of permutations of the multiset $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$ is the *multinomial coefficient*

$$\binom{N}{m_1, m_2, \ldots, m_n} = \frac{N!}{m_1! m_2! \cdots m_n!}. \tag{2.1}$$

We conclude this section by giving our definition of multiset version of a combinatorial identity dealing with the permutations of a finite set.

Definition 2.10. For a given combinatorial identity related to the cyclic decomposition of permutations, if we replace the permutations of a finite set A of cardinality n with the permutations of a finite multiset $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$ then the resulting new combinatorial identity is called the *multiset version* of the given identity.

3 Lyndon words and The Coin Arrangements Lemma

This section briefly reviews the combinatorics of Lyndon words and its connection with the coin arrangements lemma. The reader may consult the book [2]. Let A be a finite set. The set of all finite sequences of elements (letters) of A will be denoted by $A^\star$. Each element of $A^\star$ is called a *word* over the alphabet A. In particular, an empty sequence of letters is also an element of $A^\star$ which is called an *empty word* and is denoted by 1. The *length* of a word w is defined as the number of its letters and is dented by $|w|$. The set of all nonempty words over A will be denoted by A^+:

$$A^+ = A^\star - 1.$$

A word v is called a *right factor* of a word w if there exists a third word u such that $w = uv$. If u is a nonempty word then the right factor v is called the *proper* right factor of the word w. Two words x and y are said to be *conjugate* if there exists words $u, v \in A^\star$ such that

$$x = uv, \quad y = vu.$$

This is an *equivalence relation* on $A^\star$ since x is conjugate of y if and only if y can be obtained by a *cyclic permutation* of the letters of x. It partitions the $A^\star$ into disjoint conjugate classes. A word is *primitive* if it is not a power another word. We also recall that a *Lyndon word* is a primitive word that is minimal, with respect to lexicographic order, in its conjugate class. The set of Lyndon words will be denoted by $\mathcal{L}$. We also need some classic results about Lyndon words. For proofs see the book [2]. We have the following characterization for the set of the Lyndon words.

Proposition 3.1. *A word $w \in A^+$ is a Lyndon word if and only if it is strictly smaller than any its proper right factor:*

$$w \in \mathcal{L} \Leftrightarrow \{\forall v \in A^+,\ w \in A^+ v \Rightarrow w \prec v\}.$$

The following result gives a second characterization of the Lyndon words.

Proposition 3.2. *A word $w \in A^+$ is a Lyndon word if and only if $w \in A$ or $w = lm$ with $l, m \in \mathcal{L}$. More precisely, if there exists a pair (l, m) with $w = lm$ such that $m, w \in \mathcal{L}$ and m of maximal length, then $l \in \mathcal{L}$ and $l \prec lm \prec m$.*

Definition 3.1. For $w \in \mathcal{L} - A$ a Lyndon word consisting of more than a single letter, the pair (l, m) with $w = lm$ such that $l, m \in \mathcal{L}$ and m of maximal length is called the *standard factorization* of the Lyndon w.

We also recall the following fundamental result which is well-known as the *Lyndon factorization theorem.*

Theorem 3.3. *Any word $w \in A^+$ may be written uniquely as a nonincreasing product of Lyndon words:*

$$w = l_1 l_2 \cdots l_n,\ l_i \in \mathcal{L},\quad l_1 \succeq l_2 \succeq \cdots \succeq l_n.$$

Recall that the *Stirling cycle number* denoted by $S_1(n,k)$ is the number of permutations of the finite set $[n]$ with exactly k cycles in their cyclic decompositions into disjoint cycles. By convention, we define $S_1(0,0) = 1$. There are many interesting combinatorial identities related to the Stirling cycle numbers of which we can mention the following:

$$\begin{aligned} \sum_{k=1}^{n} S_1(n,k) &= n!, \qquad (n \geq 1), \\ \sum_{k=1}^{n} (-1)^k S_1(n,k) &= 0, \qquad (n \geq 2). \end{aligned} \tag{3.1}$$

The multiset version of the identity (3.1) is called the *coin arrangements lemma*. To find the multiset version of (3.1), we only need to find the multiset version of the Stirling cycle number. We first need the following definition.

Definition 3.2. The system $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$ of distinct Lyndon words over alphabet $A = \{1, 2, \ldots, n\}$ is called a *Lyndon system* over the finite multiset $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$, if γ is a permutation of M and the following condition holds:

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_k| = N,$$

where $|\gamma_j|$ denotes the usual length of the word γ_j. We let the *Lyndon system index* of γ, denoted by $i_{ls}(\gamma)$, be equal to k.

Note that in Definition 3.2, we do not require $\gamma_1 \preceq \gamma_2 \preceq \cdots \preceq \gamma_k$. Thus, we arrive at the following multiset version of the Stirling cycle numbers.

Definition 3.3. Let $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a finite multiset of cardinality $N = \sum_{i=1}^{n} m_i$. We define the generalized Stirling cycle number $S_1(N, k; m_1, m_2, \ldots, m_k)$ as the number of permutations γ of M with $i_l(\gamma) = k$, $k \neq 0$. By convention, we set $S_1(N, 0; m_1, m_2, \ldots, m_k) = 0$ for any integer number $N \geq 1$ and $S_1(0, 0; m_1, m_2, \ldots, m_k) = 1$.

Now, we are at the position to state the multiset version of the identity (3.1), which is known as the coin arrangements lemma [3].

Theorem 3.4. *For any integer $N > 1$, we have*

$$\sum_{k=1}^{N} (-1)^k S_1(N, k; m_1, \ldots, m_n) = 0.$$

It is worth noting that the original form of the coin arrangements lemma, due to Sherman [3] is as follows:
Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind, ..., and m_n of n-th kind. Let $b_{N,k}$ be the number of exhaustive unordered arrangements of these symbols into k disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. Then, we have

$$\sum_{k=1}^{N} (-1)^k b_{N,k} = 0, \qquad (N > 1).$$

Clearly those numbers $b_{N,k}$ are exactly the generalized Stirling cycle numbers.

4 Cycle Covers

In this section, we briefly review the graph-theoretical interpretation of the determinant. The definitions and the results of this section are mostly from the book [7], but we will present them in a way that is more suitable for our purposes. For brevity, we will use the notation $[n]$ to denote the finite set $\{1, 2, \ldots, n\}$.

Definition 4.1. Let $\sigma = j_1 j_2 \cdots j_n$ be a permutation. An *inversion* of σ is a pair k, l of integers with $1 \leq k < l \leq n$ such that $j_k > j_l$. We will denote the number of inversions of σ by $inv(\sigma)$.

Definition 4.2. Let $\sigma = j_1 j_2 \cdots j_n$ be a permutation of $[n]$. Let $A = (a_{ij})$ be a square matrix of order n. The sign $sgn(\sigma)$ of the permutation σ is defined to be $sgn(\sigma) = (-1)^{inv(\sigma)}$. The weight $wt(\sigma)$ of the permutation σ is defined as $wt(\sigma) = a_{1j_1} a_{2j_2} \cdots a_{nj_n}$.

Now, we are ready to give the classical definition of a determinant for the case that indices of rows and columns form the same finite set $[n]$. We will denote the set of all permutations of the set $[n]$ by S_n.

Definition 4.3. Let $A = (a_{ij})$ be a square matrix of order n. Then we define the determinant of A denoted by $det(A)$, as follows

$$det(A) = \sum_{\sigma=j_1 j_2 \cdots j_n \in S_n} sgn(\sigma) wt(\sigma),$$

where σ runs over the set of all permutations of the set $[n]$.

Before continuing, we need to recall some basic definitions and results from digraph theory. For more information, reader can consult on [8]. Let us remind that an *Eulerian digraph* is a digraph such that each vertex has the same *in-degree* and *out-degree*. We also recall the following classical result for Eulerian digraphs.

Theorem 4.1. *Every Eulerian digraph can be decomposed as the edge-disjoint union of (directed) cycles.*

Now, suppose $A = (a_{ij})$ is a square matrix of order n. We associate a digraph $D = D(A)$ with the matrix A which plays an essential role in the graph-theoretical interpretations of the determinant of the matrix A. Formally, we have the following definition.

Definition 4.4. For a given square matrix $A = (a_{ij})$ of order n, its *associated digraph* $D = D(A)$ is defined as a digraph with the vertex set $[n]$ and there is an arc from vertex i to vertex j, if $a_{ij} \neq 0$. Whenever $a_{ii} \neq 0$, we have a loop at vertex i.

Definition 4.5. A *linear subdigraph* L of a digraph D is a spanning subdigraph of D in which each vertex has in-degree 1 and out-degree 1. Thus, by Theorem 4.1, a linear subdigraph L consists of a spanning collection of pairwise vertex-disjoint cycles. The number of cycles contained in L is denoted by $c(L)$. For a given square matrix $A = (a_{ij})$, the weight $wt(L)$ of a linear subdigraph L of $D(A)$ is defined as the product of the weights of its arcs and the weight of any arc (i, j) is defined to be a_{ij}.

Definition 4.6. Let $A = (a_{ij})$ be a square matrix of order n and $D = D(A)$ its associated digraph. The sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of vertex-disjoint cycles of D such that $\sum_{i=1}^{k} |\gamma_i| = n$ where $|\gamma_j|$ is the length of the cycle γ_j is called a *cycle cover* of D. The sign $sgn(\gamma)$ of a cycle cover γ is defined to be $sgn(\gamma) = (-1)^{n-k}$. We also define the weight $wt(\gamma)$ of γ as the product of weights of its cycles and the weight $wt(\gamma_j)$ of each cycle $\gamma_j = (j_1 j_2 \cdots j_{t-1} j_t)$ of length t is defined as $wt(\gamma_j) = a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{t-1} j_t} a_{j_t j_1}$.

We also need the following technical lemma due to Cauchy (see [7]). We denote the *number of cycles* in the cyclic decomposition of σ by $c(\sigma)$.

Lemma 4.2. *Let $\sigma = j_1 j_2 \cdots j_n$ be a permutation of $\{1,\ldots,n\}$. Then $inv(\sigma)$ and $n - c(\sigma)$ have the same parity. Therefore,*

$$(-1)^{n-c(\sigma)} = (-1)^{inv(\sigma)}.$$

As a direct consequence of the Cauchy lemma and the cyclic decomposition of permutations, we have the following two equivalent graph-theoretical interpretations of the determinant of the matrix A.
Observation 4.1 *Let $A = (a_{ij})$ be a square matrix of order n. Then*

$$det(A) = \sum_{L \in Lin(A)} (-1)^{n-c(L)} wt(L) = \sum_{\gamma \in \Gamma} sgn(\gamma) wt(\gamma),$$

where $Lin(A)$ is the set of all linear subdigraphs of $D(A)$ and Γ is the set of all cycle covers of $D(A)$.

Proposition 4.3 (A Proof of the Identity (3.1))**.** *For any integer $n > 1$,*

$$\sum_{k=1}^{n} (-1)^k S_1(n,k) = 0.$$

Proof. In Observation 4.1, put $A = J$, where J is all one matrix. $\square$

Remark 4.1. The above result shows that the identity (3.1) is indeed equivalent to the determinantal identity $det(J) = 0$.

5 Lyndon Covers

Let $\mathcal{A} = \{1,\ldots,n\}$ be our alphabet. We will consider a multiset $M = \{1^{m_1},\ldots,n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$.

Definition 5.1. Let $M = \{1^{m_1},\ldots,n^{m_n}\}$ be a multiset of cardinality $N = \sum_{i=1}^{n} m_i$. We associate a function f_M with M which we call the *set-indicator* function of M, as follows

$$f_M : \{1,\ldots,N\} \mapsto \{1,\ldots,n\}$$
$$f_M(i_l) = l, \quad i_l \in [\Big(\sum_{j=1}^{l-1} m_j\Big) + 1, \sum_{j=1}^{l} m_j], \quad (1 \le l \le n),$$

where by convention we set $\sum_{j=1}^{0} m_j = 0$ and for positive integers a and b, by $[a,b]$ we mean the set of all integers between a and b including themselves.

Example 5.1. *For $M = \{1^2, 2^2\}$ of cardinality 4, we have*

$$f_M : \{1,2,3,4\} \mapsto \{1,2\}$$
$$f_M(1) = 1, f_M(2) = 1, f_M(3) = 2, f_M(4) = 2.$$

Next, we define a matrix over a multiset.

Definition 5.2. Let $M = \{1^{m_1}, \dots, n^{m_n}\}$ be a multiset of cardinality $N = \sum_{i=1}^{n} m_i$ over alphabet $[n]$. We say that a square matrix A of order N is a matrix *indexed by multiset* M if there is a square matrix A' of order n so that for all $i, j \in \{1, \dots, N\}$,

$$(A)_{ij} = (A')_{f_M(i) f_M(j)}.$$

Example 5.2. *Let $\mathcal{A} = \{1,2\}$ be our alphabet. For the multiset $M = \{1^2, 2^2\}$ of cardinality 4, a matrix $A = (a_{ij})$ indexed by M in general has the following form*

$$A = \begin{bmatrix} a_{11} & a_{11} & a_{12} & a_{12} \\ a_{11} & a_{11} & a_{12} & a_{12} \\ a_{21} & a_{21} & a_{22} & a_{22} \\ a_{21} & a_{21} & a_{22} & a_{22} \end{bmatrix}.$$

We have the following simple observation.

Observation 5.1. *Let $\widetilde{A} = (\tilde{a}_{ij})$ be a square matrix of order N. Let $A = (a_{ij})$ be a square matrix indexed by the multiset M of cardinality N. Define the transformation S, as follows*

$$S : \tilde{a}_{ij} \mapsto a_{f_M(i) f_M(j)}.$$

Then, we have

$$\widetilde{A}|_S = A.$$

As we already mentioned, our main goal in this section is to find a multiset version of the graph-theoretical interpretation of determinants. The first step is to find the appropriate multiset version of the classical combinatorial definition of a determinant. Since the recursive definition of the classic determinant starting from a 2-by-2 matrix implies that the determinant of a matrix of order n in general is the sum of $n!$ monomials in the entries of that matrix, the multiset version of the determinant has to have an expansion as a sum of $N!$ monomials in the entries of the matrix A indexed by the multiset M of cardinality N. Having this fact in mind and also motivated by the idea of Rota [9], we will associate with each permutation σ of the set $[N] = \{1, \ldots, N\}$ a word system. We will also denote the set of all permutations of $[N]$ by S_N. Also recall that a word $a_1 \cdots a_n$ is called a *circular* word if a_1 is regarded as following a_n, where $a_1 \cdots a_n, a_2 \cdots a_n a_1$ and all other *cyclic shifts* (rotations) of $a_1 \cdots a_n$ are regarded as the same word. We will use the notations w and (w) to distinguish between a word and the circular word obtained from it.

Definition 5.3. Let $M = \{1^{m_1}, \ldots, n^{m_n}\}$ be a multiset of cardinality $N = \sum_{i=1}^{n} m_i$ with set-indicator function f_M and let $A = (a_{ij})$ be a matrix indexed by M. Assume that $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ is the cyclic decomposition of $\sigma \in S_N$. We define the *word system* associated with the permutation σ as a multiset $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$ of circular words, as follows. For each i $(i = 1, 2, \ldots, k)$, we associate with the cycle $\sigma_i = (i_1 i_2 \cdots i_t)$ of σ a circular word $\gamma_i = (f_M(i_1) f_M(i_2) \cdots f_M(i_t))$. The sign $sgn(\gamma)$ of the word system $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$, is defined to be $sgn(\gamma) = (-1)^{N-k}$. The weight $wt_M(\gamma)$ of γ, is defined as the products of the weights of its circular words and the weight $wt_M(\gamma_j)$, $j = 1, 2, \ldots, k$, of the circular word $\gamma_j = (j_1 j_2 \cdots j_t)$ is defined to be $wt_M(\gamma_j) = a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{t-1} j_t} a_{j_t j_1}$.

Example 5.3. *For the multiset $M = \{1^2, 2^2\}$ and the permutation $\sigma = (1)(23)(4)$ of S_4, the word system γ associated with σ is $\gamma = \langle (1), (12), (2) \rangle$.*

We will denote the multiset of all $N!$ word systems associated with the set of all $N!$ permutations of $[N]$ by Γ_{sys}. Hence, we arrive at the multiset version of the classical combinatorial definition of a determinant of a matrix, as follows.

Definition 5.4. Let $A = (a_{ij})$ be a square matrix indexed by the multiset $M = \{1^{m_1}, \ldots, n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$. Then we define the determinant of A denoted by $\overline{det}(A)$, as follows

$$\overline{det}(A) = \sum_{\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle \in \Gamma_{sys}} sgn(\gamma) wt_M(\gamma),$$

where γ runs over the multiset of all $N!$ word systems associated with S_N.

The following observation is key in the proof of the main result of this section.

Observation 5.2. *Consider the same matrices $A, \widetilde{A}$ as in Observation 5.1. Then for any integer $N > n$, we have*

$$det(\widetilde{A}|_S) = \overline{det}(A) = 0.$$

Here is the argument. Clearly for $N > n$, the matrix $\widetilde{A}|_S$ has two identical rows. Hence, $det(\widetilde{A}|_S) = 0$. By Observation 5.1, we have

$$det(\widetilde{A}) = \sum_{\widetilde{\gamma}=\langle\widetilde{\gamma}_1,\widetilde{\gamma}_2,\ldots,\widetilde{\gamma}_k\rangle\in\widetilde{\Gamma}} sgn(\widetilde{\gamma})wt_M(\widetilde{\gamma}),$$

where $\widetilde{\Gamma}$ is the set of all cycle covers of $D(\widetilde{A})$. Now, we apply the transformation S to the both sides of the above identity. By Definition 5.3, it is straight forward that $\widetilde{\gamma}|_S$ is a word system associated with the permutation $\sigma = \widetilde{\gamma}_1\widetilde{\gamma}_2\cdots\widetilde{\gamma}_k$ of S_N. Thus, we get

$$det(\widetilde{A}|_S) = \sum_{\widetilde{\gamma}|_S\in\Gamma_{sys}} sgn(\widetilde{\gamma}|_S)wt_M(\widetilde{\gamma}|_S) = \overline{det}(A).$$

Example 5.4. *For the multiset $M = \{1^2, 2^1\}$, $N = 3$, we have*

cycle notation	*word system*	*weight*
(1)(2)(3)	$\langle(1),(1),(2)\rangle$	$a_{11}a_{11}a_{22}$
(1)(23)	$\langle(1),(12)\rangle$	$a_{11}a_{12}a_{21}$
(12)(3)	$\langle(11),(2)\rangle$	$a_{11}a_{11}a_{22}$
(123)	$\langle(112)\rangle$	$a_{11}a_{12}a_{21}$
(132)	$\langle(121)\rangle$	$a_{12}a_{21}a_{11}$
(13)(2)	$\langle(1),(12)\rangle$	$a_{11}a_{12}a_{21}$

Hence, we get

$$\begin{aligned}\overline{det}(A) &= 2!\Big((-1)^{3-1}wt_M(\langle(112)\rangle) + (-1)^{3-2}wt_M(\langle(1),(12)\rangle)\Big)\\ &+ \Big((-1)^{3-3}wt_M(\langle(1),(1),(2)\rangle) + (-1)^{3-2}wt_M(\langle(11),(2)\rangle)\Big).\end{aligned}$$

Remark 5.1. Not that in Example 5.4 the word systems inside the parentheses in the first sum are consisting of only distinct circular *primitive words*, i.e., those which are not powers of any smaller words. It is also straight forward to see that the second sum is zero. This fact is not a mere coincidence. It leads to the multiset version of the graph-theoretical interpretation of determinants.

For simplicity of arguments, since the γ_i's in Definition 5.3 are circular words, we can replace each circular primitive word γ_i with the circular word obtained as a minimal element (with respect to lexicographic order) in the class of all cyclic shifts of γ_i. We call the resulting circular word the *Lyndon representation* of γ_i or *circular Lyndon word* corresponding to γ_i. The whole process is called the *replacement process.*

Definition 5.5. A word system $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$ is a *Lyndon cover* of the multiset M if all γ_i's are circular Lyndon words and distinct. The weight and the sign of a Lyndon cover are defined in the same way that they are defined for a word system.

We need the following observation in the proof of our main result.

Observation 5.3. *There is 1-to-$(m_1!)(m_2!)\cdots(m_n!)$ correspondence between the multiset of all word systems of Γ_{sys} which are consisting of only distinct primitive circular words and the set of all Lyndon covers of M.*

Here is the argument. Let $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$ be an arbitrary Lyndon cover of M. Now, let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be the permutation of S_N whose associated word system is γ. The multiset of the letters of γ_i's has m_j letters of j, $(j = 1, 2, \ldots, n)$. Set $A_j = \{i \in [N] : f_M(i) = j\}$. Since γ_i's are distinct circular Lyndon words, if we only permute the elements of A_j in σ, we obtain $m_j!$ distinct permutations of S_N whose associated word systems are γ after applying the replacement process. Thus, by product rule, there is 1-to-$(m_1!)(m_2!)\cdots(m_n!)$ correspondence between the multiset of all word systems of Γ_{sys} which are consisting of only distinct primitive circular words and the set of all Lyndon covers of M.

Now, we are at the situation to state and prove the main result of this chapter.

Theorem 5.1. *For any square matrix $A = (a_{ij})$ indexed by the multiset $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ of cardinality $N = \sum_{i=1}^{n} m_i$, we have*

$$\overline{det}(A) = (m_1!)(m_2!)\cdots(m_n!) \sum_{\gamma \in \Gamma_{lc}} sgn(\gamma) wt_M(\gamma),$$

where Γ_{lc} is the set of all Lyndon covers of the multiset M.

Proof. Considering Observation 5.3, we can rewrite $\overline{det}(A)$, as follows

$$\begin{aligned}\overline{det}(A) &= (m_1!)(m_2!)\cdots(m_n!) \sum_{\gamma \in \Gamma_{lc}} sgn(\gamma) wt_M(\gamma) \\ &+ \sum_{\gamma \in \Gamma'} sgn(\gamma) wt_M(\gamma),\end{aligned}$$

where Γ' is the multiset of those word systems which are not Lyndon covers. Therefore, we proceed by showing that the contribution of the word systems that are not Lyndon covers, the second sum, is zero.
It is straight forward to see that if $m_1 = m_2 = \cdots = m_n = 1$, then the second sum is automatically zero. If there is i with $m_i > 1$, then $N > n$ and by Observation 5.2 we know that $\overline{det}(A) = 0$. Hence, theorem follows if we show that the second sum is zero even when $N > n$.
We let $N(M) = \{(1, f_M(1)), (2, f_M(2)), \ldots, (N, f_M(N))\}$. If σ is a permutation of $[N]$, then we denote by $\sigma(M)$ the corresponding permutation of $N(M)$. Now, we recall Definition 5.3. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a permutation of $[N]$ and let $\gamma = \langle\gamma_1, \gamma_2, \ldots, \gamma_k\rangle$ be the corresponding word system. If $\sigma_i = (\sigma_{i,1}\sigma_{i,2}\cdots\sigma_{i,t_i})$, then $\gamma_i = (f_M(\sigma_{i,1})f_M(\sigma_{i,2})\cdots f_M(\sigma_{i,t_i}))$. We note that the permutation $\sigma(M)$ may be written as $\langle\sigma_1(M), \ldots, \sigma_k(M)\rangle$ where $\sigma_i(M) = ((\sigma_{i,1}, f_M(\sigma_{i,1}))\cdots(\sigma_{i,t_i}, f_M(\sigma_{i,t_i}))$.
We introduce the following notation. If $\sigma(M)$ is a permutation of $N(M)$ as above, then we denote by $f(\sigma(M))$ the permutation of M written as

$$\langle(f_M(\sigma_{1,1})\cdots f_M(\sigma_{1,t_1})), (f_M(\sigma_{2,1})\cdots f_M(\sigma_{2,t_2})), \ldots, (f_M(\sigma_{k,1})\cdots f_M(\sigma_{k,t_k}))\rangle.$$

Clearly, f is a sign-preserving bijection from the set of the permutations of $N(M)$ to the multiset of the permutations of M. We let $\Gamma'(M)$ be the set of all permutations $\gamma(M)$ of $N(M)$ so that $f(\gamma(M)) \in \Gamma'$. In order to prove Theorem 5.1, we need to show:
Claim 1.

$$\sum_{\gamma(M)\in\Gamma'(M)} sgn(\gamma(M))wt_M(f(\gamma(M))) = 0.$$

We note that the summation in the Claim 1 is over a set, not over a multiset as before.
Proof of Claim 1.
Let $\gamma(M) = \langle\gamma_1(M), \ldots, \gamma_k(M)\rangle \in \Gamma'(M)$ with $\gamma_i(M) = (g_{i,1}, \ldots, g_{i,t_i})$ and $g_{i,j} = (\sigma_{i,j}, \gamma_{i,j})$. First we define, for each $\gamma_i(M)$, its circular word $P(\gamma_i(M))$ as follows:

1. If $\gamma_i = (\gamma_{i,1}\cdots\gamma_{i,t_i})$ is primitive, then $P(\gamma_i(M)) = \gamma_i$.
2. If $\gamma_i = (\gamma_{i,1}\cdots\gamma_{i,t_i}) = (w^k)$, $k > 1$, w primitive, then we let $P(\gamma_i(M)) = (w)$.

We say that $\gamma_i(M)$ is *problematic* if $\gamma_i = (\gamma_{i,1}\cdots\gamma_{i,t_i})$ is not a primitive word or there exists $j \neq i$ so that $P(\gamma_i(M)) = P(\gamma_j(M))$. Next, let $P(\gamma(M)) =$

$min_{\preceq}(P(\gamma_i(M)); \;\; \gamma_i(M) \;$ problematic), where $\preceq$ is the lexicographic order on Lyndon representation of circular words.
Next we define type $T(\gamma(M))$ of a permutation $\gamma(M) = \langle \gamma_1(M), \ldots, \gamma_k(M) \rangle$ of $\Gamma'(M)$:

1. $T(\gamma(M)) = \bigcup_{i=1}^{k} T(\gamma_i(M))$.
2. If $P(\gamma_i(M)) \neq P(\gamma(M))$, then $T(\gamma_i(M)) = \{\gamma_i(M)\}$.
3. If $P(\gamma_i(M)) = P(\gamma(M))$ and $\gamma_i = (\gamma_{i,1} \cdots \gamma_{i,t_i})$ is primitive, then $T(\gamma_i(M))$
   $= \{\gamma_i(M)\}$.
4. If $\gamma_i = (\gamma_{i,1} \cdots \gamma_{i,t_i}) = (w^k)$, $k > 1$, w primitive, then $\gamma_i(M)$ may be written as $\gamma_i(M) = (w_1(M), \ldots, w_k(M))$, where if we denote $w_i(M) = w_{i,1}(M) \cdots w_{i,r_i}(M)$, $w_{i,j}(M) = (x_{i,j}, y_{i,j})$, then for each i, $(w) = (y_{i,1} \cdots y_{i,r_i})$.
   Then, we let $T(\gamma_i(M)) = \{(w_i(M)) : \;\; i = 1, \ldots, k\}$.

This finishes the definition of the type of $\gamma(M)$. The following observation is simple.
If $T(\gamma_1(M)) = T(\gamma_2(M))$, then

$$wt_M(f(\gamma_1(M))) = wt_M(f(\gamma_2(M))).$$

Hence, Claim 1 follows from the following Claim 2.

Claim 2.
Let $\gamma_0(M)$ be problematic. Then

$$\sum_{\gamma(M):T(\gamma(M))=T(\gamma_0(M))} sgn(\gamma(M)) = 0.$$

Proof of Claim 2.
$T(\gamma_0(M))$ is a set of cyclic words of biletters; we write $T(\gamma_0(M)) = \{v_1, \ldots, v_l\}$, $v_i = (v_{i,1} \cdots v_{i,s_i})$, $v_{i,j} = (a_{i,j}, b_{i,j})$. Let $c(\gamma_0(M)) = \{v_i : \ (b_{i,1}, ..., b_{i,s_i}) = P(\gamma_0(M))\}$. Note that $|c(\gamma_0(M))| > 1$ by definition of $P(\gamma_0(M))$, $\gamma_0(M)$ problematic. Claim 2 now follows, since the number of even permutations of $c(\gamma_0(M))$ is the same as the number of odd permutations of $c(\gamma_0(M))$.
This finishes the proof of Claim 2, Claim 1 and Theorem 5.1 ☐

Corollary 5.2. *For any integer $N > n$, we have*

$$\sum_{\gamma \in \Gamma_{lc}} sgn(\gamma) wt_M(\gamma) = 0.$$

Proof. This is a direct consequence of Observation 5.2 and Theorem 5.1. ☐

Remark 5.2. It is worth noting that the above corollary can be viewed as a multivariate generalization of the coin arrangements lemma.

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