Cover quasi-uniformities in frames

Jorge Picado*† Aleš Pultr‡

Abstract

Quasi-uniformities (not necessarily symmetric uniformities) are usually studied via entourages (special neighbourhoods of the diagonal in $X \times X$) where one can simply forget about the symmetry requirement. This has been done successfully in the point-free context as well, but there is a demand for a covering approach, a.o. because the point-free representation of the square $X \times X$ is not without difficulties. Based on the (spatial) ideas from [9], a cover type quasi-uniformity was developped in [6] and other papers using biframes, the point-free variant of bitopologies. In this paper we show that this can be avoided and present a cover type quasi-uniformity structure enriching that of frame directly.

Introduction

In the classical context the structure of uniformity on a space is approached, basically, in two different ways. There is the Weil's definition that goes as far back as 1937 ([25]). In this approach, the uniformity is given as a suitable system of neighbourhoods of the diagonal (entourages), expressing something like uniformly similar distances of points. In the other one, not much younger (Tukey [24], 1940) the uniformity is given as a system of covers, each individual of them expressing similarity of sizes of neighbourhoods of

*CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal, picado@mat.uc.pt

†Thanks go to the Centre for Mathematics of the University of Coimbra (CMUC/FCT), the Ministry of Education and Science of Spain and FEDER under grant MTM2009-12872-C02-02 and to the project 1M0545 of the Ministry of Education of the Czech Republic.

‡Department of Applied Mathematics and ITI, MFF, Charles University, Malostranské nám. 24, 11800 Praha 1, Czech Republic, pultr@kam.ms.mff.cuni.cz

distinct points. It should be noted that in the first decades, topologists preferred the former one, and it was the celebrated Isbell's monograph [10] (1964) that brought the cover definition to a deserved focus of interest. Anyway, both of the definitions are very natural and intuitively satisfactory, and they are equivalent, which is very easy to prove.

That is, the two definitions are equivalent, but the Weil's one allows for a very useful generalization (desirable for instance when dealing with non-commutative topological groups, but not only for that). Namely, one can, without any complications, drop the symmetry, and investigate thus obtained structure (Nachbin [14], [15]; see also [13]) which has some surprising features, for instance, does not require complete regularity of the underlying space. The cover definition, however, is inherently symmetric. A non-symmetric modification presented by Gantner and Steinlage in 1972 ([9]) has not found a general response, probably because in the classical setting (unlike in the point-free one, as we will see shortly) there has not been much demand for it.

In the point-free context the uniformity appeared first in the cover variant (Isbell [11], and then in a simpler form [20], [2], and others). Here the structure is perhaps (if possible) even more desirable then in the classical context, and the cover approach is the first that comes to the mind (granulation of the generalized space, suitable definition of "arbitrarily small" when defining points as filters containing arbitrarily small non-zero elements). But of course the entourage definition is also of a considerable interest ([16], [17], [3], see also [4]). Again it can be proved that the cover and entourage definitions are equivalent. But unlike in the classical case this is not such a commonplace fact (and may be indeed thought of as a surprise): the square of the underlying locale (coproduct of the frame with itself) is, first, by far not such a simple structure as the square of spaces $X \times X$, and, second and more important, in the spatial case it does not generally correspond to the space $X \times X$.

Again, one can very naturally extend the entourage definition to obtain not necessarily symmetric quasi-uniformities. But in the point-free context, the cover approach is in a sense much more natural than the entourage one and hence, unlike in the classical case it is now worthwhile to try to modify it for nonsymmetry and pay the price of some complications. The problem was successfully attacked by Frith ([6], [7]) using the analogue of Gantner and Steinlage paircovers [9]. There is a drawback, however: unlike the entourage quasi-uniformity that is defined on a frame as desired, this happens on a more involved carrier, a biframe. In this paper we endeavour to overcome this obstacle and present (and analyze) a variant of paircover definition avoiding the need of the biframe structure, and thus being genuinely equivalent to the entourage definition while in the same time keeping to the "natural cover" structure of a frame (locale).

The paper is divided into four sections. In the first one, Preliminaries, we recall the definitions of concepts necessary for studying uniformities and quasi-uniformities in the point-free context. The second one contains the new definition and proofs of technical properties. In the third one it is confronted with the biframe approach, and in the last one with the situation in classical spaces.

1 Preliminaries

1.1. Frames and biframes. Recall that a *frame* is a complete lattice satisfying the distributivity law $(\bigvee_{i\in J} a_i)\wedge b = \bigvee_{i\in J}(a_i)\wedge b)$ and that *frame homomorphisms* preserve all joins and finite meets. A *biframe* [1] is a triple (L, L_1, L_2) in which L_0 is a frame, L_1 and L_2 are subframes of L_0 and $L_1 \cup L_2$ generates L_0 (by joins of finite meets); a *biframe homomorphism* $h : (L, L_1, L_2) \to (M, M_1, M_2)$ is a frame homomorphism from L_0 to M_0 such that the image of L_i $(i = 1, 2)$ under h is contained in M_i. Biframes and biframe homomorphisms are the objects and morphisms of the category BiFrm. [12] and [22]. For more about frames the reader can consult [12] or [22], for biframes see [1] and [23].

1.2. (Cover) uniformities. A *cover* of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$. A cover A *refines* a cover B and we write

$$A \leq B$$

if for each $a \in A$ there is $b \in B$ such that $a \leq b$. Further one defines

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\};$$

note that it is a common refinement of A and B, maximal in the preorder $\leq$ of covers.

For a cover A of L and an element $x \in L$ set

$$Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\} \quad \text{and for two covers } A, B \text{ set} \quad AB = \{Ab \mid b \in B\}.$$

Let $\mathcal{A}$ be a system of covers on L. The relation $\stackrel{\mathcal{A}}{\triangleleft}$ on L is defined by setting

$$x \stackrel{\mathcal{A}}{\triangleleft} y \quad \equiv \quad \exists A \in \mathcal{A},\ Ax \leq y.$$

A system of covers $\mathcal{A}$ of L is said to be *admissible* if

$$\forall x \in L, \quad x = \{y \mid y \stackrel{\mathcal{A}}{\triangleleft} x\}.$$

1.2.1. A *(cover) uniformity* [20] on a frame L is an admissible system of covers $\mathcal{A}$ such that

(Uc1) $A \in \mathcal{A}$ and $A \leq B \quad \Rightarrow \quad B \in \mathcal{A}$,

(Uc2) $A, B \in \mathcal{A} \quad \Rightarrow \quad A \wedge B \in \mathcal{A}$,

(Uc3) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $BB \leq A$.

A (cover) uniform frame is a couple $(L, \mathcal{A})$ where L is a frame and $\mathcal{A}$ is a uniformity on L. Let $(L, \mathcal{A})$, $(M, \mathcal{B})$ be uniform frames. A frame homomorphism $h : A \to B$ is *uniform* if for each $A \in \mathcal{A}$, $h[A] \in \mathcal{B}$.

1.3. (Entourage) uniformities. An *entourage* [16] of a frame L is an element E of the coproduct $L \oplus L$ (tsuch elements are special downsets in $X \times X$, see e.g. [22]) for which $\bigvee\{a \in L \mid (a, a) \in E\} = 1$. The inverse of E is the entourage $E^{-1} = \{(b, a) \mid (a, b) \in E\}$.

For two entourages E, F set

$$E \circ F = \bigvee\{a \oplus c \in L \oplus L \mid \exists b \in L \smallsetminus \{0\} : (a, b) \in E, (b, c) \in F\}$$

and for a system $\mathcal{E}$ of entourages on L define the relation $\stackrel{\mathcal{E}}{\triangleleft}$ by setting

$$x \stackrel{\mathcal{E}}{\triangleleft} y \quad \equiv \quad \exists E \in \mathcal{E},\ E \circ (x \oplus x) \subseteq y \oplus y.$$

A system of entourages $\mathcal{E}$ is said to be *admissible* if

$$\forall x \in L, \quad x = \{y \mid y \stackrel{\mathcal{E}}{\triangleleft} x\}.$$

1.3.1. A *(entourage) uniformity* [16] on a frame L is an admissible system of entourages $\mathcal{E}$ such that

(Ue1) $E \in \mathcal{E}$ and $E \subseteq F \quad \Rightarrow \quad F \in \mathcal{E}$,

(Ue2) $E, F \in \mathcal{E} \quad \Rightarrow \quad E \cap F \in \mathcal{E}$,

(Ue3) $E \in \mathcal{E} \quad \Rightarrow \quad E^{-1} \in \mathcal{E}$,

(Ue4) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \leq E$.

A (entourage) uniform frame is a couple $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is an entourage uniformity on L. Let $(L, \mathcal{E})$, $(M, \mathcal{F})$ be uniform frames. A frame homomorphism $h : A \to B$ is *uniform* if for each $E \in \mathcal{E}$, $(h \oplus h)(E) \in \mathcal{F}$, where $h \oplus h : L \oplus L \to M \oplus M$ is the unique morphism $L \oplus L \to M \oplus M$ such that $(h \oplus h) \cdot u_i^L = u_i^M \cdot h$ ($i = 1, 2$; u_i^L and u_i^M denote the coproduct injections of $L \oplus L$ and $M \oplus M$ respectively).

1.3.2. Note. Similarly like in the classical case, the two structures – cover and entourage uniformity – are equivalent, see [16, 17]. The resulting categories are concretely isomorphic.

1.4. Asymmetric uniformities with entourages. Dropping the symmetry condition (Ue3) in 1.3.1 we obtain the notion of a *quasi-uniform frame*.

Because of the lack of symmetry, instead of $\overset{\mathcal{E}}{\lhd}$ we have two distinct order relations

$$x \overset{\mathcal{E}}{\lhd}_1 y \quad \equiv \quad \exists E \in \mathcal{E},\ E \circ (x \oplus x) \subseteq y \oplus y,$$

$$x \overset{\mathcal{E}}{\lhd}_2 y \quad \equiv \quad \exists E \in \mathcal{E},\ (x \oplus x) \circ E \subseteq y \oplus y,$$

which yields two admissible subframes of L, namely

$$L_i(\mathcal{E}) = \{x \in L \mid x = \bigvee \{y \in L \mid y \overset{\mathcal{E}}{\lhd}_i x\}\}, \quad i = 1, 2.$$

Now, a system of entourages $\mathcal{E}$ is said to be *admissible* if $(L, L_1(\mathcal{E}), L_2(\mathcal{E}))$ is a biframe (or, equivalently and without the biframes, if $x = \bigvee \{y \mid y \overset{\overline{\mathcal{E}}}{\lhd} x\}$ for every $x \in L$, where $\overline{\mathcal{E}}$ denotes the filter of entourages of L generated by $\mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$).

1.4.1. A *(entourage) quasi-uniformity* [17, 18] on a frame L is an admissible system of entourages $\mathcal{E}$ such that

(QUe1) $E \in \mathcal{E}$ and $E \subseteq F \quad \Rightarrow \quad F \in \mathcal{E}$,

(QUe2) $E, F \in \mathcal{E} \quad \Rightarrow \quad E \cap F \in \mathcal{E}$, (QUe3) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \leq E$.

1.5. Asymmetric uniformities with paircovers. The asymmetric extension of a cover uniformity is not so straightforward since the symmetry is implicit in the definition. Frith's treatment [6, 7] made use of the approach due to Gantner and Steinlage [9] via conjugate pairs of covers.

Let (L, L_1, L_2) be a biframe. A subset A of $L_1 \times L_2$ is a *paircover* of a biframe (L, L_1, L_2) if

$$\bigvee\{a_1 \wedge a_2 \mid (a_1, a_2) \in A\} = 1.$$

A paircover A of (L, L_1, L_2) is *strong* if, for any $(a_1, a_2) \in A$, $a_1 \vee a_2 = 0$ whenever $a_1 \wedge a_2 = 0$ (that is, $(a_1, a_2) = (0, 0)$ whenever $a_1 \wedge a_2 = 0$). For two paircovers A and B of (L, L_1, L_2) one writes $A \leq B$ (and say that A *refines* B) if

for any $(a_1, a_2) \in A$ there is $(b_1, b_2) \in B$ with $a_1 \leq b_1$ and $a_2 \leq b_2$.

Further we set $A \wedge B = \{(a_1 \wedge b_1, a_2 \wedge b_2) \mid (a_1, a_2) \in A, (b_1, b_2) \in B\}$. It is obvious that $A \wedge B$ is a paircover of (L, L_1, L_2). For $x \in L_0$ and A a paircover of (L, L_1, L_2), let

$$\mathsf{st}_1(x, A) = \bigvee\{a_1 \mid (a_1, a_2) \in A \text{ and } a_2 \wedge x \neq 0\},$$

$$\mathsf{st}_2(x, A) = \bigvee\{a_2 \mid (a_1, a_2) \in A \text{ and } a_1 \wedge x \neq 0\}$$

(these are analogues of the Ax's above) and

$$AA = \{(\mathsf{st}_1(a_1, A), \mathsf{st}_2(a_2, A)) \mid (a_1, a_2) \in A\}.$$

A system of paircovers $\mathcal{A}$ of (L, L_1, L_2) is *admissible* if

$$\forall x \in L_i,\ x = \bigvee\{y \in L_i \mid \mathsf{st}_i(y, A) \leq x \text{ for some } A \in \mathcal{A}\} \quad (i = 1, 2).$$

1.5.1. A *(cover) quasi-uniformity* [6] on a biframe (L, L_1, L_2) is an admissible system of paircovers $\mathcal{A}$ such that

(QUc1) $A \in \mathcal{A}$ and $A \leq B \quad \Rightarrow \quad B \in \mathcal{A}$,

(QUc2) $A, B \in \mathcal{A} \quad \Rightarrow \quad \exists$ strong $C \in \mathcal{A} : C \leq A \wedge B$,

(QUc3) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $BB \leq A$.

The couple $((L, L_1, L_2), \mathcal{A})$ is called a *quasi-uniform biframe* [8] ((cover) *quasi-uniform frame* in the original [6]). If $((L, L_1, L_2), \mathcal{A})$ and $((M, M_1, M_2), \mathcal{B})$ are quasi-uniform biframes, a biframe homomorphism $h : (L, L_1, L_2) \to (M, M_1, M_2)$ is *uniform* if for every $A \in \mathcal{A}$, $h[A] = \{(h(a_1), h(a_2)) \mid (a_1, a_2) \in A\} \in \mathcal{B}$. The resulting category will be denoted by

$$\mathsf{QUBiFrm}.$$

1.5.2. Note. Again the $\mathsf{QUBiFrm}$ is concretely isomorphic to the category of (entourage) quasi-uniform frames and uniform homomorphisms (see [17, 18]). In other words, the structures of entourage quasi-uniformities and cover quasi-uniformities are equivalent.

2 Quasi-uniformities without biframes

2.1. Paircovers of a frame. Let L be a frame. In analogy with [6] we call a subset $C \subseteq L \times L$ a *paircover* of L if $\bigvee\{c_1 \wedge c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover C of L is *strong* if, for any $(c_1, c_2) \in C$, $c_1 \vee c_2 = 0$ whenever $c_1 \wedge c_2 = 0$. For any $C, D \subseteq L \times L$ we write $C \leq D$ (and say that C *refines* D) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further we write

$$C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\};$$

obviously it is a is a paircover again.

For $a \in L$ and $C, D \subseteq L \times L$, we set

$$\begin{aligned}
&\mathsf{st}_1(a, C) = \bigvee\{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \wedge a \neq 0\},\\
&\mathsf{st}_2(a, C) = \bigvee\{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \wedge a \neq 0\},\\
&C^{-1} = \{(c_2, c_1) \mid (c_1, c_2) \in C\}, \text{and}\\
&\mathsf{st}(D, C) = \{(\mathsf{st}_1(d_1, C), \mathsf{st}_2(d_2, C)) \mid (d_1, d_2) \in D\}
\end{aligned}$$

and we write briefly C^* for $\mathsf{st}(C, C)$.

2.2. Proposition. *Let $U, V, W \subseteq L \times L$ and $a, b \in L$. We have the following facts:*

(a) If $a \leq b$ then $\mathsf{st}_i(a, U) \leq \mathsf{st}_i(b, U)$.

(b) *If* $U \leq V$ *then* $\mathsf{st}_i(a,U) \leq \mathsf{st}_i(a,V)$.

(c) $a \wedge \mathsf{st}_1(b,U) = 0$ *iff* $b \wedge \mathsf{st}_2(a,U) = 0$.

(d) *If* U *is a paircover then* $a \leq \mathsf{st}_i(a,U)$ *and* $U \leq U^*$.

(e) *If* U *is a paircover then* $\mathsf{st}_i(\mathsf{st}_i(a,U),U) \leq \mathsf{st}_i(a,U^*)$.

(f) $\mathsf{st}_i(a,U^{-1}) = \mathsf{st}_j(a,U)$ $(j \neq i)$.

(g) *If* U *is a paircover then* $V \leq \mathsf{st}(V,U)$.

(h) *If* U *and* V *are paircovers and* V *is strong then* $V \leq \mathsf{st}(U,V)$.

(i) *If* $V \leq W$ *then* $\mathsf{st}(U,V) \leq \mathsf{st}(U,W)$ *and* $\mathsf{st}(V,U) \leq \mathsf{st}(W,U)$.

(j) *If* U *is a paircover then* $\mathsf{st}(\mathsf{st}(V,U),U) \leq \mathsf{st}(V,U^*)$.

(k) *For any frame homomorphism* $h : L \to M$, $\mathsf{st}_i(h(a),h[U]) \leq h(\mathsf{st}_i(a,U))$.

(l) *For any frame homomorphism* $h : L \to M$, $h[U]^* \leq h[U^*]$.

Proof. (a), (b), (c) and (f) are obvious and (e) is an immediate consequence of (b) and (c). The first assertion of (i) follows from (b) while the second one follows from (a). The statement (j) follows from (e).

(d): For each $a \in L$ we have

$$\begin{aligned} a = a \wedge 1 = a \wedge \bigvee\{u_1 \wedge u_2 \mid (u_1,u_2) \in U\} &= \\ = \bigvee\{a \wedge u_1 \wedge u_2 \mid (u_1,u_2) \in U, a \wedge u_1 \wedge u_2 \neq 0\} &\leq \\ \leq \bigvee\{u_1 \mid (u_1,u_2) \in U, a \wedge u_2 \neq 0\}&. \end{aligned}$$

Thus, $a \leq \mathsf{st}_1(a,U)$ and the case $i = 2$ is similar. Hence $U \leq U^*$.

(g) follows immediately from (d).

(h): Let $(v_1,v_2) \in V$. If $v_1 \wedge v_2 = 0$ then $(v_1,v_2) = (0,0)$ and so there is obviously a $(u_1,u_2) \in U$ such that $v_i \leq \mathsf{st}_i(u_i,V)$ $(i = 1,2)$, otherwise $v_1 \wedge v_2 \neq 0$. Then, since U is a paircover, there is some $(u_1,u_2) \in U$ such that $u_1 \wedge u_2 \wedge v_1 \wedge v_2 \neq 0$, and $v_i \leq \mathsf{st}_i(u_i,V)$ $(i = 1,2)$.

(k) is obvious, since $h(a) \neq 0$ implies $a \neq 0$ for every frame homomorphism h, and (l) is an immediate consequence of (k). $\square$

2.3. Induced subframes. Given a non-empty family $\mathcal{U}$ of paircovers of L, we write $a \overset{\mathcal{U}}{\lhd_i} b$ $(i = 1, 2)$ whenever $\mathsf{st}_i(a, U) \leq b$ for some $U \in \mathcal{U}$, and define

$$L_i(\mathcal{U}) = \{a \in L \mid a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\lhd_i} a\}\} \quad (i = 1, 2).$$

2.4. Proposition. *Let $\mathcal{U}$ be a basis for a filter of paircovers of L. Then, for $i = 1, 2$, the relations $\overset{\mathcal{U}}{\lhd_i}$ are sublattices of $L \times L$, both stronger than $\leq$, and we have that*

(a) for any $a, b, c, d \in L$, $a \leq b \overset{\mathcal{U}}{\lhd_i} c \leq d$ implies $a \overset{\mathcal{U}}{\lhd_i} d$,

(b) for any $a, b \in L$, $a \overset{\mathcal{U}}{\lhd_i} b$ implies $a \prec b$ (that is, $a^* \vee b = 1$), and

(c) $L_i(\mathcal{U})$ are subframes of L.

Proof. The fact that each $\overset{\mathcal{U}}{\lhd_i}$ is stronger than $\leq$ follows from 2.2(d). Clearly $0 \overset{\mathcal{U}}{\lhd_i} 0$ and $1 \overset{\mathcal{U}}{\lhd_i} 1$. If $\mathsf{st}_i(a_1, U_1) \leq b_1$ and $\mathsf{st}_i(a_2, U_2) \leq b_2$ with $U_1, U_2 \in \mathcal{U}$ then immediately $\mathsf{st}_i(a_1 \wedge a_2, U_1 \wedge U_2) \leq \mathsf{st}_i(a_1, U_1) \wedge \mathsf{st}_i(a_2, U_2) \leq b_1 \wedge b_2$. Since $\mathcal{U}$ is a filter basis, there exists a $V \in \mathcal{U}$ such that $V \leq U_1 \wedge U_2$. Hence, using 2.2(b) we conclude that $a_1 \wedge a_2 \overset{\mathcal{U}}{\lhd_i} b_1 \wedge b_2$. On the other hand, as can be easily checked, $\mathsf{st}_i(a_1 \vee a_2, U_1 \wedge U_2) \leq \mathsf{st}_i(a_1, U_1) \vee \mathsf{st}_i(a_2, U_2) \leq b_1 \vee b_2$. Thus, $a_1 \vee a_2 \overset{\mathcal{U}}{\lhd_i} b_1 \vee b_2$.

(a) follows from 2.2(a), and (c) is an immediate consequence of the fact that each $\overset{\mathcal{U}}{\lhd_i}$ is a sublattice of $L \times L$, stronger than $\leq$, and of (a).

(b): Let $i, j \in \{1, 2\}$ with $i \neq j$. Assume $a \overset{\mathcal{U}}{\lhd_i} b$, that is, $\mathsf{st}_i(a, U) \leq b$ for some $U \in \mathcal{U}$. Let $u_1 \wedge u_2$ with $(u_1, u_2) \in U$. If $u_j \wedge a = 0$ then $u_1 \wedge u_2 \leq u_j \leq a^*$; otherwise, $u_1 \wedge u_2 \leq u_i \leq \mathsf{st}_i(a, U) \leq b$. Hence $1 = \bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} \leq a^* \vee b$. ☐

2.5. First step to a definition of quasi-uniformities. For a non-empty family $\mathcal{U}$ of paircovers of L consider the following requirements.

(QU1) For any $U \in \mathcal{U}$ and any paircover V with $U \leq V$, then $V \in \mathcal{U}$.

(QU2) For any $U, V \in \mathcal{U}$ there exists a strong $W \in \mathcal{U}$ such that $W \leq U \wedge V$.

(QU3) For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^* \leq U$.

(QU4′) $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ is a biframe.

These requirements will become a definition of quasi-uniformity shortly. So far, (QU4′) depends on the notion of biframe; it will be our task in the remainder of this section to mend it.

2.6. A technical lemma. Let $\mathcal{U}$ satisfy (QU1), (QU2), (QU3) and (QU4′). Define a relation $\overset{\mathcal{U}}{\Subset}$ on $\mathcal{P}(L \times L)$ by

$$C \overset{\mathcal{U}}{\Subset} D \quad \equiv \quad \mathsf{st}(C, U) \leq D \text{ for some } U \in \mathcal{U}$$

By (g) in 2.2 it is stronger than $\leq$. Further, we will need the following interior operator on $\mathcal{P}(L \times L)$:

$$\mathrm{int}(C) = \bigcup \{D \subseteq L \times L \mid D \overset{\mathcal{U}}{\Subset} C\}.$$

Lemma. *Let $\mathcal{U}$ satisfy* (QU1), (QU2), (QU3) *and* (QU4′). *For each $U \in \mathcal{U}$ we have:*

(a) $int(U) \leq U \leq int(U^*)$.

(b) For every $a \in L$, $\mathsf{st}_i(a, int(U)) \in L_i(\mathcal{U})$ $(i = 1, 2)$.

Proof. (a) The inequality $\mathrm{int}(U) \leq U$ is trivial and the other one follows from the obvious fact that for every $U \in \mathcal{U}$, $U \overset{\mathcal{U}}{\Subset} U^*$.

(b) We only prove the case $i = 1$ (the case $i = 2$ is similar). We need to show that

$$\mathsf{st}_1(a, \mathrm{int}(U)) \leq \bigvee \{y \in L \mid y \overset{\mathcal{U}}{\lhd}_1 \mathsf{st}_1(a, \mathrm{int}(U))\}.$$

By definition, $\mathsf{st}_1(a, \mathrm{int}(U)) = \bigvee \{d_1 \mid (d_1, d_2) \in \mathrm{int}(U), d_2 \wedge a \neq 0\}$. Thus, let $(d_1, d_2) \in \mathrm{int}(U)$ such that $d_2 \wedge a \neq 0$. Then $(d_1, d_2) \in D \subseteq L \times L$ and there exists $V \in \mathcal{U}$ such that $\mathsf{st}(D, V) \subseteq U$. We need to show that $d_1 \overset{\mathcal{U}}{\lhd}_1 \mathsf{st}_1(a, \mathrm{int}(U))$. To see this consider $W \in \mathcal{U}$ such that $W^* \leq V$. Now suffices to prove that $\mathsf{st}_1(d_1, W) \leq \mathsf{st}_1(a, \mathrm{int}(U))$.

By (i) and (j) in 2.2 we have

$$\mathsf{st}(\mathsf{st}(D, W), W) \leq \mathsf{st}(D, W^*) \leq \mathsf{st}(D, V) \leq U,$$

which shows that $\mathsf{st}(D, W) \overset{\mathcal{U}}{\Subset} U$. Thus, $\mathsf{st}(D, W) \subseteq \mathrm{int}(U)$ and we only need to check that $\mathsf{st}_1(d_1, W) \leq \mathsf{st}_1(a, \mathsf{st}(D, W))$. This is easy, $(\mathsf{st}_1(d_1, W), \mathsf{st}_2(d_2, W)) \in \mathsf{st}(D, W)$ and $\mathsf{st}_2(d_2, W) \wedge a \geq d_2 \wedge a \neq 0$. $\square$

2.7. Combined admissibility and a definition of quasi-uniformity. Let $\overline{\mathcal{U}}$ be the filter of paircovers of L generated by $\{U \wedge U^{-1} \mid U \in \mathcal{U}\}$. Then, by 2.2(f), $\mathsf{st}_1(a, U \wedge U^{-1}) = \mathsf{st}_2(a, (U \wedge U^{-1})^{-1})$. Since $(U \wedge U^{-1})^{-1} = U \wedge U^{-1}$ we have

$$\overset{\overline{\mathcal{U}}}{\lhd}_1 = \overset{\overline{\mathcal{U}}}{\lhd}_2 \, . \tag{2.7.1}$$

We will denote this relation on L by

$$\overset{\overline{\mathcal{U}}}{\lhd} \, .$$

Proposition. *Let $\mathcal{U}$ be a non-empty family of paircovers of L satisfying axioms* (QU1), (QU2) *and* (QU3). *Then $\mathcal{U}$ satisfies* (QU4′) *if and only if*

(QU4) For each $a \in L$, $a = \bigvee\{b \in L \mid b \overset{\overline{\mathcal{U}}}{\lhd} a\}$.

Proof. $\Rightarrow$: For each $a \in L$ we may write $a = \bigvee_{i \in I}(a_i^1 \wedge a_i^2)$ for some

$$\{a_i^1 \mid i \in I\} \subseteq L_1(\mathcal{U}) \quad \text{and} \quad \{a_i^2 \mid i \in I\} \subseteq L_2(\mathcal{U}).$$

Taking into account that, for any $i \in I$,

$$a_i^1 = \{b \in L \mid b \overset{\mathcal{U}}{\lhd}_1 a_i^1\} \quad \text{and} \quad a_i^2 = \{b \in L \mid b \overset{\mathcal{U}}{\lhd}_2 a_i^2\},$$

it suffices to show that $b_1 \wedge b_2 \overset{\overline{\mathcal{U}}}{\lhd} a_1 \wedge a_2$ whenever $b_1 \overset{\mathcal{U}}{\lhd}_1 a_1$ and $b_2 \overset{\mathcal{U}}{\lhd}_2 a_2$. This property is an immediate consequence of (2.7.1) and (a) and (b) of 2.2.

$\Leftarrow$: By 2.4, each $L_i(\mathcal{U})$ $(i = 1, 2)$ is a subframe of L. It remains to show that each $a \in L$ is a join of finite meets in $L_1(\mathcal{U}) \cup L_2(\mathcal{U})$.

Let $a \in L$. Then $a = \bigvee S$ where $S = \{b \in L \mid b \overset{\overline{\mathcal{U}}}{\lhd} a\}$. For each $b \in S$ there exists $\overline{U}_b \in \overline{\mathcal{U}}$ and $U_b \in \mathcal{U}$ such that $\mathsf{st}_1(b, \overline{U}_b) \leq a$ and $U_b \wedge U_b^{-1} \leq \overline{U}_b$. Consider $V_b \in \mathcal{U}$ such that $V_b^* \leq \overline{U}_b$. Then $\overline{V}_b = V_b \wedge V_b^{-1} \in \overline{\mathcal{U}}$ and $\overline{V}_b^* \leq \overline{U}_b$. Therefore $\mathrm{int}(\overline{V}_b^*) \leq \mathrm{int}(\overline{U}_b)$. Thus

$$\begin{aligned} a = \bigvee S &\leq \bigvee_{b \in S} (\mathsf{st}_1(b, \overline{V}_b) \wedge \mathsf{st}_2(b, \overline{V}_b)) \leq \bigvee_{b \in S} (\mathsf{st}_1(b, \mathrm{int}(\overline{V}_b^*)) \wedge \mathsf{st}_2(b, \mathrm{int}(\overline{V}_b^*))) \leq \\ &\leq \bigvee_{b \in S} (\mathsf{st}_1(b, \mathrm{int}(\overline{U}_b)) \wedge \mathsf{st}_2(b, \mathrm{int}(\overline{U}_b))) \leq \mathsf{st}_1(b, \overline{U}_b) \leq a. \end{aligned}$$

Hence

$$a = \bigvee_{b \in S} (\mathsf{st}_1(b, \mathrm{int}(\overline{U}_b)) \wedge \mathsf{st}_2(b, \mathrm{int}(\overline{U}_b)))$$

and, by Lemma 2.6, $\mathsf{st}_i(b, \mathrm{int}(\overline{U}_b)) \in L_i(\mathcal{U})$ $(i = 1, 2)$. $\square$

2.7.1. Quasi-uniformity. A (non-void) class $\mathcal{U}$ of paircovers on L satisfying the requirements (QU1), (QU2), (QU3) and (QU4) (equivalent, as we now know, to (QU1), (QU2), (QU3) and (QU4$'$) is called a *quasi-uniformity* on L, and the pair $(L,\mathcal{U})$ is called a *quasi-uniform frame*. $\mathcal{B} \subseteq \mathcal{U}$ is a *basis* for $\mathcal{U}$ if, for each $U \in \mathcal{U}$, there is a $B \in \mathcal{B}$ such that $B \leq U$.

Let $(L,\mathcal{U})$ and $(M,\mathcal{V})$ be quasi-uniform frames. A frame homomorphism $h : L \to M$ is *uniform* if $h[U] \in \mathcal{V}$ for every $U \in \mathcal{U}$. The resulting category will be denote by

$$\mathsf{QUFrm}.$$

We will be now heading to the main aim of this paper which is to the prove that this category is isomorphic to QUBFrm.

2.8. Remark. Note that this definition contains as a special case the standard uniform structures defined by covers, represented as those quasi-uniformities that have a basis consisting of pairs of coinciding covers (i.e., of the form (U,U) where U is a cover of L). Note that each such paircover is strong, that $(U,U)^* = (U^*,U^*)$, where U^* is the usual UU of 1.2, and that the relations $\overset{\overline{\mathcal{U}}}{\lhd}$ and $\overset{\mathcal{U}}{\lhd}$ coincide.

3 Confronting the biframe approach

3.1. Viewing biframe quasi-uniformities as quasi-uniformities.

3.1.1. Proposition. *Let $((L,L_1,L_2),\mathcal{C})$ be an object of* $\mathsf{QUBiFrm}$*. Then $(L,\mathcal{C})$ is a quasi-uniform frame.*

Proof. Every element of $\mathcal{C}$, being a paircover of (L,L_1,L_2), is a paircover of L_0. In addition, $\mathcal{C}$ satisfies axioms (QU1), (QU2) and (QU3) trivially so that it suffices to check (QU4). By 2.4, each $L_i(\mathcal{C})$ is a subframe of L_0 and hence we need to show that $L_i \subseteq L_i(\mathcal{C})$ for $i = 1,2$. Let $a \in L_i$. Then

$$\begin{aligned} a = \bigvee\{b \in L_i \mid \mathsf{st}_i(b,C) \leq a \text{ for some } C \in \mathcal{C}\} \leq \\ \leq \bigvee\{b \in L_0 \mid b \overset{\mathcal{C}}{\lhd}_i a \text{ for some } C \in \mathcal{C}\} \leq a. \end{aligned}$$ □

Concerning maps, the following is obvious.

3.1.2. Proposition. *Let* $h : ((L, L_1, L_2), \mathcal{C}) \to ((M, M_1, M_2), \mathcal{D})$ *be a morphism of* $\mathsf{QUBiFrm}$*. Then* $h : (L, \mathcal{C}) \to (M, \mathcal{D}) \in \mathsf{QUFrm}$. □

The functor established in 3.1.1 and 3.1.2 will be denoted by

$$\Phi : \mathsf{QUBiFrm} \to \mathsf{QUFrm}.$$

3.2. ***U*-small elements.** Let $U \subseteq L \times L$ and $a \in L$. In the following, we will write

$$\mathsf{st}(a, U) = \bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U, u_1 \wedge u_2 \wedge a \neq 0\}$$

It is obvious that for every $a \in L$ and every paircover U of L,

$$a \leq \mathsf{st}(a, U) \leq \mathsf{st}_1(a, U) \wedge \mathsf{st}_2(a, U).$$

Given a paircover U of a frame L, we declare an element a to be of L U*-small* if $a \leq \mathsf{st}(b, U)$ whenever $a \wedge b \neq 0$. Note that, for any $(u_1, u_2) \in U$, $u_1 \wedge u_2$ is U-small.

Further we set

$$C_U = \{(\mathsf{st}_1(a, \mathrm{int}(U)), \mathsf{st}_2(a, \mathrm{int}(U))) \mid a \text{ is an } U\text{-small member of } L\}.$$

Lemma. *Let* $(L, \mathcal{U}) \in \mathsf{QUFrm}$*. For each* $a \in L$ *and* $U, V \in \mathcal{U}$ *we have that*

(a) each C_U *is a strong paircover of the biframe* $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$,

(b) $C_{U \wedge V} \leq C_U \wedge C_V$,

(c) $\mathsf{st}_i(a, C_U) \leq \mathsf{st}_i(a, U^{**})$ $(i = 1, 2)$,

(d) $\mathsf{st}_i(a, U) \leq \mathsf{st}_i(a, C_{U^*})$ $(i = 1, 2)$*, and*

(e) $(C_U)^* \leq C_{U^{***}}$.

Proof. (a): By 2.6(b) each C_U is a subset of $L_1(\mathcal{U}) \times L_2(\mathcal{U})$. It is a paircover since

$$\bigvee\{\mathsf{st}_1(a, \mathrm{int}(U)) \wedge \mathsf{st}_2(a, \mathrm{int}(U)) \mid a \text{ is } U\text{-small}\} \geq \bigvee\{a \in L \mid a \text{ is } U\text{-small}\} \geq$$
$$\geq \bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} = 1.$$

Finally, it is strong: if $\mathsf{st}_1(a, \mathrm{int}(U)) \vee \mathsf{st}_2(a, \mathrm{int}(U)) \neq 0$ then $a \neq 0$ and hence $\mathsf{st}_1(a, \mathrm{int}(U)) \wedge \mathsf{st}_2(a, \mathrm{int}(U)) \geq a \neq 0$.

(b) is trivial.

(c): Fix $i \in \{1, 2\}$ and let $j \in \{1, 2\}$ with $j \neq i$. By definition,

$$\mathsf{st}_i(a, C_U) = \bigvee\{\mathsf{st}_i(b, \mathrm{int}(U)) \mid b \text{ is } U\text{-small}, \mathsf{st}_j(b, \mathrm{int}(U)) \wedge a \neq 0\}.$$

By 2.2(c), $\mathsf{st}_j(b, \mathrm{int}(U)) \wedge a \neq 0$ is equivalent to $\mathsf{st}_i(a, \mathrm{int}(U)) \wedge b \neq 0$ and, since b is U-small, this implies that $b \leq \mathsf{st}(\mathsf{st}_i(a, \mathrm{int}(U)), U) \leq \mathsf{st}_i(\mathsf{st}_i(a, U), U) \leq \mathsf{st}_i(a, U^*)$ (using 2.2(d)). Hence

$$\mathsf{st}_i(b, \mathrm{int}(U)) \leq \mathsf{st}_i(\mathsf{st}_i(a, U^*), \mathrm{int}(U)) \leq \mathsf{st}_i(\mathsf{st}_i(a, U^*), U^*) \leq \mathsf{st}_i(a, U^{**}).$$

(d): We have $\mathsf{st}_1(a, U) = \bigvee\{u_1 \mid (u_1, u_2) \in U, u_2 \wedge a \neq 0\}$ and for each such u_1, we can write $u_1 = \bigvee\{u_1 \wedge d_1 \wedge d_2 \mid (d_1, d_2) \in U^*, u_1 \wedge d_1 \wedge d_2 \neq 0\}$ (since U^* is a paircover).

Now for each $(d_1, d_2) \in U^*$, $u_1 \wedge d_1 \wedge d_2$ is U^*-small. Indeed, if $y \wedge u_1 \wedge d_1 \wedge d_2 \neq 0$ then $u_1 \wedge d_1 \wedge d_2 \leq \mathsf{st}(y, U^*)$ (since $u_1 \wedge d_1 \wedge d_2 \leq d_1 \wedge d_2$, $(d_1, d_2) \in U^*$ and $d_1 \wedge d_2 \wedge y \geq y \wedge u_1 \wedge d_1 \wedge d_2 \neq 0$). Therefore

$$(\mathsf{st}_1(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)), \mathsf{st}_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*))) \in C_{U^*}.$$

It only remains to prove that $a \wedge \mathsf{st}_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0$ which is easy:

$$\begin{aligned}
&a \wedge \mathsf{st}_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) = \\
&\quad = \bigvee\{a \wedge c_2 \mid (c_1, c_2) \in \mathrm{int}(U^*), c_1 \wedge a \wedge \mathsf{st}_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0\} \\
&\quad \geq \bigvee\{a \wedge c_2 \mid (c_1, c_2) \in U, c_1 \wedge a \wedge \mathsf{st}_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0\} \geq \\
&\quad \geq a \wedge u_2 \neq 0.
\end{aligned}$$

(e): Let a be an U-small element of L and consider

$$(\mathsf{st}_1(\mathsf{st}_1(a, \mathrm{int}(U)), C_U), \mathsf{st}_2(\mathsf{st}_2(a, \mathrm{int}(U)), C_U)) \in C_U^*.$$

For $i = 1, 2$ and $j \in \{1, 2\}, j \neq i$, we have

$$\begin{aligned}
&\mathsf{st}_i(\mathsf{st}_i(a, \mathrm{int}(U)), C_U) = \\
&\quad = \bigvee\{\mathsf{st}_i(b, \mathrm{int}(U)) \mid b \text{ is } U\text{-small}, \mathsf{st}_i(a, \mathrm{int}(U)) \wedge \mathsf{st}_j(b, \mathrm{int}(U)) \neq 0\}. \quad (*)
\end{aligned}$$

Now by 2.2(c),

$$\mathsf{st}_i(a, \mathrm{int}(U)) \wedge \mathsf{st}_j(b, \mathrm{int}(U)) \neq 0 \Leftrightarrow b \wedge \mathsf{st}_i(\mathsf{st}_i(a, \mathrm{int}(U)), \mathrm{int}(U)) \neq 0.$$

Hence, by the U-smallness of b,

$$b \leq \mathsf{st}(\mathsf{st}_i(\mathsf{st}_i(a, \mathrm{int}(U)), \mathrm{int}(U)), U) \leq \mathsf{st}_i(\mathsf{st}_i(\mathsf{st}_i(a, U), U), U)$$

so that each element in the join $(*)$ satisfies

$$\mathsf{st}_i(b, \mathrm{int}(U)) \leq \mathsf{st}_i(\mathsf{st}_i(\mathsf{st}_i(\mathsf{st}_i(a, U), U), U), U).$$

Finally, applying 2.2(e) (twice) and 2.6(a), we obtain

$$\mathsf{st}_i(b, \mathrm{int}(U)) \leq \mathsf{st}_i(a, U^{**}) \leq \mathsf{st}_i(a, \mathrm{int}(U^{***}))$$

(note that, of course, a is $U^{***}-$small). $\square$

3.3. Translating in the other direction. The following three facts will yield a functor

$$\Psi : \mathsf{QUFrm} \to \mathsf{QUBiFrm}.$$

3.3.1. Proposition. *Let $(L,\mathcal{U})$ be an object of* QUFrm. *Then $\{C_U \mid U \in \mathcal{U}\}$ is a basis for a quasi-uniformity $\mathcal{C}_{\mathcal{U}}$ on the biframe $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$.*

Proof. By 3.2 each C_U is a strong paircover of the biframe $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$. Let us check that $\mathcal{C}_{\mathcal{U}}$ satisfies (QUc1), (QUc2), (QUc3) and (QUc4):

(QUc1) is trivial.

(QUc2) is in 3.2(b).

(QUc3): Let $C \in \mathcal{C}_{\mathcal{U}}$. Then there exists $U \in \mathcal{U}$ such that $C_U \leq C$. Take $V \in \mathcal{U}$ satisfying $V^{***} \leq U$. By 3.2(e), $(C_V)^* \leq C_{V^{***}} \leq C_U \leq C$.

(QUc4): Let $a \in L_i(\mathcal{U})$ $(i = 1, 2)$. We need to prove that

$$a = \bigvee\{b \in L_i(\mathcal{U}) \mid \mathsf{st}_i(b, C) \leq a \text{ for some } C \in \mathcal{C}_{\mathcal{U}}\}.$$

By hypothesis, $a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\triangleleft}_i a\}$. Therefore it suffices to show that $b \overset{\mathcal{U}}{\triangleleft}_i a$ implies the existence of $b' \in L_i(\mathcal{U})$ such that $b \leq b' \leq \mathsf{st}_i(b', C_U) \leq a$ for some $U \in \mathcal{U}$.

Let $b \overset{\mathcal{U}}{\lhd}_1 a$. Then there is $U \in \mathcal{U}$ satisfying $\mathsf{st}_1(b, U) \leq a$. Let $V \in \mathcal{U}$ such that $V^{**} \leq U$ and consider also $W \in \mathcal{U}$ such that $W^{****} \leq V$. By 2.6, $b \leq \mathsf{st}_1(b, \mathrm{int}(W)) \in L_1(\mathcal{U})$. Let us show that $\mathsf{st}_1(b, \mathrm{int}(W))$ is the required $b' \in L_1(\mathcal{U})$, by checking that $\mathsf{st}_1(\mathsf{st}_1(b, \mathrm{int}(W)), C_{W^*}) \leq a$:

By 3.2(d) we have

$$\mathsf{st}_1(\mathsf{st}_1(b, \mathrm{int}(W)), C_{W^*}) \leq \mathsf{st}_1(\mathsf{st}_1(b, W), C_{W^*}) \leq \mathsf{st}_1(\mathsf{st}_1(b, C_{W^*}), C_{W^*}).$$

Then, by 2.2(e) and 3.2(e),

$$\mathsf{st}_1(\mathsf{st}_1(b, \mathrm{int}(W)), C_{W^*}) \leq \mathsf{st}_1(b, (C_{W^*})^*) \leq \mathsf{st}_1(b, C_{W^{****}}) \leq \mathsf{st}_1(b, C_V).$$

Finally, using 3.2(c) we can conclude that

$$\mathsf{st}_1(\mathsf{st}_1(b, \mathrm{int}(W)), C_{W^*}) \leq \mathsf{st}_1(b, V^{**}) \leq \mathsf{st}_1(b, U) \leq a.$$

The proof for $i = 2$ is similar. $\square$

3.3.2. Lemma. *Let $h : (L, \mathcal{U}) \to (M, \mathcal{V})$ be a morphism in* QUFrm*, $a, b \in L$ and let $U \in \mathcal{U}$. Then*

(a) if $b \overset{\mathcal{U}}{\lhd}_i a$ then $h(b) \overset{\mathcal{V}}{\lhd}_i h(a)$, $(i = 1, 2)$, and

(b) $C_{h[U]} \leq h[C_{U^{**}}]$.

Proof. (a): Let $b \overset{\mathcal{U}}{\lhd}_i a$. Then $\mathsf{st}_i(b, U) \leq a$ for some $U \in \mathcal{U}$. Consider $V = h[U] \in \mathcal{V}$. Using 2.2(k) we conclude that $\mathsf{st}_i(h(b), V) \leq h(\mathsf{st}_i(b, U)) \leq h(a)$ so that $h(b) \overset{\mathcal{V}}{\lhd}_i h(a)$.

(b): We have to show that for each $h[U]$-small element b of M there exists an U^{**}-small element $a \in L$ such that

$$\mathsf{st}_i(b, \mathrm{int}(h[U])) \leq h(\mathsf{st}_i(a, \mathrm{int}(U^{**}))) \quad (i = 1, 2).$$

Thus, let $b \neq 0$ be $h[U]$-small. Since $h[U]$ is a paircover of M, there exists $(u_1, u_2) \in U$ for which $b \wedge h(u_1) \wedge h(u_2) \neq 0$. Then, by the $h[U]$-smallness of b, $b \leq \mathsf{st}(h(u_1) \wedge h(u_2), h[U])$. Denote $u_1 \wedge u_2$ by a; we have $b \leq \mathsf{st}_i(h(a), h[U])$. Then

$$\mathsf{st}_i(b, \mathrm{int}(h[U])) \leq \mathsf{st}_i(b, h[U]) \leq \mathsf{st}_i(\mathsf{st}_i(h(a), h[U]), h[U]) \leq \mathsf{st}_i(h(a), h[U]^*).$$

Using 2.2(l),(k) we obtain $\mathsf{st}_i(b, \mathrm{int}(h[U])) \leq \mathsf{st}_i(h(a), h[U^*]) \leq h(\mathsf{st}_i(a, U^*))$. Hence, by Lemma 2.6, $\mathsf{st}_i(b, \mathrm{int}(h[U])) \leq h(\mathsf{st}_i(a, \mathrm{int}(U^{**})))$. Finally, a is U^{**}-small because $a = u_1 \wedge u_2$ and $(u_1, u_2) \in U \leq U^{**}$. $\square$

3.3.3. Proposition. *Let $h : (L,\mathcal{U}) \to (M,\mathcal{V})$ be a morphism of* QUFrm*. Then h is a* $\mathsf{QUBiFrm}$*-morphism $((L, L_1(\mathcal{U}), L_2(\mathcal{U})), \mathcal{C}_{\mathcal{U}}) \to ((M, M_1(\mathcal{V}), M_2(\mathcal{V})), \mathcal{C}_{\mathcal{V}})$.*

Proof. First we check that the frame homomorphism $h : (L,\mathcal{U}) \to (M,\mathcal{V})$ is indeed a biframe homomorphism from $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ into $(M, M_1(\mathcal{V}), M_2(\mathcal{V}))$. Let $a \in L_i(\mathcal{U})$ $(i = 1, 2)$. We have to show that $h(a) \in M_i(\mathcal{V})$. Since $a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\lhd_i} a\}$, we may conclude by 3.3.2(1) that

$$h(a) = \bigvee\{h(b) \mid b \in L, b \overset{\mathcal{U}}{\lhd_i} a\} \leq \bigvee\{m \in M \mid m \overset{\mathcal{V}}{\lhd_i} h(a)\} \leq h(a)$$

which makes sure that $h(a) = \bigvee\{m \in M \mid m \overset{\mathcal{V}}{\lhd_i} h(a)\}$ and hence $h(a) \in M_i(\mathcal{V})$.

Finally, it remains to show that $h[C] \in \mathcal{C}_{\mathcal{V}}$ for every $C \in \mathcal{C}_{\mathcal{U}}$. Let $C \in \mathcal{C}_{\mathcal{U}}$ and $U \in \mathcal{U}$ such that $C_U \leq C$. Consider $V \in \mathcal{U}$ satisfying $V^{**} \leq U$. By 3.3.2(b), $C_{h[V]} \leq h[C_{V^{**}}] \leq h[C_U] \leq h[C]$. Since $h[V] \in \mathcal{V}$ we have $h[C] \in \mathcal{C}_{\mathcal{V}}$. □

3.4. The concrete isomorphism $\mathsf{QUFrm} \cong \mathsf{QUBiFrm}$.

Lemma. *Let $(L,\mathcal{U})$ be a quasi-uniform frame. For any paircover U in $\mathcal{U}$ we have that*

(a) if U is strong then $U \leq C_{U^*}$, and

(b) $C_U \leq U^{**}$.

Proof. (a): Let $(u_1, u_2) \in U$. Since U is strong we can assume that $u_1 \wedge u_2 \neq 0$. Since U^* is a paircover then $u_1 \wedge u_2 = \bigvee\{u_1 \wedge u_2 \wedge v_1 \wedge v_2 \mid (v_1, v_2) \in U^*\}$. Therefore there exists $(v_1, v_2) \in U^*$ for which $u_1 \wedge u_2 \wedge v_1 \wedge v_2 \neq 0$. This immediately implies that

$$u_1 \leq \mathsf{st}_1(v_1 \wedge v_2, U) \leq \mathsf{st}_1(v_1 \wedge v_2, \mathrm{int}(U^*)) \quad \text{and}$$
$$u_2 \leq \mathsf{st}_2(v_1 \wedge v_2, U) \leq \mathsf{st}_2(v_1 \wedge v_2, \mathrm{int}(U^*)).$$

But $v_1 \wedge v_2$ is U^*-small (because $(v_1, v_2) \in U^*$) and hence

$$(\mathsf{st}_1(v_1 \wedge v_2, \mathrm{int}(U^*)), \mathsf{st}_2(v_1 \wedge v_2, \mathrm{int}(U^*))) \in C_{U^*}$$

as desired.

(b): Let $a \neq 0$ be an U-small element of L. Since U is a paircover, there exists $(u_1, u_2) \in U$ such that $a \wedge u_1 \wedge u_2 \neq 0$. Then, by the U-smallness of a, $a \leq \mathsf{st}(u_1 \wedge u_2, U) \leq \mathsf{st}_1(u_1, U) \wedge \mathsf{st}_2(u_2, U)$. Consequently, $\mathsf{st}_i(a, \mathrm{int}(U)) \leq \mathsf{st}_i(\mathsf{st}_i(u_i, U), U) \leq \mathsf{st}_i(u_i, U^*)$, for $i = 1, 2$. Of course, there exists $(v_1, v_2) \in U^*$ for which $u_i \leq v_i$ $(i = 1, 2)$. Hence $\mathsf{st}_i(a, \mathrm{int}(U)) \leq \mathsf{st}_i(v_i, U^*)$ which guarantees that $C_U \leq U^{**}$. □

Now we are ready for the main theorem.

Theorem. *The functors* Φ *and* Ψ *establish a concrete isomorphism between the concrete categories* QUFrm *and* $\mathsf{QUBiFrm}$.

Proof. We want to show that $\Psi\Phi = \mathrm{Id}_{\mathsf{QUBiFrm}}$ and $\Phi\Psi = \mathrm{Id}_{\mathsf{QUFrm}}$. After 3.1.2 and 3.3.4 there is nothing left to prove for morphisms.

Now for the objects. We have

$$\Psi\Phi(((L, L_1, L_2), \mathcal{D})) = \Psi((L, \mathcal{D})) = ((L, L_1(\mathcal{C}), L_2(\mathcal{C})), \mathcal{C}_{\mathcal{D}}) \quad \text{and}$$
$$\Phi\Psi((L, \mathcal{U})) = \Phi((L, L_1(\mathcal{U}), L_2(\mathcal{U})), \mathcal{C}_{\mathcal{U}}) = (L, \mathcal{C}_{\mathcal{U}}),$$

so that we need to prove (a) $L_i(\mathcal{C}) = L_i$ $(i = 1, 2)$, (b) $\mathcal{C}_{\mathcal{D}} = \mathcal{D}$ and (c) $\mathcal{C}_{\mathcal{U}} = \mathcal{U}$.

(a): By hypothesis, $((L, L_1, L_2), \mathcal{D})$ is an object of $\mathsf{QUBiFrm}$ so $L_i \subseteq L_i(\mathcal{C})$ $(i = 1, 2)$. The reverse inclusion follows the same way as the satisfaction of (QUc3) in 3.3.1.

(b): Take a $D \in \mathcal{D}$ and consider an $E \in \mathcal{D}$ such that $E^{**} \leq D$. By 3.4, $C_E \leq E^{**} \leq D$ so that $D \in \mathcal{C}_{\mathcal{D}}$. Conversely, let $U \in \mathcal{C}_{\mathcal{D}}$. Then there exists $D \in \mathcal{D}$ such that $C_D \leq U$. Take a strong $E \in \mathcal{D}$ such that $E^* \leq D$. Then by 3.4 we obtain $E \leq C_{E^*} \leq C_D \leq U$ and $U \in \mathcal{D}$, as required.

(c) is an immediate consequence of 3.4, as in (b). ☐

3.4.1. Remarks. (1) It should be noted that the space quasi-uniformities in [9] used bitopologies but ultimately avoided them. The previous efforts modifying the approach for the point-free context ([6], [7]) used the intermediate state of bitopologies and naturally worked with biframes. Thus, our approach is, after all, a more consequent exploitation of the Gantner's and Steinlage's ideas.

(2) Note that there is no requirement on the frame to be quasi-uniformizable (unlike the complete regularity for uniformizability), simply because we have the equivalence with $\mathsf{QUBiFrm}$ where there is none.

4 Confrontation with spaces: quasi-uniform spectrum

4.1. The functor Ω. Denote by QUnif the category of quasi-uniform spaces and uniformly continuous maps [5].

Let (X, μ) be a quasi-uniform space. To explain the notation and terminology, recall that μ induces two topologies $\mathfrak{T}_1(\mu)$ and $\mathfrak{T}_2(\mu)$ on X in the following manner:

> $A \subseteq X$ is $\mathfrak{T}_i(\mu)$-open $(i = 1, 2)$ if for every $a \in A$ there exists $\mathcal{U}_a \in \mu$ such that $\mathsf{st}_i(a, \mathcal{U}_a) \subseteq A$;

Take a $\mathcal{U} \in \mu$. We say that $\mathcal{U}$ is an *open* paircover of (X, μ) if for each $(U_1, U_2) \in \mathcal{U}$, U_1 is $\mathfrak{T}_1(\mu)$-open and U_2 is $\mathfrak{T}_2(\mu)$-open. Set

$$\Omega(X, \mu) = (\mathfrak{T}_1(\mu) \vee \mathfrak{T}_2(\mu), \mathcal{C}_\mu)$$

where $\mathcal{C}_\mu$ is the set of all open paircovers of (X, μ). It is not hard to check that $\Omega(X, \mu)$ is a quasi-uniform space. Furthermore, if $f : (X, \mu) \to (Y, \nu)$ is uniformly continuous then $\Omega(f) : \Omega(Y, \nu) \to \Omega(X, \mu)$, defined by $\Omega(f)(B) = f^{-1}(B)$ for any $B \in \mathfrak{T}_1(\nu) \vee \mathfrak{T}_2(\nu)$, is a uniform frame homomorphism. Thus, Ω is a contravariant functor from QUnif into QUFrm.

4.2. The spectrum functor Σ. On the other hand we consider for each frame L, the *spectrum*, that is, the topological space

$$(\Sigma L, \{\Sigma_a \mid a \in L\}),$$

where ΣL is as usual the set of all frame homomorphisms $p : L \to \{0, 1\}$ and $\Sigma_a = \{p \in \Sigma L \mid p(a) = 1\}$. Now let $(L, \mathcal{U})$ be a quasi-uniform frame. For each $U \in \mathcal{U}$ let Σ_U be the system $\{(\Sigma_{u_1}, \Sigma_{u_2}) \mid (u_1, u_2) \in U\}$ and let $\Sigma_\mathcal{U}$ be the filter of paircovers of ΣL generated by $\{\Sigma_U \mid U \in \mathcal{U}\}$.

4.2.1. Proposition. *Let $(L, \mathcal{U})$ be a quasi-uniform frame. Then $\Sigma(L, \mathcal{U}) = (\Sigma L, \Sigma_\mathcal{U})$ is a quasi-uniform space.*

Proof. (1) Take a $U \in \mathcal{U}$. Then

$$\bigcup\{\Sigma_{u_1} \cap \Sigma_{u_2} \mid (u_1, u_2) \in U\} = \bigcup\{\Sigma_{u_1 \wedge u_2} \mid (u_1, u_2) \in U\} =$$
$$= \Sigma_{\bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\}} = \Sigma_1 = \Sigma L.$$

(2) Let $U, V \in \mathcal{U}$. Trivially $U \leq V$ implies $\Sigma_U \leq \Sigma_V$ and $\Sigma_{U \wedge V} = \Sigma_U \wedge \Sigma_V$.

(3) Let $U^* \leq V$. Then $(\Sigma_U)^* \leq \Sigma_V$. Indeed: for each $(u_1, u_2) \in U$ there exists $(v_1, v_2) \in V$ satisfying $\mathsf{st}_1(u_1, U) \leq v_1$ and $\mathsf{st}_2(u_2, U) \leq v_2$; then

$$\begin{aligned}\mathsf{st}_1(\Sigma_{u_1}, \Sigma_U) &= \bigcup\{\Sigma_{u_1'} \mid \Sigma_{u_2'} \cap \Sigma_{u_1} \neq \emptyset, (u_1', u_2') \in U\} = \\ &= \bigcup\{\Sigma_{u_1'} \mid \Sigma_{u_2' \wedge u_1} \neq \emptyset, (u_1', u_2') \in U\} \subseteq \\ &\subseteq \bigcup\{\Sigma_{u_1'} \mid u_2' \wedge u_1 \neq 0\} = \Sigma_{\mathsf{st}_1(u_1, U)} \subseteq \Sigma_{v_1}.\end{aligned}$$

Similarly, $\mathsf{st}_2(\Sigma_{u_2}, \Sigma_U) \subseteq \Sigma_{v_2}$. ☐

For a uniform homomorphism $h : (L, \mathcal{U}) \to (M, \nu)$ define $\Sigma h : \Sigma(M, \nu) \to \Sigma(L, \mathcal{U})$ by $\Sigma h(p) = ph$. It is easy to check that $\Sigma h \in \mathsf{QUnif}$. Thus we obtain a contravariant functor $\Sigma : \mathsf{QUFrm} \to \mathsf{QUnif}$.

4.2.2. Remark. It is also easy to check that the topologies $\mathfrak{T}_1(\Sigma_{\mathcal{U}})$ and $\mathfrak{T}_2(\Sigma_{\mathcal{U}})$ induced by the quasi-uniformity $\Sigma_{\mathcal{U}}$ on ΣL coincide with the spectral topologies of the spectra of $L_1(\mathcal{U})$ and $L_2(\mathcal{U})$ respectively.

4.3. Theorem. *The two above contravariant functors Ω and Σ constitute a dual adjunction, with units $\eta_{(X,\mu)} : (X, \mu) \to \Sigma\Omega(X, \mu)$ and $\xi_{(L,\mathcal{U})} : (L, \mathcal{U}) \to \Omega\Sigma(L, \mathcal{U})$ given by $\eta_{(X,\mu)}(a)(U) = 1$ iff $a \in U$ and $\xi_{(L,\mathcal{U})}(a) = \Sigma_a$.*

Proof. Checking that each $\eta_{(X,\mu)}$ is uniformly continuous and that each $\xi_{(L,\mathcal{U})}$ is a uniform frame homomorphism can be left to the reader.

The formulas define natural transformations $\eta : \mathrm{Id}_{\mathsf{Unif}} \stackrel{\cdot}{\to} \Sigma\Omega$ and $\xi : \mathrm{Id}_{\mathsf{QUFrm}} \stackrel{\cdot}{\to} \Omega\Sigma$. Indeed, consider the following diagrams with arbitrary $f : (X, \mu) \to (Y, \nu)$ and $h : (L, \mathcal{U}) \to (M, \mathcal{V})$:

$$\begin{tikzcd}(X,\mu) \arrow[r, "\eta_{(X,\mu)}"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & \Sigma\Omega(X,\mu) \arrow[d, "\Sigma\Omega(f)"] \\ (Y,\nu) \arrow[r, "\eta_{(Y,\nu)}"'] & \Sigma\Omega(Y,\nu)\end{tikzcd} \qquad \begin{tikzcd}(L,\mathcal{U}) \arrow[r, "\xi_{(L,\mathcal{U})}"] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & \Omega\Sigma(L,\mathcal{U}) \arrow[d, "\Omega(\Sigma h)"] \\ (M,\mathcal{V}) \arrow[r, "\xi_{(M,\mathcal{V})}"'] & \Omega\Sigma(M,\mathcal{V})\end{tikzcd}$$

For an $x \in X$,

$$\Sigma\Omega(f)(\eta_{(X,\mu)}(x)) = \eta_{(X,\mu)}(x)\Omega(f)$$

is the map $F : \Omega(Y, \nu) \to \{0, 1\}$ given by $F(B) = 1$ iff $x \in f^{-1}(B)$. Since $x \in f^{-1}(B)$ iff $f(x) \in B$, this is precisely the map $\eta_{(Y,\nu)}(f(x))$ and the diagram (1) commutes.

On the other hand, for an $a \in L$,

$$\Omega\Sigma h(\xi_{(L,\mathcal{U})}(a)) = \Omega\Sigma h(\Sigma_a) = (\Sigma h)^{-1}(\Sigma_a) = \{p \in \Sigma M \mid \Sigma h(p) \in \Sigma_a\}.$$

Since $\Sigma h(p) \in \Sigma_a$ iff $ph \in \Sigma_a$ iff the $p(h(a)) = 1$ iff $p \in \Sigma_{h(a)}$, the diagram (2) commutes as well.

Finally, $\eta : \mathrm{Id}_{\mathsf{Unif}} \stackrel{\cdot}{\to} \Sigma\Omega$ and $\xi : \mathrm{Id}_{\mathsf{QUFrm}} \stackrel{\cdot}{\to} \Omega\Sigma$ satisfy the adjunction identities $\Omega(\eta_{(X,\mu)}) \cdot \xi_{\Omega(X,\mu)} = 1$ and $\Sigma(\xi_{(L,\mathcal{U})}) \cdot \eta_{\Sigma(L,\mathcal{U})} = 1$. Indeed, for each A,

$$(\Omega(\eta_{(X,\mu)}) \cdot \xi_{\Omega(X,\mu)})(A) = \Omega(\eta_{(X,\mu)})(\Sigma_A) = \eta^{-1}_{(X,\mu)}(\Sigma_A)$$

and $x \in \eta^{-1}_{(X,\mu)}(\Sigma_A)$ iff $\eta_{(X,\mu)}(x) \in \Sigma_A$ iff $\eta_{(X,\mu)}(x)(A) = 1$ iff $x \in A$,

and for each $p : L \to \{0,1\}$ in ΣL,

$$(\Sigma(\xi_{(L,\mathcal{U})}) \cdot \eta_{\Sigma(L,\mathcal{U})})(p) = q \cdot \xi_{(L,\mathcal{U})}$$

where $q : \Omega\Sigma(L,\mathcal{U}) \to \{0,1\}$ maps A into 1 iff $p \in A$. But $(q \cdot \xi_{(L,\mathcal{U})})(a) = q(\Sigma_a)$ is equal to 1 iff $p \in \Sigma_a$, that is, iff $p(a) = 1$. Hence $q \cdot \xi_{(L,\mathcal{U})} = p$ as required. □

References

[1] B. Banaschewski, G. C. L. Brümmer and K. A. Hardie, Biframes and bispaces, *Quaest. Math.* 6 (1983) 13–25.

[2] B. Banaschewski and A. Pultr, Samuel compactification and completion of uniform frames, *Math. Proc. Camb. Phil. Soc.* 108 (1990) 63–78.

[3] M. J. Ferreira and J. Picado, Functorial quasi-uniformities on frames, *Appl. Categ. Structures* 13 (2005) 281–303.

[4] Fletcher P., Hunsaker W., Entourage uniformities for frames, *Monatsh. Math.* 112 (1991) 271–279.

[5] P. Fletcher and W. Lindgren, *Quasi-uniform Spaces*, Marcel Dekker, New York, 1982.

[6] J. Frith, *Structured frames*, PhD thesis, University of Cape Town, 1987.

[7] J. Frith, The category of quasi-uniform frames, *Research Reports*, Department of Mathematics, University of Cape Town, 1991.

[8] J. Frith and A. Schauerte, The Samuel compactification for quasi-uniform biframes, *Topol. Appl.* 156 (2009) 2116–2122.

[9] T. E. Gantner and R. C. Steinlage, Characterizations of quasi-uniformities, *J. London Math. Soc. (2)* 5 (1972) 48–52.

[10] J. R. Isbell, *Uniform Spaces*, Math. Surveys, vol. 12, Amer. Math. Soc., 1964.

[11] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* 31 (1972) 5–32.

[12] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Math., vol. 3, Cambridge Univ. Press, Cambridge, 1982.

[13] H.-P. Künzi, *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, in: C. E. Aull, R. Lowen (Eds.), *Handbook of the History of General Topology*, vol. 3, Kluwer, 2001, pp. 853–968.

[14] L. Nachbin, Sur les espaces uniformes ordonnés, *C. R. Acad. Sci. Paris* 226 (1948) 774–775.

[15] L. Nachbin, *Topology and Order*, D. van Nostrand, Princeton, 1965.

[16] J. Picado, Weil uniformities for frames, *Comment. Math. Univ. Carolin.* 36 (1995) 357–370.

[17] J. Picado, Frame quasi-uniformities by entourages, in: *Symposium on Categorical Topology (Univ. of Cape Town 1994)*, Department of Mathematics, Univ. of Cape Town, 1999, pp. 161–175.

[18] J. Picado, Structured frames by Weil entourages, *Appl. Categ. Structures* 8 (2000) 351–366.

[19] J. Picado and A. Pultr, *Locales mostly treated in a covariant way*, Textos de Matemática, vol. 41, University of Coimbra, 2008.

[20] A. Pultr, Pointless uniformities I. Complete regularity, *Comment. Math. Univ. Carolinae* 25 (1984) 91–104.

[21] A. Pultr, Pointless uniformities II. (Dia)metrization, *Comment. Math. Univ. Carolinae* 25 (1984) 105–120.

[22] A. Pultr, *Frames*, in: M. Hazewinkel (Ed.), *Handbook of Algebra*, vol. 3, Elsevier, 2003, pp. 791–858.

[23] A. Schauerte, Biframe compactifications, *Comment. Math. Univ. Carolin.* 34 (1993) 567–574.

[24] J. W. Tukey, *Convergence and uniformity in topology*, Ann. of Math. Studies 2, Princeton University Press, 1940.

[25] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Hermann, Paris, 1938.