Bass' Identity and The Coin Arrangements Lemma

Martin Loebl* and Hossein Teimoori Faal*

Abstract

Lemma on coin arrangements is an important trick in Sherman's proof of Feynman conjecture on two dimensional Ising model. Here, we show that the coin arrangements lemma is indeed equivalent to the Bass' identity.

1 Introduction

There is no doubt that the Riemann zeta function is a function of great significance in number theory because of its connection to the distribution of prime numbers. The Ihara-zeta function was first introduced by Yasutaka Ihara [1] in the 1960s in the context of discrete subgroup of the two-by-two p-adic special linear group. Jean-Pierre Serre suggested in his book "Trees" that Ihara's original definition can be reinterpreted in graph-theoretical setting. It was Toshikazu Sunada [2] who put this suggestion into practice in 1985. Indeed, the Ihara-zeta function is a zeta function associated with a finite graph which is used to relate closed walks in a graph to the spectrum of the adjacency matrix of the line graph of its symmetric digraph. This connection is via an identity which is well-known as Bass' identity [3]. It is shown in the book [4] that using this identity and some other ideas from linear algebra one can come up with the graph-theoretical analogue of the prime number theorem. Therefore, it seems that the Bass' identity plays an essential role in the new theory of combinatorial zeta function of a finite graph. Since the orginal work of Bass [3], his identity has been proven several times in an inspiring way. We mention at least [13] and one lecture of

*Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI), Charles University, Malostranske Namesti 25, 11800 Praha 1, Czech Republic

Xavier Viennot at Newton Institute, Cambridge University in 2008.
In this paper we try to add one more understanding of this Bass' identity. First, while reviewing the basic concepts of the theory of zeta function of finite graphs we show how one can derive a new proof of Witt identity, using the Bass identity. In the next step, we will work in another direction by presenting a new proof of the Bass identity by only slight modifications to the approach that has been developed by Feynman and Sherman as the *path method* for combinatorial solution of two dimensional *Ising problem* $[5-8]$. Lemma on coin arrangements [7] which is indeed equivalent to the Witt identity [9] is the key in this new proof.

2 Witt Identity and The Coin Arrangements Lemma

We first briefly recall some definitions from the combinatorics of words. The reader may consult the book [10]. Let A be a finite set, as our alphabet. The set of all finite sequences of elements (letters) of A will be denoted by $A^\star$. Each element of $A^\star$ is called a *word* over the alphabet A. In particular, an empty sequence of letters is also an element of $A^\star$ which is called an *empty word* and is denoted by 1. The length of a word w is defined as the number of its letters and is dented by $|w|$. The set of all nonempty words over A will be denoted by A^+:

$$A^+ = A^\star - 1.$$

It is called the *free semigroup* over A (recall that a semigroup is a set with an associative binary operation). Sometimes, for the set $A = \{a_1, \dots, a_n\}$, it is also called a free semigroup generated by the generators $a_1, \dots, a_n$. A word v is called a *right factor* of a word w if there exists a third word u such that $w = uv$. If u is a nonempty word then the right factor v is called the *proper* right factor of the word w. Recall that two words x and y are said to be *conjugate* if there exists words $u, v \in A^\star$ such that

$$x = uv, \quad y = vu.$$

This is an *equivalence relation* on $A^\star$ since x is conjugate of y if and only if y can be obtained by a *cyclic permutation* of the letters of x. It partitions the $A^\star$ into disjoint conjugate classes. A word is said to be *primitive* if it is not a power of another word. We also recall that a *Lyndon word* is a primitive word that is minimal, with respect to lexicographic order, in its conjugate class. The set of all Lyndon words will be denoted by $\mathcal{L}$. We also need to recall some properties of Lyndon words. For proofs see the book [10]. We have the following characterization for the set of the Lyndon words.

Proposition 2.1. *A word $w \in A^+$ is a Lyndon word if and only if it is strictly smaller than any its proper right factor:*

$$w \in \mathcal{L} \Leftrightarrow \{\forall v \in A^+,\ w \in A^+ v \Rightarrow w \prec v\}.$$

We have the following fundamental result which is known as the *Lyndon factorization theorem.*

Theorem 2.2. *Any word $w \in A^+$ may be written uniquely as a nonincreasing product of Lyndon words:*

$$w = l_1 l_2 \cdots l_n,\ l_i \in \mathcal{L},\quad l_1 \succeq l_2 \succeq \cdots \succeq l_n.$$

We also need few more definitions and results from the theory of numbers [11]. Recall that the classical Möbius function $\mu : \mathbb{N} \mapsto \{-1, 0, 1\}$ is defined as follows:

- $\mu(1) = 1$;
- $\mu(n) = 0$ if n has a squared factor;
- $\mu(p_1 p_2 \cdots p_k) = (-1)^k$ if all the primes $p_1, p_2, \ldots, p_k$ are different.

We also recall the following classical number theoretic Möbius inversion formula.

Proposition 2.3 (Möbius inversion formula)**.** *If $g(n)$ and $f(n)$ are functions on natural numbers satisfying*

$$g(n) = \sum_{d|n} f(d) \quad \text{for every integer } n \geq 1,$$

then

$$f(n) = \sum_{d|n} \mu(\frac{n}{d}) g(d) = \sum_{d|n} \mu(d) g(\frac{n}{d}) \quad \text{for every integer } n \geq 1.$$

Before stating the Witt identity, we need a key lemma. We remind that a word $a_1 \cdots a_n$ is called a *circular* or *cyclic* word if a_1 is regarded as following a_n, where $a_1 \cdots a_n$, $a_2 \cdots a_n a_1$ and all other cyclic shifts (rotations) of $a_1 \cdots a_n$ are regarded as the same word. Also recall that a circular word of length n may be considered as a word given by repeating a segment of d letters $\frac{n}{d}$ times, where d is a divisor of n. Then we say that word is of *period* d. Each word has unique smallest period; *minimal period*. For the sake of completeness we also include the proof of the key lemma [9].

Lemma 2.4. *Let $A = \{a_1, \ldots, a_k\}$ be our alphabet. Let $M(m_1, \ldots, m_k)$ be the number of circular words of length $m = m_1 + \cdots + m_k$ and of minimal period m (often called primitive words) such that a_i occurs m_i times, $i = 1, 2, \ldots, k$. Then, we have*

$$M(m_1, \ldots, m_k) = \frac{1}{m} \sum_{d | gcd(m_1, \ldots, m_k)} \mu(d) \binom{\frac{m}{d}}{\frac{m_1}{d}, \cdots, \frac{m_k}{d}}. \tag{2.1}$$

Proof. It is clear that the total number of ordinary words such that each a_i occurs m_i times is equal to the multinomial coefficient $\binom{m}{m_1, \cdots, m_k}$. Now, since every word w has unique smallest period, we get

$$\binom{m}{m_1, \cdots, m_k} = \sum_{d | gcd(m_1, \ldots, m_k)} \frac{m}{d} M(\frac{m_1}{d}, \ldots, \frac{m_k}{d}).$$

Thus, it follows by Möbius inversion formula of Proposition 2.3 that

$$M(m_1, \ldots, m_k) = \frac{1}{m} \sum_{d | gcd(m_1, \ldots, m_k)} \mu(d) \binom{\frac{m}{d}}{\frac{m_1}{d}, \cdots, \frac{m_k}{d}},$$

as required. □

Note that it is clear by the definition of circular words of length and period m and the definition of Lyndon words of length m that $M(m_1, \ldots, m_k)$ also equals to the number of Lyndon words of length $m = m_1 + \cdots + m_k$ over the alphabet A with exactly m_i copies of a_i, $i = 1, 2, \ldots, k$.
Now, we state the Witt identity which is an algebraic identity in the context of the Lyndon words [9]. For the sake of completeness, we also include a short proof, using the Lyndon factorization theorem (see [9]).

Proposition 2.5 (The Witt Identity). *Let $a_1, \ldots, a_k$ generate a free semigroup. Let $M(m_1, \ldots, m_k)$ be the number of Lyndon words with m_1 occurrences of a_1, m_2 occurrences of a_2, etc. Let $z_1, \ldots, z_k$ be commuting variables. Then*

$$\prod_{m_1,\ldots,m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1,\ldots,m_k)} = 1 - z_1 - \cdots - z_k. \tag{2.2}$$

Proof. Using the geometric series formula and the Lyndon factorization theorem, we have

$$\begin{aligned} \frac{1}{1 - z_1 - \cdots - z_k} &= \sum_{\omega \in \{z_1,\ldots,z_k\}^\star} \omega = \prod_{l \in \mathcal{L}} \frac{1}{1-l} \\ &= \frac{1}{\prod_{m_1,\ldots,m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1,\ldots,m_k)}}. \end{aligned}$$

□

In the combinatorial solution of the two dimensional Ising problem via path method, the following coin arrangements lemma [7] plays a key role.

Proposition 2.6. *Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind, $\ldots$, and m_n of n-th kind. Let $b_{N,k} = b(N, k; m_1, \ldots, m_n)$ be the number of exhaustive unordered arrangements of these symbols into k disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. Then, we have*

$$\sum_{k=1}^{N} (-1)^k b_{N,k} = 0, \qquad (N > 1).$$

There are many interesting proofs of the coin arrangements lemma [12]. Here, we present the original proof of Sherman [7] which uses the Witt identity. Indeed, it shows that that the coin arrangements lemma and the Witt identity are equivalent.

Proof. (The Coin Arrangements Lemma) If we expand the left hand side of (2.2), we get
$\prod_{m_1,\ldots,m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1,\ldots,m_k)}$
$= 1 - \sum_{N \geq 1} \sum_{c_1,\ldots,c_k \geq 0 : \sum c_i = N} \left(\sum_{i=1}^{N} (-1)^i b(N, i; c_1, \ldots, c_k) \right) z_1^{c_1} \cdots z_k^{c_k}.$

By comparing the corresponding coefficients in both sides of the Witt identity, we obtain the desired identity. □

3 Primes in Graphs and Combinatorial Zeta Function

In the combinatorics of words, the Lyndon words are primes of the words. The Lyndon factorization theorem says that any word can be written uniquely as a non-increasing product (concatenation) of the Lyndon words.

Definition 3.1. For a given graph $G = (V, E)$, we define its symmetric digraph $\vec{G}$ by replacing each undirected edge e with two arcs a_e and a_e^{-1} which are in opposite directions. We denote by $E(\vec{G})$ the edge-set of $\vec{G}$.

Definition 3.2. A *path* or a *walk* $C = a_{e_1} \cdots a_{e_t}$ in $\vec{G}$, where a_{e_i} is a an oriented edge of G, is said to be *closed* if the starting vertex is the same as the terminal vertex.

Definition 3.3. A prime in a finite graph G is an equivalence class $[C]$ of the closed walk $C = a_{e_1} \cdots a_{e_t}$, in $\vec{G}$ such that:

- it has no backtracking: which means no arc is followed immediately by the arc with the same ends but in the opposite direction, i.e., there is no $j = 1, \ldots, t-1$ with $a_{e_{j+1}} = a_{e_j}^{-1}$.
- it has no tail: which means the starting and the ending arcs are not two arcs with the same ends and just opposite directions, i.e., $a_{e_t} \neq a_{e_1}^{-1}$.
- it is primitive: which means there is no other closed walk C_0 with $C = C_0^l$ for some integer $l > 1$.
- orientation counts: which means the oriented walk C and the walk obtained from it by reversing all the directions are not in the same equivalence class.
- starting point does not count: which means the equivalence class is with respect to cyclic permutations of the vertices of the closed walk C.

Definition 3.4. The *size* (length) of a prime is defined, as follows

$$|C| = \nu(C) = \nu([C]) = \text{the number of arcs of } C.$$

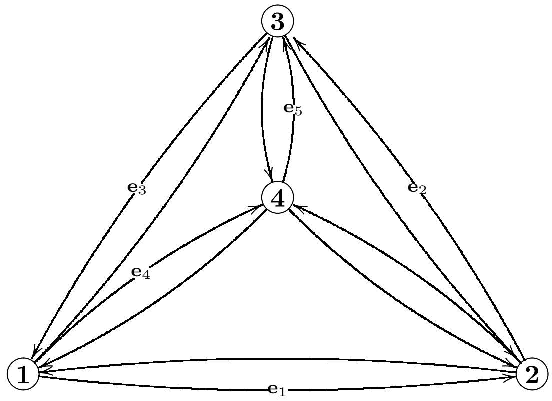

Fig1. Examples of Primes in K_4

Example 3.1. *Consider the symmetric digraph of the complete graph on* 4 *vertices* K_4*, as shown in Fig 1. Two examples of primes are:*

$$[C_1] = [e_1e_2e_3] = \{e_1e_2e_3, e_3e_1e_2, e_2e_3e_1\},$$

and

$$[C_2] = [e_1e_2e_3e_4e_5e_3].$$

Definition 3.5. Ihara-zeta function $\zeta(u, G)$ of a finite graph $G = (V, E)$ is defined, as follows

$$\zeta(u, G) = \prod_{[C]:\ prime} \left(1 - u^{\nu(C)}\right)^{-1},$$

where u is a formal variable.

Definition 3.6. The *adjacency matrix of a digraph* (with loops and multiple arcs) is the integer-valued matrix with rows and columns corresponding to the digraph vertices, where a nondiagonal entry a_{ij} is the number of arcs from vertex i to vertex j, and the diagonal entry a_{ii} is the number of loops at vertex i.

Definition 3.7. The *edge matrix* of a finite graph $G = (V, E)$, is a matrix T of order $E(\overrightarrow{G}) \times E(\overrightarrow{G})$ with (i, j)-entry equal to 1 if edge e_i feeds edge e_j, i.e., the end vertex of e_i is the starting vertex of e_j and $a_{e_i} \neq a_{e_j}^{-1}$. Otherwise the (i, j)-entry is 0.

Definition 3.8. The line digraph of the symmetric digraph of G denoted by $L(\overrightarrow{G})$ is a digraph which has one vertex for each arc of $\overrightarrow{G}$. Two vertices representing the arcs xy and uv are connected by an arc from xy to uv in the line graph when $y = u$, provided that $xy \neq uv^{-1}$.

Note that it is clear by the Definition 3.8 that the edge matrix T of G is merely the adjacency matrix of the graph $L(\overrightarrow{G})$. Now, we are at the position to state Bass' theorem for Ihara-zeta function of a finite graph.

Theorem 3.1 (Bass' Identity)**.**

$$\zeta^{-1}(u, G) = det(I - uT).$$

There are many proofs of the Bass' identity in the literatures. In [13] three different combinatorial proofs, all based on the algebra of Lyndon words have been given.
Next, we introduce a weighted version of the Ihara-zeta function of a finite graph which is called the edge zeta function [4]. From now on, we also extend the graphs we consider.

Definition 3.9. From now on, we will allow loops in the edge-set of graph G. We assume each loop e has only one orientation a_e; hence, such a_e has no opposite orientation. In the symmetric digraph $\overrightarrow{G}$ we add a_e for each loop e to $E(\overrightarrow{G})$.

Definition 3.10. The *weighted edge matrix* of the finite graph $G = (V, E)$, is a matrix W of order $E(\overrightarrow{G}) \times E(\overrightarrow{G})$ with (i, j)-entry equal to $w_{a_i a_j}$ if edge a_i feeds edge a_j, otherwise the (i, j)-entry is 0. Here, w_{ab} is a formal variable corresponding to the pair of arcs (a, b).

Definition 3.11. Let $C = a_1 \cdots a_m$ be a closed walk in finite graph $G = (V, E)$. We define the *edge norm* $N_E(C)$ of C, as follows

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{m-1} a_m} w_{a_m a_1}.$$

Now, we are ready to give the weighted version of the Ihara-zeta function of a finite graph G.

Definition 3.12. The edge zeta function $\zeta_E(W, G)$ of a finite graph $G = (V, E)$ is defined, as follows

$$\zeta_E(W, G) = \prod_{[P]:\ prime} (1 - N_E(P))^{-1}.$$

Remark 3.1. If we set $w_{ab} = u$ for every pair of arcs (a, b), we get the (standard) Ihara-zeta function of G.

We define the multivariate zeta function, as a specialization of the edge-zeta function (see [13]).

Definition 3.13. The *multivariate edge matrix* of the finite graph $G = (V, E)$, is a matrix W_{mult} of order $E(\overrightarrow{G}) \times E(\overrightarrow{G})$ with (a, b)-entry equal to $\sqrt{w_a}\sqrt{w_b}$ if arc a feeds arc b. Here,each w_a is a formal variable corresponding to the weight of the arc a.

Definition 3.14. The multivariate zeta function $\zeta_E(W_{mult}, G)$ of a finite graph $G = (V, E)$ is defined, as follows

$$\zeta_E(W_{mult}, G) = \prod_{[P=a_1\cdots a_m]:\ prime} (1 - w_{a_1} \cdots w_{a_m})^{-1}.$$

We arrive at the following weighted versions of the Bass' identity (see [13]).

Theorem 3.2. *(Weighted Bass' Identities)*

$$\zeta_E^{-1}(W, G) = det(I - W),$$

$$\zeta_E^{-1}(W_{mult}, G) = det(I - W_{mult}).$$

We conclude this section by noting that we can rewrite the multivariate Ihara-zeta function in the following form

$$\zeta_E(W_{mult}, G) = \prod_{m_1,\ldots,m_k \geq 0} (1 - w_1^{m_1} \cdots w_k^{m_k})^{-\pi_G(m_1,\ldots,m_k)}, \tag{3.1}$$

where $\pi_G(m_1, \ldots, m_k)$ is the number of primes of G of length $|m| = m_1 + \cdots + m_k$ with exactly m_i copies of the arc with weight w_i, $i = 1, 2, \ldots, k$.

4 Witt Identity Via Bass' Identity

In this section we show how one can get the Witt identity from the weighted Bass' identity. First we need to fix some notations. Here, we will use the idea of *multi-index notation* [14].

Definition 4.1. An m-dimensional *multi-index* is a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ of non-negative integers. For multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^m$, where $\mathbb{N}_0$ denotes the set of non-negative integers and $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ we define:

Scalar multiplication $r\boldsymbol{\alpha} = (r\alpha_1, \ldots, r\alpha_m)$, where r is a real scalar.

Sum of components (absolute value) $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_m$.

Factorial $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_m!$.

Power $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Multinomial coefficient $\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} = \frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}$.

Thus the *multinomial formula* can be read, as follows

$$(x_1 + \cdots + x_m)^k = \sum_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|=k} \binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}. \tag{4.1}$$

We first need few definitions. Here, we denote the set $\{1, \ldots, n\}$ by $[n]$.

Definition 4.2. Let $A = (a_{ij})$ be a square matrix of order n. The *associated digraph* of the matrix A denoted by $D(A)$ is a digraph with vertex set $[n]$ and there is an arc from vertex i to vertex j $(i, j \in [n])$, if $a_{ij} \neq 0$. For vertex i with $a_{ii} \neq 0$, we have a loop at this vertex.

Definition 4.3. Let $W = a_1 \cdots a_t$ be a walk in $D(A)$. The weight of the walk W is defined as the product of the weights of its arcs. The weight of an arc (i, j) is defined as a_{ij}.

Lemma 4.1. *Let $A = (a_{ij})$ be a square matrix of order n. For each positive integer k, the entry a_{ij}^k of A^k equals to the sum of the of the weights of all walks in $D(A)$ of length k from vertex i to vertex j.*

Proof. (Witt Identity) We start with the multivariate Ihara-zeta function $\zeta_E(W_{mult}, G)$ of G. Now, by putting $A = W_{mult}$ in Lemma 4.1, we get

$$\sum_{\boldsymbol{m}:|\boldsymbol{m}|=m} N(\boldsymbol{m})\boldsymbol{w}^{\boldsymbol{m}} = Tr(W_{mult}^m), \tag{4.2}$$

where $N(\boldsymbol{m})$ is the number of the closed walks (not necessarily prime) of length $|\boldsymbol{m}| = m$ without backtracking or tails that we count both starting point and orientation with exactly m_i copies of the arc with weight w_i, $i = 1, 2, \ldots, k$. It can be easily seen that by the similar argument as in the proof of Lemma 2.4, we get

$$N(\boldsymbol{m}) = \sum_{d|gcd(m_1,\ldots,m_k)} \left|\frac{\boldsymbol{m}}{d}\right| \pi_G\Big(\frac{\boldsymbol{m}}{d}\Big), \tag{4.3}$$

where $\pi_G(\boldsymbol{m})$ denotes the number of primes in G of length $|\boldsymbol{m}| = m_1 + \cdots + m_k$ with exactly m_i copies of the arc with weight w_i, $i = 1, 2, \ldots, k$. Therefore, the number theoretic Möbius inversion formula implies

$$\pi_G(\boldsymbol{m}) = \frac{1}{|\boldsymbol{m}|} \sum_{d|gcd(m_1,\ldots,m_k)} \mu(d) N\Big(\frac{\boldsymbol{m}}{d}\Big). \tag{4.4}$$

We use

$$Tr(W_{mult}^m) = \sum_{\lambda:\text{ eigenvalue of } W_{mult}} \lambda^m, \tag{4.5}$$

$$det(I - W_{mult}) = \prod_{\lambda:\text{ eigenvalue of } W_{mult}} (1-\lambda). \tag{4.6}$$

Now we apply the machinery to particular graph G consisting of 1 vertex and k loops $e_1, \ldots, e_k$ attached to this vertex. Let a_i denote the directed loop a_{e_i} and let w_i be the weight of a_i.

Lemma 4.2. *The only non-zero eigenvalue of $W_{mult}(G)$ is $(w_1 + \cdots + w_k)$.*

Proof. For our special graph G the entries a_{ij} of $W_{mult}(G)$ satisfy $a_{ij} = \sqrt{w_i w_j}$. Let $\sqrt{w}$ denote the vector $(\sqrt{w_1}, \ldots, \sqrt{w_k})$. We have $W_{mult}(G) = \sqrt{w}\sqrt{w}^T$. The dimension of the kernel of $W_{mult}(G)$ is $k-1$ and thus the multiplicity of the eigenvalue $\lambda_1 = 0$ is $k-1$. Finally, for the last eigenvalue we have $W_{mult}(G)\sqrt{w} = (\sqrt{w}\sqrt{w}^T)\sqrt{w} = \sqrt{w}(\sqrt{w}^T\sqrt{w}) = \lambda\sqrt{w}$ where $\lambda = \sqrt{w}^T\sqrt{w} = (w_1 + \cdots + w_k)$. $\square$

As a corollary, we have

$$Tr(W_{mult}(G)^m) = \sum_{\lambda:\text{ eigenvalue of } W_{mult}(G)} \lambda^m = (w_1 + \cdots + w_k)^m, \tag{4.7}$$

$$det(I - W_{mult}(G)) = \prod_{\lambda:\text{ eigenvalue of } W_{mult}} (1-\lambda) = 1 - (w_1 + \cdots + w_k). \tag{4.8}$$

for all positive integers m. Considering formulas (4.2) and (4.7), we get

$$\sum_{\boldsymbol{m}:|\boldsymbol{m}|=m} N(\boldsymbol{m})\boldsymbol{w}^{\boldsymbol{m}} = (w_1 + \cdots + w_k)^m.$$

Now, by multinomial formula (4.1), we get the identity

$$\sum_{\boldsymbol{m}:|\boldsymbol{m}|=m} N(\boldsymbol{m})\boldsymbol{w}^{\boldsymbol{m}} = \sum_{\boldsymbol{m}:|\boldsymbol{m}|=m} \binom{|\boldsymbol{m}|}{\boldsymbol{m}} \boldsymbol{w}^{\boldsymbol{m}},$$

which implies that

$$N(\boldsymbol{m}) = \binom{|\boldsymbol{m}|}{\boldsymbol{m}}.$$

Note that we can also obtain $N(\boldsymbol{m})$ by a simple counting argument on our special graph G. Now, formula (4.4) yields

$$\pi_G(\boldsymbol{m}) = \frac{1}{|\boldsymbol{m}|} \sum_{d|gcd(m_1,\cdots,m_k)} \mu(d) \binom{|\frac{\boldsymbol{m}}{\boldsymbol{d}}|}{\frac{\boldsymbol{m}}{\boldsymbol{d}}}. \tag{4.9}$$

Note that by formula (2.1), $\pi_G(\boldsymbol{m})$ is indeed the number of Lyndon words over the alphabet $A = \{w_1, \ldots, w_k\}$ of length $|\boldsymbol{m}| = m_1 + \cdots + m_k$, with exactly m_i copies of the letter w_i, $i = 1, 2, \ldots, k$. Finally using the multivariate Bass' identity for zeta function in formula (3.1), formula (4.8) and formula (4.9), we get the following identity

$$\prod_{m_1,\ldots,m_k \geq 0} (1 - w_1^{m_1} \cdots w_k^{m_k})^{\pi(m_1,\ldots,m_k)} = 1 - w_1 - \cdots - w_k,$$

where $\pi(m_1, \ldots, m_k) = \frac{1}{m} \sum_{d|gcd(m_1,\ldots,m_k)} \mu(d) \binom{\frac{m}{d}}{\frac{m_1}{d}, \cdots, \frac{m_k}{d}}$. This is exactly the Witt identity, as desired. $\square$

5 Bass' Identity Using the Coin Arrangements Lemma

The first step is to rewrite the product form of the inverse of the Ihara-zeta function as an (infinite) sum, as follows

$$\prod_{[C]:\ prime} \left(1 - u^{\nu(C)}\right) = \sum_{\gamma \in \Gamma} (-1)^r u^{\sum |\gamma_i|}$$

where Γ is the set of all $\gamma = (\gamma_1, \dots, \gamma_r)$ in which γ_i's are primes in G and $|\gamma_i|$ is the length of γ_i.
In the next step, we divide the above sum into two individual sums. Namely,

$$\sum_{\gamma \in \Gamma} (-1)^r u^{\sum |\gamma_i|} = \sum_{\gamma \in \Gamma_1} (-1)^r u^{\sum |\gamma_i|} + \sum_{\gamma \in \Gamma_2} (-1)^r u^{\sum |\gamma_i|},$$

where now, Γ_1 is the finite set of all $\gamma = (\gamma_1, \dots, \gamma_r)$ in which each directed edge of $\overrightarrow{G}$ is traversed by γ *at most once* and Γ_2 is the set of all $\gamma = (\gamma_1, \dots, \gamma_r)$ where at least one of the directed edges in $\overrightarrow{G}$ is traversed by γ *at least twice*.
In the third step, we show that the first sum in the right-hand side of the above equality is indeed equal to the determinant appearing in the Bass' identity. To do this, we need the following observations. Recall the definition of the line graph $L(\overrightarrow{G})$.
Observation 5.1

$$\sum_{\gamma \in \Gamma_1} (-1)^r u^{\sum |\gamma_i|} = \sum_{\gamma' \in \Gamma'_1} (-1)^r u^{\sum |\gamma'_i|},$$

where Γ'_1 is the finite set of all $\gamma' = (\gamma'_1, \dots, \gamma'_r)$ which are the disjoint union of (directed) cycles in $L(\overrightarrow{G})$.
The reason is that clearly a cycle in $L(\overrightarrow{G})$ is a closed walk with distinct edges.
Observation 5.2

$$\sum_{\gamma' \in \Gamma'_1} (-1)^r u^{\sum |\gamma'_i|} = det(I - uT),$$

where Γ'_1 is the set of all $\gamma' = (\gamma'_1, \dots, \gamma'_r)$ which are the disjoint union of directed cycles in $L(\overrightarrow{G})$.
Indeed, the above observation is the special case the following well-known graph-theoretical interpretation of determinants which is a direct consequence of the cyclic decomposition of permutations. Here, $D(A)$ is the associated digraph of matrix A as in the Definition 4.2.
Observation 5.3

$$det(I - A) = \sum_{\gamma \in \Omega} sgn(\gamma) wt(\gamma),$$

where Ω is the set of all $\gamma = (\gamma_1, \ldots, \gamma_r)$ which are the disjoint union of directed cycles of $D(A)$, $sgn(\gamma) = (-1)^r$ and $wt(\gamma)$ is defined as the product of weights of its directed cycles and the weight of any directed cycle $\gamma_i = i_1 i_2 \cdots i_{l-1} i_l$ of length l is defined as $wt(\gamma_i) = a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{l-1},i_l} a_{i_l,i_1}$, in which $a_{i,j}$ is the (i,j)-entry of the matrix A.

Remark 5.1. For a generalization of Observation 5.3 , see the paper [15].

Now it is clear that if we choose $A = uT$ and consider the fact that T is merely the adjacency matrix of the line graph $L(\overrightarrow{G})$, then we obtain Observation 5.2 as a direct consequence of Observation 5.3. Therefore, we have

$$\prod_{[C]:\ prime} \left(1 - u^{\nu(C)}\right) = det(I - uT) + \sum_{\gamma \in \Gamma_2} (-1)^r u^{\sum |\gamma_i|}.$$

Thus, the last step of the proof is to show that $\sum_{\gamma \in \Gamma_2} (-1)^r u^{\sum |\gamma_i|} = 0$.

We associate an indeterminate variable x_a with each arc a of $\overrightarrow{G}$. We define the weight of circular word $p = v_1, a_1, v_2, a_2, \ldots, a_n, v_{n+1} = v_1$ which is a prime in G, denoted by $X(p)$, to be $X(p) = \prod_{i=1}^{n} x_{a_i}$. Note that weight is invariant under cyclic permutation. We prove that if $\prod_{[p]:\ \mathrm{prime}} (1 - X(p))$ is expanded as a sum of monomials, any monomial with factor $x_a^k,\ k > 1$, has a zero coefficient. Let a_1 be a directed edge of $\overrightarrow{G}$. We set A_1:= the set of all primes p such that a_1 appears in p. Then, we claim that:

C1. $\prod_{p \in A_1} (1 - X(p)) = 1 - x_{a_1} d_{11}$,

where d_{11} is a formal (possibly infinite) sum of monomial summands none of which has x_{a_1} as a factor.

Proof. (of C1). Each prime p can be uniquely decomposed into closed walks $(W_1, \ldots, W_r)$ each of which starts with a_1 and has no other appearance of a_1. Now, let $B(p)$ denote the multiset $(W_1, \ldots, W_r)$. We call these walks *stones*. In the sum below, Ω denotes a finite set of primes in A_1.

$$\begin{aligned}
\prod_{p \in A_1} (1 - X(p)) &= \sum_{\Omega} (-1)^{|\Omega|} \prod_{p \in \Omega} X(p) \\
&= \sum_{\Omega} (-1)^{|\Omega|} \prod_{p \in \Omega} \cdot \prod_{W \in B(p)} X(W) \\
&= \sum_{\mathfrak{S}} \alpha(\mathfrak{S}) \prod_{W \in \mathfrak{S}} X(W),
\end{aligned}$$

where $\mathfrak{S}$ is a finite multiset of stones and $\alpha(\mathfrak{S}) = \sum_{R} (-1)^{|R|}$ where the sum is over all unordered arrangements of the stones of $\mathfrak{S}$ into collections of distinct primes. Now, it follows form the coin arrangements lemma that $\alpha(\mathfrak{S}) = 0$ whenever $|\mathfrak{S}| > 1$. This proves $C1$. $\square$

As a corollary of $C1$, we have the following:
If $\prod_{[p]:\ \mathrm{prime}}(1 - X(p))$ is expanded as a sum of monomials, there is no monomial having factor $x_{a_1}^k$, $k > 1$. This completes the proof since the arc a_1 was chosen arbitrarily.

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