Why some fuzzyfications are easier then others

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In honour of Petr Hájek on the occassion of his 70th birthday.

Abstract

When building a fuzzy variant of a theory of Gabriel-Ulmer type (such as a variety of algebras, a relational system, classical automata, etc.) it does not matter whether one models the theory directly in the universe of fuzzy sets or takes the corresponding crisp theory and fuzzifies it ex post.

Introduction

The concept of fuzzy set naturally led to fuzzy variants of crisp theories (fuzzy algebras, fuzzy automata, fuzzy spaces, etc.). Some of such efforts brought deep results, some others, however, appeared to be rather straightforward. In this paper we present a partial explanation of this phenomenon. Namely, it turns out that if the fuzzyfication is based on a value lattice without an additional structure (such as an additional operation, say residuation), and if the theory in question is locally presentable (varieties of algebras, relational systems, automata, etc., see Section 2) it does not matter whether one treats the theory in the fuzzy context, or takes the ready crisp theory and fuzzyfies it afterwards (4.4 below). Thus, the message is that one can expect more interesting results either with enriched value systems, or when discussing theories that are not locally presentable (like for instance spaces).

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I use the technique of category theory, but I wish the text to be easily readable for everyone who just knows the basic concepts (category, functor, transformation) – or is willing to look for them in the first pages of a textbook. Therefore, although the requirements are not very demanding, I am rather explicit in presenting the necessary (known) facts. This makes the text slightly longer, but not really much so: I do not think I could save much more than a page or two (and certainly I would not like to make cuts in the examples in Section 2).

A reader wishing for more about categories can consult, e.g., [6].

1 Preliminaries

1.1. If $\mathfrak{C}$ is a category and a, b some objects of $\mathfrak{C}$ we write

$$\mathfrak{C}(a,b) \tag{$*$}$$

for the set of all morpghisms $\phi : a \to b$ in $\mathfrak{C}$. It will be indeed always assumed a set, while the system $|\mathfrak{C}|$ of all the objects of $\mathfrak{C}$ may be a proper class. If $|\mathfrak{C}|$ is a set we speak of a *small category.* As a rule, small categories will be denote by roman capitals.

The identical morphism of an object a will be denoted by 1_a.

1.2. The opposite (dual) category of $\mathfrak{C}$, that is, the category obtained by keeping $|\mathfrak{C}|$ and formally reversing the directions of morphisms, will be denoted by

$$\mathfrak{C}^{\mathrm{op}}.$$

The category of all sets and mappings will be denoted by **Set**.

1.2.1 With a category $\mathfrak{C}$ we associate a functor

$$\mathfrak{C}(-,-) : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathbf{Set}$$

(the cartesian product $\mathfrak{A} \times \mathfrak{B}$ is structured by setting $1_{(a,b)} = (1_a, 1_b)$ and $(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', \beta\beta')$, of course) defined by

$$\mathfrak{C}(\alpha,\beta) = (\phi \mapsto \beta\phi\alpha) : \mathfrak{C}(a,b) \to \mathfrak{C}(a',b') \quad \text{for} \quad \alpha : a' \to \alpha, \beta : b \to b'$$

($\mathfrak{C}(a,b)$ as in ($*$) above). This functor will be often restricted to just one of the variables, say $\mathfrak{C}(a,-) : \mathfrak{C} \to \mathbf{Set}$. Then we write $\mathfrak{C}(a,\beta)$ for $\mathfrak{C}(1_a,\beta)$ which will hopefully create no confusion. Similarly, $\mathfrak{C}(-,b)$.

1.3. A *diagram* in $\mathfrak{C}$ is a functor $D : K \to \mathfrak{C}$ with K small. A *lower bound* of such D is a collection of morphisms $(\phi_k : x \to D(k))_{k\in|K|}$ such that for each $\kappa : k \to l$ in K, $D(\kappa)\phi_k = \phi_l$. A lower bound $(\lambda_k : a \to D(k))_{k\in|K|}$ is a *limit* of D if for each lower bound $(\phi_k : x \to D(k))_{k\in|K|}$ there is precisely one $\phi : x \to a$ such that $\lambda_k\phi = \phi_k$ for all $k \in |K|$.

A category $\mathfrak{C}$ is said to be *complete* if each diagram in $\mathfrak{C}$ has a limit.

1.4. Fact. *Each* $\mathfrak{C}(c, -) : \mathfrak{C} \to \mathbf{Set}$ *preserves all the existing limits.*

(Indeed, let $\lambda = (\lambda_k : a \to D(k))_k$ be a limit of $D : K \to \mathfrak{C}$ and let $(f_k : X \to \mathfrak{C}(c, D(k))_k$ be a lower bound of $\mathfrak{C}(c, D(-))$ Then for a $\kappa : k \to l$ in K and $x \in X$, $D(\kappa) \cdot f_k(x) = (\mathfrak{C}(c, \kappa)f_k)(x)$ and since λ is a limit there is exactly one $f(x) : c \to a$ such that $\lambda_k \cdot f(x) = f_k(x)$ for all k. This yields a mapping $f : X \to \mathfrak{C}(c, a)$ satisfying $\mathfrak{C}(c, \lambda_k) \cdot f = f_k$. $\square$)

1.5. If A is a small category and $\mathfrak{C}$ any category we denote by

$$\mathfrak{C}^A$$

the caqtegory of all functors $F : A \to \mathfrak{C}$ (as objects) and all transformations $\tau : F \to G$ between them as morphisms.

1.5.1. Proposition. *Let* $\mathfrak{C}$ *be a complete category. Then each* $\mathfrak{C}^A$ *is a complete category with the limits given componentwise, that is, a limit* $(\lambda^r : L \to D(r))_r$ *is obtained from the limits* $(\lambda^r_a : L(a) \to D(r)(a))_r$ *taken individually for all* $a \in |A|$.

Proof. For a diagram $D : R \to \mathfrak{C}^A$ consider the $D(-)(a) : R \to \mathfrak{C}$ and for each of them choose a limit

$$(\lambda^r_a : L(a) \to D(r)(a))_r. \tag{$*$}$$

For a morphism $\alpha : a \to b$ in A we have $D(\rho)_b \cdot D(r)(\alpha) \cdot \lambda^r_a = D(s)(\alpha) \cdot D(\rho)_a \cdot \lambda^r_a = D(s)(\alpha) \cdot \lambda^s_a$; hence, applying $(*)$ for $L(b)$, we obtain a unique $L(\alpha) : L(a) \to L(b)$ such that

$$\lambda^r_b \cdot L(\alpha) = D(r)(\alpha) \cdot \lambda^r_a.$$

This (with the unicity) makes L a functor, and all the $\lambda^r = (\lambda^r_a : L \to D(r))_a$ natural transformations.

Now let $\tau^r : F \to D(r))_r$ be a lower bound of D. Then for each $a \in |A|$, $(\tau^r_a : F(a) \to D(r)(a))_r$ is a lower bound of $D(-)(a)$ and hence there is precisely one $\theta_a : F(a) \to L(a)$ such that $\lambda^r_a \cdot \theta_a = \tau^r_a$ for all r. Now

$\lambda_b^r \cdot L(\alpha) \cdot \theta_a = D(r)(\alpha) \cdot \lambda_a^r \cdot \theta_a = D(r)(\alpha) \cdot \tau_a^r = \tau_b^r \cdot F(\alpha) = \lambda_b^r \cdot \theta_b \cdot F(\alpha)$, and since $(\lambda_b^r)_r$ is a limit we obtain $L(\alpha) \cdot \theta_a = \theta_b \cdot F(\alpha)$; thus, $\theta : F \to L$ is a transformation, unique such that $\lambda_-^r \cdot \theta = \tau^r$ for all r. $\square$

1.5.2. Observation. *For the product (as in 1.2.1) of small categories there is a natural isomorphism of categories*

$$\mathfrak{C}^{A\times B} \cong (\mathfrak{C}^A)^B.$$

(Namely, for a functor $F : A \times B \to \mathfrak{C}$ define $\widetilde{F} : A \to \mathfrak{C}^B$ by setting $\widetilde{F}(a)(b) = F(a, b)$, $\widetilde{F}(a)(\beta) = F(1_a, \beta)$ and $\widetilde{F}(\alpha)_b = F(\alpha, 1_b)$; for $G : A \to \mathfrak{C}^B$ and $\alpha : a \to a', \beta : b \to b'$ set $\widehat{G}(a, b) = G(a)(b)$, $\widehat{G}(\alpha, \beta) = G(\alpha)_{a'} \cdot G(a)(\beta) = G(a')(\beta) \cdot G(\alpha)_b$.)

1.6. A partially ordered set $L = (X, \leq)$ can be viewed as a category with $|L| = X$ and morphisms

$$(x \leq y) : x \to y$$

(thus, for any x, y there is at most one morphism $x \to y$).

The category L^{op} as in 1.2 then coincides with thus interpreted reversely ordered poset. In L^{op} we will often write

$$(x \geq y) : x \to y$$

using the original order of L.

Note that the limits in $L = (X, \leq)$ are the infima, and hence L is complete as a category iff it is a complete lattice.

1.7. The Yoneda embedding. Recall 1.2.1. For a small category A define a functor

$$\mathsf{Y} : A^{\mathrm{op}} \to \mathbf{Set}^A$$

by setting $\mathsf{Y}(a) = A(a, -)$ and $\mathsf{Y}(\alpha) = (\mathsf{Y}(\alpha)_c = A(\alpha, 1_c))_c : \mathsf{Y}(a) \to \mathsf{Y}(b)$ for $\alpha : b \to a$ (hence $\mathsf{Y}(\alpha)_c(\phi) = \phi\alpha$). $\mathsf{Y}(\alpha)$ is obviously a natural transformation since $A(\alpha, 1_c)A(1_a, \phi) = A(\alpha, \phi) = A(1_b, \phi)A(\alpha, 1_d)$.

1.7.1. Fact. Y *is an embedding of* A^{op} *into* $\mathbf{Set}^A$ *as a full subcategory.*

(Trivially $\mathsf{Y}(a) \neq \mathsf{Y}(b)$ if $a \neq b$ and if $\alpha \neq \beta$, $\alpha, \beta : b \to a$ then $\mathsf{Y}(\alpha)_b(1) = \alpha \neq \beta = \mathsf{Y}(\beta)_b(1)$; hence Y is one-one. Now let $\tau : \mathsf{Y}(a) \to \mathsf{Y}(b)$ be a natural transformation. Set $\alpha = \tau_a(1_a) \in \mathsf{Y}(b)(a) = A(b,a)$. Then since for any $\phi : a \to c$ the diagram

$$\begin{tikzcd}
\mathsf{Y}(a)(c) \arrow[r, "\tau_c"] & \mathsf{Y}(b)(c) \\
\mathsf{Y}(a)(a) \arrow[u, "A(1_a,\phi)"] \arrow[r, "\tau_a"] & \mathsf{Y}(b,a) \arrow[u, "A(1_b,\phi)"']
\end{tikzcd}$$

commutes we have $\tau_c(\phi) = \tau_c(A(1,\phi)(1)) = A(1,\phi)(\tau_a(1)) = \phi\alpha = \mathsf{Y}(\alpha)_c(\phi)$, and the embedding is full. $\square$)

2 Gabriel-Ulmer theories

2.1. Examples: Multigraphs and graphs. A multigraph $X = (A, V, b, e)$ consists of a set of arrows A, a set of vertices V, and two maps $b, e : A \to V$ (the beginning and the end of an arrow). A multigraph homomorphism consists of two maps $f_a : A_1 \to A_2$ and $f_v : V_1 \to V_2$ respecting the beginnings and ends, that is,

$$b_2(f_a(x)) = f_v(b_1(x)) \quad \text{and} \quad e_2(f_a(x)) = f_v(e_1(x)). \tag{2.1.1}$$

Thus we see that we can view a multigraph as a functor $X : M \to \mathbf{Set}$ where M has two objects a and v and just two nontrivial morphisms $\beta, \epsilon : a \to v$, as indicated in the following picture:

$$\begin{tikzcd}
a \arrow[d, bend right, "\beta"'] \arrow[d, bend left, "\epsilon"] & & X(a) \arrow[d, bend right, "X(\beta)"'] \arrow[d, bend left, "X(\epsilon)"] \\
v \arrow[r, phantom, ""] & & X(v)
\end{tikzcd}
\qquad \overset{X}{\dashrightarrow}$$

The conditions (2.1.1) say nothing else but that a multigraph homomorphism $f : X \to Y$ is a natural transformation between the functors X, Y: we have the commuting diagram

$$\begin{tikzcd}
X(a) \arrow[r, "f_a"] \arrow[d, shift right, "X(\beta)"'] \arrow[d, shift left, "X(\epsilon)"] & Y(a) \arrow[d, shift right, "X(\beta)"'] \arrow[d, shift left, "X(\epsilon)"] \\
X(v) \arrow[r, "f_v"] & Y(v)\ .
\end{tikzcd}$$

Thus we have represented the category of multigraphs as

$$\mathbf{Set}^M.$$

Now if we would wish to represent the category of (directed) graphs instead, we can restrict ourselves to the functors X in which the pair $X(\beta), X(\epsilon)$ is collectionwise monomorphic, that is, if $X(\beta)g = X(\beta)h$ and $X(\epsilon)g = X(\epsilon)h$ then $g = h$. This can be expressed as a limit-preserving condition. Note that (m_1, m_2) is a collectionwise monomorphic pair iff in the diagram

$$\begin{tikzcd}
A \arrow[r, dotted, "\mathrm{id}"] \arrow[d, dotted, "\mathrm{id}"'] & A \arrow[dr, "m_1"] \arrow[dd, "m_2"] & \\
A \arrow[rr, "m_1"] \arrow[dr, "m_2"] & & B_1 \\
& B_2 &
\end{tikzcd}$$

the dotted arrows constitute a limit of the rest.

2.2. Example. (Mealy) automata. Consider the category A described by the diagram

$$\begin{tikzcd}
p \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\pi_s"'] \arrow[d, "\pi_a"'] & s \\
a\ .
\end{tikzcd}$$

The category of Mealy automata can be represented as a full subcategory of $\mathbf{Set}^A$ determined by the functors $X : A \to \mathbf{Set}$ that send

$$\begin{tikzcd} p \arrow[r, "\pi_s"'] \arrow[d, "\pi_a"'] & s \\ a & \end{tikzcd} \tag{2.2.1}$$

to a product diagram. For such functors one sees easily that in a transformation $\tau : X \to Y$ one has $\tau_p = \tau_s \times \tau_a$ and the commutativity of the diagram

$$\begin{tikzcd} X(s) \times X(a) \arrow[r, "X(\pi_a)"] \arrow[d, "X(\pi_s)"'] \arrow[d, bend left, "X(\sigma)"] \arrow[drr, dotted, "\tau_p = \tau_s \times \tau_a"] & X(a) \arrow[drr, dotted, "\tau_a"] & & \\ X(s) \arrow[drr, dotted, "\tau_s"] & & Y(s) \times Y(a) \arrow[r, "Y(\pi_a)"'] \arrow[d, "Y(\pi_s)"'] \arrow[d, bend left, "Y(\sigma)"] & Y(a) \\ & & Y(s) & \end{tikzcd}$$

is what one knows as the definition of a homomorphism of automata.

2.3. Lawvere's representation of varieties of algebras. Construct a category A as follows. First, take the sets $n = \{0, 1, \ldots, n-1\}$ and all the maps between them, obtaining a category B. Note that in B, n is a coproduct of n-times 1, with the coproduct injections $\pi_j = (0 \mapsto j)$. Now

(1) add new formal morphism taking care that n remains the coproduct ${}^n1$ as before (which creates further additional morphism with each new one), obtaining a category C, and finally

(2) define A as the dual of C; thus, in A we have products

$$\pi_j : n = 1^n \to 1. \tag{2.3.1}$$

Now consider the subcategory of $\mathbf{Set}^A$ determined by all the functors that preserve all the products (2.3.1).

What it amounts to: First, the values $X(n)$ are determined by that of $X(1)$, namely as $X(1)^n$, with $X(\pi_j)$ the standard projections. Second, we have new mappings $X(\omega) : X(1)^n \to X(1)$ obtained from the extras $\omega : n \to 1$, the n-ary operations in the theory, making X to an algebraic structure. Finally, the compositions in A give rise to equalities of an algebraic theory. Now the fact of life is that each variety of algebras of a finite type can be represented this way.

2.4. Gabriel - Ulmer theories and locally presentable categories. A *Gabriel-Ulmer theory* $\mathbb{A} = (A, \mathcal{A})$ (further often just *theory*) consists of

- a small category A, and
- a system $\mathcal{A}$ of diagrams endowed with fixed lower bounds.

Let $\mathfrak{C}$ be a complete category. A model of a theory $\mathbb{A} = (A, \mathcal{A})$ in $\mathfrak{C}$ is a functor $X : A \to \mathfrak{C}$ such that it sends all the $D \in \mathcal{A}$ into limits in $\mathfrak{C}$. The category of models of $\mathbb{A}$ in $\mathfrak{C}$ will be denoted by

$$\mathfrak{C}^{\mathbb{A}}.$$

The categories $\mathbf{Set}^{\mathbb{A}}$ are often referred to as *locally presentable categories.*

2.4.1. Notes. A concept of the theory of the type presented above appeared, first, in an unpublished paper by Gabriel in the sixties. About the same time, Lawvere dealt with varieties of algebras in this vein ([5]). An early treatment can be found in [2] (where the term "locally presentable" occurs). For a reader wishing for more on the subject we can recommend the modern monograph [1].

It should be noted that important relevant ideas also appeared in the Isbell's paper [4].

2.5. Notes. 1. Thus, we have seen in 2.1 the theory of multigraphs as $(M, \emptyset)$ and that of (directed) graphs as $(M, \mathcal{A})$ where $\mathcal{A}$ consists of the single lower-bound diagram

$$\begin{tikzcd}
a \arrow[r, dotted, "\mathrm{id}"] \arrow[d, dotted, "\mathrm{id}"'] & a \arrow[dd, "\epsilon"] \arrow[dr, "\beta"] & \\
a \arrow[rr, "\beta"] \arrow[dr, "\epsilon"'] & & v \\
& v &
\end{tikzcd}$$

In 2.2 we have the $\mathcal{A}$ consisting of the lower-bound diagram (2.2.1), and in 2.3 the $\mathcal{A}$ consists of all the coproduct diagrams (2.3.1).

2. A simpler fact that will be used in the sequel is that $\alpha : A \to b$ is a monomorphism iff

$$\begin{tikzcd} a \arrow[r, dotted, "1_a"] \arrow[d, dotted, "1_a"'] & a \arrow[d, "\alpha"] \\ a \arrow[r, "\alpha"] & b \end{tikzcd}$$

is a limit (if it is and $\alpha\beta = \alpha\gamma$ then there is exactly one δ such that $1_a\delta = \beta$ and $1_a\delta = \gamma$; if α is a monomorphism and $\alpha\beta = \alpha\gamma$ then $\beta = \gamma$ and $1\beta = \beta$ and $1\beta = \gamma$).

3. With the exception of the automata in 2.2, the systems of lower-bound diagrams were themselves limit diagrams in the small categories in question. In fact, as a rule it is not hard to extend the category A in such a way that we have only limit diagrams in $\mathcal{A}$ (one can then speak of the models as the functors that preserve certain specified limits; of course such an extension can be somewhat messy).

$\mathcal{A}$ consisting of limits has its advantages. The Yoneda functor Y (recall 1.7) then embeds the dual category A^{op} into $\mathbf{Set}^{\mathbb{A}}$ as a full subcategory and the objects $Y(a)$ are typically rather important (for instance in the Lawvere representation the $Y(n)$ is the free algebra with n generators). Generally, the copy $\mathsf{Y}[A^{\mathrm{op}}]$ of A^{op} in $\mathbf{Set}^{\mathbb{A}}$ is something like a basis from which all the objects of $\mathbf{Set}^{\mathbb{A}}$ can be constructed in a canonical way (see e.g. [9],[7]).

4. All our examples concerned models in the category of sets. To give another example, consider the category **Top** of topological spaces and continuous mappings. Then for a Lawvere theory $\mathbb{A}$, $\mathbf{Top}^{\mathbb{A}}$ represents the corresponding category of topological (continuous) algebras; if, say, $\mathbb{A}$ is the theory of groups, $\mathbf{Top}^{\mathbb{A}}$ is the category of topological groups. Similarly, topological graphs, ordered semigroups, etc.

2.6. Proposition. $\mathfrak{C}^{\mathbb{A}}$ *is closed under limits in* $\mathfrak{C}^{A}$.

Proof. Let $S : R \to \mathfrak{C}^A$ be a diagram and let $(\theta_r : L \to S_r)_{r\in|R|}$ be its limit in $\mathfrak{C}^A$ (we write S_r instead of $S(r)$ to simplfy the notation). Hence (recall 1.5.1), for all $x \in |A|$

$$((\theta_r)_x : L(x) \to S_r(x))_r \;\text{ is a limit in } \mathfrak{C}. \qquad (*)$$

Let all the S_r be in $\mathfrak{C}^{\mathbb{A}}$ end let $(\alpha_k : a \to D(k))_{k\in|K|}$ be in $\mathcal{A}$. Then for all $r \in |R|$

$$(S_r(\alpha_k) : S_r(a) \to S_r(D(k)))_{k\in|K|} \text{ is a limit in } \mathfrak{C}. \qquad (**)$$

Let $\phi_k : X \to L(D(k)))_k$ be a lower bound of LD, that is, for each $\kappa : k \to l$ in K, $LD(\kappa)\phi_k = \phi_l$. Consider the system

$$((\theta_k)_{D(k)} \cdot \phi_k : X \to S_r(D(k)), \quad k \in |K|.$$

It is a lower bound of $S_r \cdot D$: indeed, for $\kappa : k \to l$,

$$S_r(D(\kappa)) \cdot (\theta_r)_{D(k)} \cdot \phi_k = (\theta_r)_{D(l)} \cdot L(D(\kappa)) \cdot \phi_k = (\theta_r)_{D(k)} \cdot \phi_l.$$

Thus, there is a $\psi_r : X \to S_r(a)$ such that $S_r(\alpha_k) \cdot \psi_r = (\theta_r)_{D(l)} \cdot \phi_k$. We will show that $(\psi_r : X \to S_r(a))_r$ is a lower bound of the functor $(S_r(a), S(\rho)_a)_{r\in|R|,\rho\in R}$. Indeed, let $\rho : r \to s$ be a morphism in S. Then since $S(\rho) \cdot \theta_r = \theta_s$ we have

$$\begin{aligned} S_r(\alpha_k) \cdot S(\rho)_a \cdot \psi_r = S(\rho)_{D(k)} \cdot S_r(\alpha_k) \cdot \psi_S = S(\rho)_{D(k)} \cdot (\theta_r)_{D(k)} \cdot \phi_k = \\ = (\theta_s)_{D(k)} \cdot \phi_k = S_s(\alpha_k)_{D(k)} \cdot \psi_s \end{aligned}$$

and since $(**)$ is a limit, $S(\rho)_a \cdot \psi_r = \psi_s$.

Thus, using the limit $(\theta_k)_a$, $r \in |S|$, from $(*)$ we obtain a $\phi : X \to L(a)$ such that $(\theta_r)_a \cdot \phi = \psi_r$ for all r. Now

$$(\theta_r)_{D(k)} \cdot L(\alpha_k) \cdot \phi = S_r(\alpha_k) \cdot (\theta_r)_a \cdot \phi = S_r(\alpha_k) \cdot \psi_r = (\theta_r)_{D(k)} \cdot \phi_k$$

and since $((\theta_r)_{D(k)} : L(D(k) \to S_r(D(k))_r$ is a limit, finally $L(\alpha_k) \cdot \phi = \phi_k$.

The unicity of such ϕ is easily proved by backtracking the equalities and can be left to the reader. $\square$

3 Fuzzification as a GU-theory

3.1. Let L be a lattice (in the original papers on fuzziness, e.g. [10], L was the unit interval, here we will have in 3.1 and 3.3 a general lattice, in 3.2 and 3.4 a complete one).

In (perhaps) the simplest form, a *fuzzy set* (more precisely an *L-fuzzy set*) is a set with fuzzy membership of elements $(X, (\in_a)_{a\in L})$ satisfying

$$a \le b \;\;\&\;\; x \in_b X \quad\Rightarrow\quad x \in_a X.$$

Thus, if we set

$$X(a) = \{x \mid x \in_a X\},$$

the fuzzy structure appears as a stratification $(X(a))_{a\in L}$ of the set X such that

$$a \leq b \quad \Rightarrow \quad X(b) \subseteq X(a).$$

A morphism $f : (X, (\in_a^X)_a) \to (Y, (\in_a^Y)_a)$ between fuzzy sets is an $f : X \to Y$ such that

$$x \in_a^X X \;\Rightarrow\; f(x) \in_a^Y Y \quad \text{(in other words,} \quad f[X(a)] \subseteq Y(a)\ )$$

for all $a \in L$.

The resulting category will be denoted by

$$L\mathbf{Fuzz}_w.$$

3.2. The definition above is rather weak and, in particular, does not produce a satisfactory system of fuzzy subsets.

Take the definition of a fuzzy map. One can think of a fuzzy subset $(Y, (\epsilon_a^Y)_a)$ of $(X, (\epsilon_a^X)_a)$ if $Y \subseteq X$ and if the embedding map is a morphism, that is, $x \in_a^Y Y \;\Rightarrow\; x \in_a^X X$. Now consider an $x \in X$ with $x \in_a X$ and the fuzzy set

$$\overline{x}_a$$

carried by $\{x\}$ to which x belongs in the degree a but not more. If the union $u = \bigcup\{\overline{x}_a \mid x \in_a X\}$ exists it is carried by $\{x\}$ again, the degree of membership is $\geq a$ for all a with $x \in_a X$, and on the other hand u is a fuzzy subset of our $(X, (\in_a)_a)$. Thus,

- for every x there is a maximum $\alpha(x)$ such that $x \in_{\alpha(x)} X$,

and the full information on the fuzziness is given by the map $\alpha : X \to L$ (that is, $x \in_a X$ iff $a \leq \alpha(x)$). The morphisms $f : (X, \alpha) \to (Y, \beta)$ are then characterized by the property $\beta(f(x)) \geq \alpha(x)$.

The category of thus represented (more special) fuzzy sets will be denoted by

$$L\mathbf{Fuzz}$$

3.2.1. Remark. L**Fuzz** is what one usually takes for the plain category of fuzzy sets (that is, without more structure – an extra operation, for instance – on L), see e.g. [3],[8]. It should be noted that here one already has a satisfactory structure of fuzzy subsets: we can take the $(X,\beta) \subseteq (X,\alpha)$ with $\beta \le \alpha$ (the $(Y,(\epsilon_a^Y)_a) \subseteq (X,(\epsilon_a^X)_a)$ with $Y \subseteq X$ above are more handily represented by extending β by 0 on $X \setminus Y$). We then have, of course, $\bigcup_{i\in J}(X,\alpha_i) = (X, \sup_{i\in J}\alpha_i)$ and $\bigcap_{i\in J}(X,\alpha_i) = (X, \inf_{i\in J}\alpha_i)$.

3.2.2. Lemma. *Let L be a complete lattice. Then the following statements on a fuzzy structure $(\in_a)_a$ on X (resulting in the stratification $(X(a))_a$) are equivalent.*

(1) *There is a mapping $\alpha : X \to L$ such that*

$$x \in_a X \quad \textit{(that is, } x \in X(a)\textit{)} \quad \textit{iff} \quad a \le \alpha(x).$$

(2) *For every $M \subseteq L$,*

$$\bigcap_{m\in M} X(m) = X(\sup M).$$

(3) $\bigcap\{X(a) \mid x \in_a X(a)\} = X(\sup\{a \mid x \in_a X\})$.

(4) *For every $x \in X$ there is a maximum $s \in L$ such that $x \in_s X$.*

Proof. (1)⇒(2): $x \in \bigcap_{m\in M} X(m)$ iff $\forall m \in M,\ m \le \alpha(x)$ iff $\sup M \le \alpha(x)$ iff $x \in X(\sup M)$.

(2)⇒(3) is trivial.

(3)⇒(1): Set $\alpha(x) = \sup\{a \mid x \in X(a)\} = \sup\{a \mid x \in_a X\}$. Then trivially $x \in_a X$ implies $a \le \alpha(x)$. On the other hand, by definition $x \in X(a)$ for $x \in_a X$ and hence $x \in \bigcap\{X(a) \mid x \in_a X(a)\} = X(\alpha(x))$, and hence if $a \le \alpha(x)$ then also $x \in X(a)$.

(3)≡(4): the statements are just reformulations of each other. Of course, $s = \sup\{a \mid x \in_a X\}$. □

3.3. Let L^{op} be the lattice L with the inverse order, taken as a small category (see 1.6). Note that all of its morphisms are monomorphisms. Set

$$\mathbb{F}w = (L^{\mathrm{op}}, \mathcal{A}')$$

where $\mathcal{A}'$ is the system of all the lower bounds

$$\begin{tikzcd} a \arrow[r, dotted, "1_a"] \arrow[d, dotted, "1_a"'] & a \arrow[d, "\alpha"] \\ a \arrow[r, "\alpha"] & b \end{tikzcd}$$

(amounting to the condition that in the functors $X \in \mathfrak{C}^{\mathbb{F}}w$ all the $X((a \geq b))$ are monomorphic – recall 2.5.2).

In particular, for $\mathfrak{C} = \mathbf{Set}$, we have all the $X((a \geq b)) : X(a) \to X(b)$ one-one.

3.3.1. For an $X \in \mathbf{Set}^{\mathbb{F}w}$ define $X' = X'(0) = X(0)$ and for general $a \in L$,

$$X'(a) = X((a \geq 0))[X(a)].$$

Let $a \geq b$ in L. We have $(a \geq 0) = (b \geq 0)(a \geq b)$ in L^{op} and hence

$$X'(a) = X((b \geq 0))[X((a \geq b))[X(a)]] \subseteq X'(b)$$

so that

$$a \geq b \quad \Rightarrow \quad X'(a) \subseteq X'(b),$$

and we have a fuzzy set as in 3.1, Further, defining $X'((a \geq b))$ as the inclusion maps we obtain another functor $X' \in \mathbf{Set}^{\mathbb{F}w}$.

3.3.2. Observation. In $\mathbf{Set}^{\mathbb{F}w}$, *$X'$ is isomorphic to X.*

(Indeed, define $\tau_a : X(a) \to X'(a)$ by $\tau_a(x) = X((a \geq a))(x)$. Obviously each τ_a is one-one onto, and it is easy to check that $\tau = (\tau_a)_a$ is a natural transformation.)

3.3.3. Corollary. $\mathbf{Fuzz}_w$ *is equivalent to* $\mathbf{Set}^{\mathbb{F}w}$. *More precisely, although* $\mathbf{Set}^{\mathbb{F}w}$ *has more objects, each of them is isomorphic to an object of* $\mathbf{Fuzz}_w$.

3.4. To represent the more satisfactory L**Fuzz** (now, L is complete), consider

$$\mathbb{F} = (L^{\mathrm{op}}, \mathcal{A})$$

where $\mathcal{A}$ is $\mathcal{A}'$ extended by all the infimum diagrams in L^{op} (that is, all the supremum diagrams in L). By 3.2.2(2) and 3.3,

*L**Fuzz** is, up to isomorphisms, represented as* $\mathbf{Set}^{\mathbb{F}}$.

4 Product of GU-theories. Application

4.1. Let A, B be small categories. For a lower bound $\mathfrak{a} = (\alpha_k : a \to D(k))_{k\in|K|}$ and an object b in B define a lower bound $\mathfrak{a} * b$ in $A \times B$ by setting

$$\mathfrak{a} * b = (\alpha_k \times 1_b : (a, b) \to (D(k), b))_{k\in|K|}.$$

Similarly we define

$$a * \mathfrak{b} = (1_a \times \beta_k : (a, b) \to (a, D(k)))_{k\in|K|}$$

for $a \in |A|$ and lower bounds $\mathfrak{b}$ in B.

Note. To avoid too many symbols we keep denoting the diagram in a lower bound by D and its domain by K. Of course they typically differ for distinct elements of $\mathcal{A}$ resp. $\mathcal{B}$.

4.1.2. Define a *product of theories*

$$\mathbb{A} \times \mathbb{B} \quad \text{as} \quad (A \times B, \mathcal{A} \oplus \mathcal{B})$$

by setting

$$\mathcal{A} \oplus \mathcal{B} = \{\mathfrak{a} * b \mid \mathfrak{a} \in \mathcal{A},\ b \in |B|\} \cup \{a * \mathfrak{b} \mid \mathfrak{b} \in \mathcal{B},\ a \in |A|\}.$$

(The word "product" is here used simply for an operation producing a new object from two others, not for a categorical product. We do not introduce a category of theories.)

4.2. Theorem. *The equivalence of categories $\mathfrak{C}^{A\times B} \cong (\mathfrak{C}^B)^A$ from 1.5.2 restricts to an equivalence*

$$\mathfrak{C}^{\mathbb{A}\times\mathbb{B}} \cong (\mathfrak{C}^{\mathbb{B}})^{\mathbb{A}}.$$

Proof. Recall the functor $\widehat{F} : A \times B \to \mathfrak{C}$ from 1.5.2 associated with $F : A \to \mathfrak{C}^B$. We have to prove that

(1) $\widehat{F}$ sends all the lower bounds from $\mathcal{A} \otimes \mathcal{B}$ to limits in $\mathfrak{C}$

if and only if

(2) each $F(a)$ sends all the lower bounds from $\mathcal{B}$ to limits in $\mathfrak{C}$, and F sends the lower bounds from $\mathcal{A}$ to limits in $\mathfrak{C}^{\mathbb{B}}$ (that is, by 2.6, limits in $\mathfrak{C}^B$).

(1)⇒(2): Consider a $(\beta_k : b \to D(k))_{k\in|K|}$ from $\mathcal{B}$. Now $(F(a)(\beta_k) : F(a)(b) \to F(a)(D(k))_k = (\widehat{F}(1_a, \beta_k) : \widehat{F}(a,b) \to \widehat{F}(1_a, D(k)))_k$ is a limit in $\mathfrak{C}$ since $((1_a, \beta_k) : (a,b) \to (1_a, D(k)))_k$ is in $\mathcal{A}{\otimes}\mathcal{B}$. Thus, $F(a)$ is in $\mathfrak{C}^{\mathbb{B}}$. Now let $(\alpha_k : a \to D(k))_k$ be in $\mathcal{A}$. The system $F(\alpha_k) : F(a) \to F(D(k)))_k$ is a limit in $\mathfrak{C}^{\mathbb{B}}$ iff each $F(\alpha_k)(1_b) : F(a)(b) \to F(D(k))(b))_k$ is a limit in $\mathfrak{C}$. But it is $(\widehat{F}(\alpha_k, 1_b) : \widehat{F}(a,b) \to \widehat{F}(D(k), b))_k$, and $(\alpha_k, 1_b) : (a,b) \to (D(k), b))_k$ is in $\mathcal{A} \otimes \mathcal{B}$.

(2)⇒(1): Consider an $(\alpha_k : a \to D(k))_k$ from $\mathcal{A}$ and any of the corresponding $(\alpha_k, 1_b) : (a,b) \to (D(k), b))_k$ in $\mathcal{A} \otimes \mathcal{B}$. We have $(\widehat{F}(\alpha_k, 1_b) : \widehat{F}(a,b) \to \widehat{F}(D(k), b))_k = (F(\alpha_k)(b) : F(a)(b) \to F(D(k))(b))_k$ limits by 1.5.1 since $(F(\alpha_k) : F(a) \to F(D(k))_k$ is a limit. Finally if $\mathfrak{b} = (\beta_k : b \to D(k))_k$ is in $\mathcal{B}$, $(\widehat{F}(1_a, \beta_k) : \widehat{F}(a,b) \to \widehat{F}(a, D(k))_k$ is a limit since $F(a)$ sends $\mathcal{B}$ to a limit. □

4.3. Corollary. $(\mathfrak{C}^{\mathbb{B}})^{\mathbb{A}} \cong (\mathfrak{C}^{\mathbb{A}})^{\mathbb{B}}$.
(Indeed, we have

$$(\mathfrak{C}^{\mathbb{B}})^{\mathbb{A}} \cong \mathfrak{C}^{\mathbb{B}\times\mathbb{A}} \cong \mathfrak{C}^{\mathbb{A}\times\mathbb{B}} \cong (\mathfrak{C}^{\mathbb{A}})^{\mathbb{B}}$$

where the middle isomorphism follows from the obvious correspondence $F'(\alpha, \beta) = F(\beta, \alpha)$.)

4.3.1. A simple example. Ordered semigroups can be studied as semigroups in the category of posets (that is, assuming the operation monotone), or as an order in the category of semigroups (that is, the relations $\leq$ are subsemigroups of the semigroups $S \times S$).

4.4. Using the representations from Section 3 we obtain

Corollary. For any Gabriel-Ulmer theory $\mathbb{A}$,

$$L\mathbf{Fuzz}_w^{\mathbb{A}} \cong (\mathbf{Set}^{\mathbb{A}})^{\mathbb{F}w} \quad \text{and} \quad L\mathbf{Fuzz}^{\mathbb{A}} \cong (\mathbf{Set}^{\mathbb{A}})^{\mathbb{F}}.$$

(Roughly speaking, treating a theory $\mathbb{A}$ in the context of fuzzy sets is the same as taking the standard crisp models of the theory and fuzzyfying them afterwards.)

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