Fractional total colourings of graphs of high girth*

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Abstract

Reed conjectured that for every $\epsilon > 0$ and Δ there exists g such that the fractional total chromatic number of a graph with maximum degree Δ and girth at least g is at most $\Delta + 1 + \epsilon$. We prove the conjecture for $\Delta = 3$ and for even $\Delta \geq 4$ in the following stronger form: For each of these values of Δ, there exists g such that the fractional total chromatic number of any graph with maximum degree Δ and girth at least g is equal to $\Delta + 1$.

1 Introduction

Total colouring and edge colouring share many common features. For instance, Vizing's theorem asserts that the chromatic index of any graph with maximum degree Δ is at most $\Delta + 1$. The total chromatic number of such

*Supported by project GAČR 201/09/0197 of the Czech Science Foundation.

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a graph is known to be at most $\Delta + C$, where C is a constant, and is conjectured to be at most $\Delta + 2$. Asymptotically, these bounds are far from the trivial upper bounds of $2\Delta - 1$ and 2Δ, respectively.

In other ways, however, the two notions behave differently. Consider their fractional versions (see below for the necessary definitions). It is known that the fractional chromatic index of a cubic bridgeless graph is equal to 3, the obvious lower bound. The analogous assertion for fractional total colouring is false, as shown by the graph K_4, whose fractional total chromatic number is 5. One might ask whether high girth makes the fractional total chromatic number arbitrarily close to $\Delta + 1$. Indeed, Reed [13] conjectured that this is exactly the case (see Conjecture 5 below). In this paper, we confirm the conjecture for $\Delta = 3$ and for even Δ, in a stronger form.

Before stating the result in detail, we introduce the relevant terminology. Let G be a graph. The vertex and edge sets of G will be denoted by $V(G)$ and $E(G)$. Let w be a function assigning each independent set I of G a real number $w(I) \in [0, 1]$. The *weight* $w[x]$ of $x \in V(G)$ with respect to w is defined as the sum of $w(I)$ over all independent sets I in G containing x. We also use this term for multisets of independent sets in G: if $\mathcal{W}$ is such a multiset, we define the *weight* of $x \in V(G)$ with respect to $\mathcal{W}$ as the number of sets in $\mathcal{W}$ containing x.

The function w is a *fractional colouring* of G if for each vertex v of G,

$$w[v] \geq 1.$$

The *size* $|w|$ of a fractional colouring w is the sum of $w(I)$ over all independent sets I. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of $|w|$ as w ranges over fractional colourings of G. It is easy to see that $\chi_f(G) \leq \chi(G)$. It is also known (see, e.g., [15, p. 42]) that $\chi_f(G)$ is rational and, although it is defined as an infimum, there exists a fractional colouring of size $\chi_f(G)$. Moreover, among the optimal fractional colourings there exists a rational-valued one.

Fractional colourings may be viewed in several ways, each of which can be useful in a different context. A basic observation concerning their equivalence is given by the following lemma.

Lemma 1. *Let G be a graph. The following are equivalent:*

(i) $\chi_f(G) \leq k$,

(ii) there exists an integer N and a multiset $\mathcal{W}$ of $k \cdot N$ independent sets in G, such that each vertex is contained in exactly N sets of $\mathcal{W}$,

(iii) there exists a probability distribution π on independent sets of G such that for each vertex v, the probability that v is contained in a random independent set (with respect to π) is at least $1/k$. $\square$

For more details on fractional colouring, we refer the reader to [15].

The *fractional chromatic index* $\chi'_f(G)$ of G is defined as the fractional chromatic number of the line graph $L(G)$. An important result concerning this parameter follows from the work of Edmonds [5] (also see Seymour [16]):

Theorem 2. *The fractional chromatic index of a bridgeless cubic graph G equals 3. Equivalently, there is a multiset of $3N$ perfect matchings in G such that each edge is contained in exactly N of them.*

The *total graph* $T(G)$ of G has vertex set $V(G) \cup E(G)$; a pair xy is an edge of $T(G)$ if one of the following holds:

- x and y are adjacent vertices of G,
- x is an edge of G and y is one of its endvertices,
- x and y are incident edges of G.

Independent sets in $T(G)$ are called *total independent sets* of G. The *total chromatic number* $\chi''(G)$ of G is defined as $\chi(T(G))$. Similarly, a *fractional total colouring* of G is simply a fractional colouring of $T(G)$, and we define the *fractional total chromatic number* $\chi''_f(G)$ of G as $\chi_f(T(G))$.

Let us stress that when applying Lemma 1 to total fractional colourings, one has to work with total independent sets. Thus, for instance, $\chi''_f(G) = k$ is equivalent to the existence of kN total independent sets in G such that each vertex and each edge are contained in N of the sets.

Behzad [2] and Vizing [17] independently conjectured the following upper bound on $\chi''(G)$:

Conjecture 3 (Total colouring conjecture)**.** *For any graph with maximum degree Δ,*

$$\chi''(G) \leq \Delta + 2.$$

Currently the best upper bound on $\chi''(G)$ in terms of the maximum degree Δ of G is due to Molloy and Reed [11] who proved that $\chi''(G)$ is bounded by $\Delta + C$ for a suitable constant C.

Kilakos and Reed [9] proved the analogue of Conjecture 3 for the fractional version of total colouring:

Theorem 4. *For any graph G with maximum degree Δ,*

$$\chi''_f(G) \leq \Delta + 2.$$

Recently, Ito et al. [8] showed that the only graphs G with $\chi''_f(G) = \Delta+2$ are K_{2n} and $K_{n,n}$ $(n \geq 1)$.

As mentioned above, Reed [13] conjectured that high girth makes the fractional total chromatic number close to $\Delta + 1$:

Conjecture 5. *For every $\varepsilon > 0$ and every integer Δ, there exists g such that the fractional chromatic number of any graph with maximum degree Δ and girth at least g is at most $\Delta + 1 + \varepsilon$.*

In the present paper, we prove the conjecture in a stronger form for $\Delta = 3$ (we call graphs G with maximum degree 3 *subcubic*). The argument also applies for even $\Delta \geq 4$.

Our first main result is the following theorem:

Theorem 6. *If G is a subcubic graph of girth at least* 15 840, *then*

$$\chi''_f(G) = 4.$$

As noted above, this confirms a particular case of Conjecture 5. In Sections 3–5, we first prove Theorem 6 for graphs G which are cubic and bridgeless. In Section 6, the result is extended to subcubic graphs. Finally, in Section 7, we prove the second main result of this paper:

Theorem 7. *For any* even *integer Δ, there exists a constant $g(\Delta)$ such that if G is a graph with maximum degree Δ and girth at least $g(\Delta)$, then $\chi''_f(G) = \Delta + 1$.*

2 Overview of the method

We now present an overview of our method, restricting our attention to the cubic bridgeless case. In the proof of Theorem 6, the required fractional total colouring is obtained indirectly, by constructing a suitable probability distribution and using Lemma 1.

To show that a cubic graph G has $\chi''_f(G) = 4$, it suffices to construct a probability distribution π on total independent sets such that each vertex and edge is included in a random total independent set with probability at least $1/4$. Consider the set Y consisting of a vertex of G and the three edges incident with it. Since any total independent set contains at most one object from Y, we must ensure that every total independent set T with $\pi(T) > 0$ contains exactly one element of Y. We arrive at the following definitions.

We will say that a set $X \subseteq V(G) \cup E(G)$ *covers* a vertex $v \in V(G)$ if $v \in X$ or v is incident with an edge in X. A set covering every vertex is *full*. The set of full total independent sets of G will be denoted by $\Phi(G)$.

For the reason outlined above, π will assign nonzero probability to full sets only. Under this provision, it is clear that if each $x \in V(G) \cup E(G)$ has the same probability of being included in a π-random total independent set, then $\chi''_f(G) = 4$.

The distribution is constructed by means of a probabilistic algorithm described in Section 4. The algorithm produces a full total independent set $\tilde{T}$ for any given choice of an (oriented) 2-factor F in G. It will be observed that for a fixed choice of F, all the edges of F have the same chance of being included in $\tilde{T}$. The probability of inclusion in $\tilde{T}$ is also constant on the edges not in F, as well as on the vertices of G. To ensure that the edges in F get the same probability as those not in F, we 'average' using Theorem 2 which guarantees the existence of a multiset $\mathcal{W}$ of perfect matchings such that every edge is contained in one third of the members of $\mathcal{W}$. By running the algorithm with F ranging over complements of all the perfect matchings from $\mathcal{W}$ and taking the average of the distributions thus produced, we indeed make the probability constant on all of $E(G)$. It will also be constant on $V(G)$, but the two constant values will not be the same. Luckily, we will observe (using the results of Section 3) that the probability of inclusion for a vertex is higher than for an edge. This will enable us to augment the distribution to the desired one, essentially by taking a weighted average with a distribution on perfect matchings obtained from Theorem 2.

In the remainder of this section, we introduce some more notation and terminology for later use.

An *oriented 2-factor* in a graph G is a 2-factor with a specified orientation of each of its cycles. Assume an oriented 2-factor is chosen. For $v \in V(G)$, v^- and v^+ denote the precedessor and successor of v on F with respect to the given orientation of F. Similarly, if $e \in E(F)$, then e^- is the edge that precedes e on F and e^+ is the edge that follows it. The *left* (*right*) end of an edge or a subpath of F is its first (last) vertex with respect to the given orientation.

A path with endvertices u and v will also be referred to as a *uv-path*.

We will occassionally need to speak about the distance between two edges e and f of G. This is defined as the distance between e and f in the total graph $T(G)$. The distance of a vertex from an edge is defined similarly. In particular, note that the distance between an edge and its endvertex is 1.

For an integer i, we define the *i-neighbourhood* $N_i(e)$ of an edge $e \in E(G)$ as the set of all the vertices of G whose distance from e is at most i, and all the edges with both endvertices in $N_i(e)$. If B is a set of edges, then $N_i(B)$ is the union of all $N_i(e)$ as e ranges over B.

3 A recurrence

The purpose of this section is to analyse two sequences of real numbers, $p_k(i)$ and $q_k(i)$, needed later in Section 4. In that section, we will present an algorithm that constructs a random total independent set T in a graph G whose vertices and edges are divided into k 'levels'. It will eventually turn out that the probability of the inclusion of a vertex (edge, respectively) x in the resulting total independent set is $q_k(i)$ ($p_k(i)$, respectively), conditioned on x being at level i. We postpone the details to the next section.

Let k be a positive integer. For $i = 1, \dots, k$, we define the values $p_k(i)$ and $q_k(i)$ by the recurrence

$$\begin{aligned} 2p_k(i) + q_k(i) &= 1, \\ q_k(i) &= p_k(i)\left(1 - \frac{1}{k} - \frac{1}{k}\sum_{j=1}^{i-1} p_k(j)\right). \end{aligned} \tag{1}$$

Observe that $p_k(1) = k/(3k-1)$ and $q_k(1) = (k-1)/(3k-1)$. We set

$$p_k^* = \sum_{i=1}^{k} \frac{p_k(i)}{k} \quad \text{and} \quad q_k^* = \sum_{i=1}^{k} \frac{q_k(i)}{k} .$$

We want to understand the values of $p_k(i)$ and $q_k(i)$ as k becomes very large. In particular, we will need to know that $q_k^* \geq 1/4$ for large enough k. It suffices to prove the following.

Lemma 8. *We have*

$$\lim_{k\to\infty} p_k^* = 3 - \sqrt{7}.$$

Proof. For each $k \geq 1$, consider the piecewise linear function $h_k(x)$ on the interval $[0, 1]$ satisfying

$$h_k\left(\frac{i-1}{k-1}\right) = p_k(i)$$

for $i = 1, \ldots, k$, and linear on each interval $[\frac{i-1}{k-1}, \frac{i}{k-1}]$. It can be shown that for fixed $x \in [0, 1]$, the sequence $(h_k(x))_{k=1}^{\infty}$ converges; we define $f(x)$ to be its limit. By the Arzelà–Ascoli theorem (see, e.g., [14, p. 169]), the resulting function f on $[0, 1]$ is continuous and the convergence of h_k to f is uniform.

The sum p_k^* can be viewed as a Riemann sum which approaches $\int_0^1 f(x)\,dx$ as k tends to infinity. Combining the two equations in (1), we obtain

$$p_k(i) = \frac{k}{3k - 1 - \sum_{j=1}^{i-1} p_k(j)},$$

which implies that consecutive values of p_k are related by the equation

$$\frac{1}{p_k(i)} - \frac{1}{p_k(i+1)} = \frac{p_k(i)}{k}.$$

From this, we compute

$$p_k(i+1) - p_k(i) = \frac{p_k(i)^3}{k - p_k(i)^2}.$$

In the limit, as $k \to \infty$, $p_k(i+1) - p_k(i)$ approximates $f'(x)/k$. Thus $f(x)$ satisfies the differential equation

$$f'(x) = \lim_{k\to\infty} \frac{f(x)^3}{1 - f(x)^2/k} = f(x)^3.$$

In view of the observation that $p_k(1) = k/(3k - 1)$, which leads to the initial condition $f(0) = 1/3$, the solution to this differential equation is $f(x) = (9 - 2x)^{-1/2}$. The result follows immediately, since

$$\lim_{k\to\infty} p_k^* = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{p_k(i)}{k} = \int_0^1 f(x)\,dx = \left[-\sqrt{9 - 2x}\right]_0^1 = 3 - \sqrt{7}.$$

□

4 An algorithm

Let G be a cubic graph. Throughout this and the following section, we assume that G has girth at least $15k\ell$, where k and ℓ are sufficiently large integers which will be determined in the proof of Lemma 17. The notation $p_k(i)$ and $q_k(i)$ of Section 3 will be abbreviated to $p(i)$ and $q(i)$ as k is fixed throughout the exposition.

Fix an oriented 2-factor F. A set $B \subseteq E(G)$ will be said to be *r-distant* (where r is an integer) if the distance between any two of its edges in G is at least r. Furthermore, B is *(F, ℓ)-sparse* if it is 4-distant and $F - B$ consists of paths whose length is at least ℓ and at most 7ℓ. Observe that by the above assumptions, each cycle of F contains at least two edges from any (F, ℓ)-sparse set.

Let $B \subseteq E(F)$ be an (F, ℓ)-sparse set of edges. In this section, we describe a probabilistic algorithm producing a full total independent set $\tilde{T} = \tilde{T}(F, B)$.

The *mate* v^* of a vertex $v \in V(G)$ is the neighbour of v in $G - E(F)$. The edges in B will be referred to as *boundary edges*.

Phase 1. *We construct an intermediate set* $T = T(F, B) \subseteq V(T(G))$, *which is not necessarily total independent.*

Make a uniformly random choice of a function $\lambda : B \to \{1, \dots, k\}$, assigning a *level* $\lambda(e)$ to each edge $e \in B$. The notion of a level is extended to each vertex or edge $x \in V(F) \cup E(F)$ by defining $\lambda(x)$ to be the level of the closest boundary edge in the direction opposite to the prescribed orientation of F. If Q is a component of $F - B$, we define $\lambda(Q)$ as the level of any vertex of Q.

Let $e^1, \dots, e^m$ be an ordering of the boundary edges such that $\lambda(e^i) \leq \lambda(e^j)$ if $i < j$.

We construct the set T in a sequence of steps, starting with $T = \emptyset$. At step i ($1 \leq i \leq m$), we process the boundary edge e^i and the path P^i of $F - B$ following e^i (with respect to the selected orientation of F). Enumerate the vertices and edges of P^i as $u^i_0, e^i_1, u^i_1, e^i_2, \dots, u^i_r$, where the order of the vertices u^i_j and the edges e^i_j is again based on the orientation of F. To make the notation more uniform, we may write $e^i = e^i_0$. In the following discussion, we drop the superscript i.

For the purpose of the description below, we consider the endvertex of e_0 different from u_0 to be a new *virtual* vertex u_{-1}, and make u_{-1} incident with a virtual edge e_{-1}. The construction will proceed along the 'path'

$e_{-1}, u_{-1}, e_0, u_0, e_1, \ldots, u_r$. The vertex u_{-1} and the edge e_{-1} are in no relation to the actual vertex and edge preceding e_0 (namely, u_0^- and e_0^-).

Let t be the level of e_0. We first make a *seed choice* for the path P^i, randomly deciding about the status of the virtual edge e_{-1} and the virtual vertex u_{-1}:

- with probability $p(t)$, we consider e_{-1} to be in T,
- with probability $q(t)$, we consider u_{-1} to be in T,
- with probability $p(t)$, neither of the above happens.

The choice is independent of the seed choices for the other paths $P^{i'}$.

The rest of the process for the path P^i is deterministic. Let $j \geq 0$. We specify whether e_j or u_j will be included in T, assuming that the status of e_s and u_s ($s < j$) has been decided.

The edge e_j will be added to T if and only if

$$e_{j-1} \notin T \text{ and } u_{j-1} \notin T. \tag{2}$$

(For $j = 0$, these events refer to the result of the seed choice.) The vertex u_j will be included in T if and only if both of the following hold:

$$u_{j-1} \notin T \text{ and } e_j \notin T, \tag{3}$$

$$(u_j^* \notin T \text{ and } \lambda(u_j^*) < \lambda(u_j)) \text{ or } \lambda(u_j^*) > \lambda(u_j). \tag{4}$$

After all of P^i is processed according to these rules, step i is completed and if $i < m$, we proceed to the boundary edge e^{i+1}.

Once we have completed all m steps, we have obtained the set T. It is not necessarily a total independent set, since the random decision on e^i and u_0^i did not take into account the real status of the edge and vertex preceding them in F. There can be a similar problem at the end of the path P^i and, furthermore, the last vertex of P^i may not be covered by T. Before we resolve these problems and construct the full total independent set $\tilde{T}$, we analyse the probability that a given vertex or edge is contained in T.

We first derive a lemma concerning the independence of certain events. Assume that P is a path from u to v in G, where $u, v \in V(G)$. We consider P as directed from u to v. We say that P is *rightward* if P contains no edge in B, and the direction of each edge of P contained in F matches the orientation of F. Given a function $\lambda : V(T(G)) \to \{1, \ldots, k\}$, we will say that P is λ-*ascending* if it is rightward and for every edge xy of P that is not contained in F, we have $\lambda(x) < \lambda(y)$. If there is a λ-ascending path from u to v, we write $u <_\lambda v$. Since B is (F, ℓ)-sparse, the length of a λ-ascending path is at most $7k\ell + k - 1$.

Lemma 9. *Let $u, v \in V(G)$ and let $s, t \in \{1, \ldots, k\}$. Assume that a 2-factor F and a set B of boundary edges are fixed and that u and v are not contained in the same component of $F - B$. If $G - u^*$ contains a rightward uv-path P_{uv} of length at most ℓ, then the following hold:*

(i) the events $u \in T$ and $\lambda(v) = s$ are conditionally independent given that $\lambda(u) = t$,

(ii) if $s < t$, then the events $u \in T$ and $v \in T$ are conditionally independent given that $\lambda(u) = t$ and $\lambda(v) = s$,

(iii) if $s < t$, then the events $uu^+ \in T$ and $v \in T$ are conditionally independent given that $\lambda(u) = t$ and $\lambda(v) = s$.

Proof. We start with an important observation. Suppose that, in our algorithm, the random choice of a function $\lambda : V(T(G)) \to \{1, \ldots, k\}$ has been made. In this situation, we can correctly decide whether a vertex $z \in V(G)$ is included in the set $T(F, B)$ based on the following information:

- the level $\lambda(w)$ of every vertex w such that $w <_\lambda z$ or $w^* <_\lambda z$ (observe that this includes the vertex z^*), and
- the result of the seed choice for every path containing a vertex w such that $w <_\lambda z$.

We now prove (i). Let

$$\mathcal{P}(z) = \{w \in V(G) : w <_\lambda z \text{ or } w^* <_\lambda z \\ \text{for some } \lambda \text{ such that } \lambda(u) = t\}.$$

We claim that in the component Q of $F - B$ containing v, there is no vertex v' such that $v' \in \mathcal{P}(u)$. Suppose the contrary. Assuming first that $v' <_\lambda u$ for some λ, we choose a λ-ascending $v'u$-path $P_{v'u}$ for a suitable λ. Observe that since $P_{v'u}$ does not contain the edge following u in F while P_{uv} does, the union $P_{v'u} \cup P_{uv} \cup Q$ contains a cycle. Furthermore, the length of the cycle is at most $\ell + (7k\ell + k - 1) + k < 15k\ell$, contradicting the girth assumption. The proof for the case $(v')^* <_\lambda u$ is similar.

Since we can decide about $u \in T$ without the knowledge of $\lambda(Q)$, and the choice of $\lambda(Q)$ is independent of all the other random choices made during the execution of the algorithm, the assertion follows.

The proofs of (ii) and (iii) are similar; we only prove (ii). For a vertex z, we define

$$\begin{aligned}\mathcal{P}'(z) = \{w \in V(G) :\ & w <_\lambda z \text{ or } w^* <_\lambda z \\ & \text{for some } \lambda \text{ such that } \lambda(u) = t \text{ and } \lambda(v) = s\}\end{aligned}$$

and note that the knowledge of the levels of vertices in $\mathcal{P}'(z)$ and the seed choices for the respective paths suffice for the decision whether $z \in T(F, B)$ under the assumption that $\lambda(u) = t$ and $\lambda(v) = s$.

We claim that $\mathcal{P}'(u) \cap \mathcal{P}'(v) = \emptyset$. Suppose the contrary. Then there exists $w \in V(G)$ such that $w <_\mu u$ and $w <_\lambda v$ for suitable functions μ and λ. Let P_{wv} be a λ-ascending path from w to v. Since $\lambda(u) > \lambda(v)$, u is not contained in P_{wv}. There is a μ-ascending path to u from either w or w^* which determines a rightward wu-path P_{wu}. Unlike P_{uv}, this path does not contain the edge of F following u, so $P_{uv} \cup P_{wu} \cup P_{wv}$ contains a cycle, the length of which is at most $\ell + 1 + 2(7k\ell + k - 1) \leq 15k\ell$ (whenever $k, \ell \geq 3$, which will be the case). This is a contradiction.

Since the sets $\mathcal{P}'(u)$ and $\mathcal{P}'(v)$ are disjoint, the events $u \in T$ and $v \in T$ depend on disjoint sets of independent random choices, and they are therefore conditionally independent under the assumption that $\lambda(u) = t$ and $\lambda(v) = s$. □

In the proof of Lemma 11 below, we will need a standard fact on conditional probability (which is easily verified by direct computation):

Lemma 10 (Rule of contraction for conditional probability)**.** *Let A, B, C, D be random events. Assume that:*

(1) A and B are conditionally independent given $C \wedge D$, and

(2) A and C are conditionally independent given D.

Then A is conditionally independent of $B \wedge C$ given D.

The following lemma is a fundamental observation on the behaviour of the algorithm described in this section.

Lemma 11. *Let $u \in V(G)$ and $e \in E(F)$ and let $t \in \{1, \ldots, k\}$. Then:*

(i) $\mathbf{P}\left(e \in T \mid \lambda(e) = t\right) = p(t)$,

(ii) $\mathbf{P}\left(u \in T \mid \lambda(u) = t \wedge \lambda(u^*) > t\right) = p(t)$,

(iii) $\mathbf{P}\left(u \in T \mid \lambda(u) = t \wedge \lambda(u^*) = t\right) = 0$,

(iv) $\mathbf{P}\left(u \in T \mid \lambda(u) = t\right) = q(t)$,

Proof. Let $u = u_j^i$ and $e = e_j^i$ in the notation introduced above. We prove all the claims simultaneously by double induction on t and j: we show that if the claims hold for any vertex $u_{j'}^{i'}$ and edge $e_{j'}^{i'}$ whose level is t', such that (t', j') precedes (t, j) in the lexicographic order, then they also hold for u and e. The base case $t = 1$ and $j = -1$ (virtual vertex or edge) follows directly from the construction.

Consider assertion (i). By the rule for the inclusion of an edge in T, $e \in T$ if and only if neither $e^- \in T$ nor $u^- \in T$. By the induction hypothesis, the latter two events occur with probability $p(t)$ and $q(t)$, respectively. (All the probabilities in this proof are relative to the condition $\lambda(u) = t$.) Since the events are disjoint, the probability that none occurs is $1 - p(t) - q(t) = p(t)$ as claimed.

The proof of (ii) is similar: given the assumption that $\lambda(u^*) > t$, the condition (4) for the inclusion of u (on page 9) is vacuously true. Thus u is included if and only if condition (3) holds, which happens with probability $1 - p(t) - q(t) = p(t)$.

Part (iii) is clear since u is never added to T if $\lambda(u) = \lambda(u^*)$.

It remains to prove (iv). Here we know that (3) again holds with probability $p(t)$. To assess the probability of (4), let us compute

$$\begin{aligned}
&\mathbf{P}\left((u_j^* \notin T \wedge \lambda(u_j^*) < \lambda(u_j)) \vee (\lambda(u_j^*) > \lambda(u_j)) \mid \lambda(u) = t\right) \\
&= \sum_{i=1}^{t-1} \mathbf{P}\left(u_j^* \notin T \wedge \lambda(u_j^*) = i \mid \lambda(u) = t\right) + \sum_{i=t+1}^{k} \mathbf{P}\left(\lambda(u_j^*) = i \mid \lambda(u) = t\right) \\
&= \sum_{i=1}^{t-1} \frac{1 - p(i)}{k} + \sum_{i=t+1}^{k} \frac{1}{k},
\end{aligned}$$

where the last equality follows from the induction hypothesis. Since $q(t)$ is just the product of the result with $p(t)$, we need to show that (3) and (4) are conditionally independent given the condition $\lambda(u) = t$. To rephrase this task, let us write

$$X_1 \equiv u^- \in T, \qquad Y_1 \equiv u^* \notin T \text{ and } \lambda(u^*) < t,$$
$$X_2 \equiv e^- \in T, \qquad Y_2 \equiv \lambda(u^*) > t,$$

so that (3) is equivalent to $\overline{X_1 \vee X_2}$ and (4) is equivalent to $Y_1 \vee Y_2$ (assuming $\lambda(u) = t$). By basic facts on probability, the above conditional independence will be established if we can show that each X_i is conditionally independent of each Y_j ($i, j \in \{1, 2\}$) given that $\lambda(u) = t$.

For $j = 2$, this follows directly from Lemma 9 (i) by expressing Y_2 as the union of disjoint events $\{\lambda(u^*) = i : i = t + 1, \ldots, k\}$. For $j = 1$, we apply Lemma 10, substituting X_i for A (where $i = 1, 2$), $u^* \notin T$ for B, $\lambda(u^*) < t$ for C and $\lambda(u) = t$ for D. The hypothesis of the Lemma is satisfied by Lemma 9 (ii) and (iii), so it follows that X_i and Y_1 are conditionally independent given $\lambda(u) = t$ as required. The proof is complete. $\square$

Lemma 11 enables us to compute the probability that any vertex of G or edge of F is in T. Note that the probabilities do not depend on G:

Observation 12. *Let $v \in V(G)$ and $e \in E(F)$. Then*

$$\mathbf{P}\left(e \in T\right) = p^* := \sum_{i=1}^{k} \frac{p(i)}{k},$$
$$\mathbf{P}\left(v \in T\right) = q^* := \sum_{i=1}^{k} \frac{q(i)}{k}.$$

Proof. The first assertion is obvious. To prove the second one, we first use Lemma 11 to compute

$$\begin{aligned}
\mathbf{P}\left(v \in T \mid \lambda(v) = t\right) &= \sum_{s \neq t} \mathbf{P}\left(x \in T \mid \lambda(v) = t \wedge \lambda(v^*) = s\right) \cdot \mathbf{P}\left(\lambda(v^*) = s\right) \\
&= \sum_{s=1}^{t-1} p(t) \cdot (1 - p(s)) \cdot \frac{1}{k} + \sum_{s=t+1}^{k} p(t) \cdot \frac{1}{k} \\
&= p(t) \left(1 - \frac{1}{k} - \sum_{s=1}^{t-1} \frac{p(s)}{k}\right) = q(t)
\end{aligned}$$

and the lemma follows. $\square$

type	situation	action on T
I	$u' \in T$ and $u_0 \in T$	replace u' and u_0 by e_0
II	$u' \in T$ and $e_0 \in T$	remove u'
IIIa	$\{e', e_0, u_1\} \subseteq T$	replace e_0 by e_1
IIIb	$\{e', e_0, u_0^*\} \subseteq T$ and $u_1 \notin T$	replace e_0 by $u_0 u_0^*$
IIIc	$\{e', e_0\} \subseteq T$ and $u_1, u_0^* \notin T$	replace e_0 by u_0
IVa	u' is not covered by T, $u_0 \in T$	replace u_0 by e_0
IVb	u' is not covered by T, $u_0 \notin T$, $u'' \in T$	replace u'' by e'
IVc	u' is not covered by T, $u_0, u'' \notin T$, $(u')^* \in T$	replace $(u')^*$ by $u'(u')^*$

Table 1: The types of conflicts.

Phase 2. *We modify T to a full total independent set $\tilde{T}$.*

Let us examine the possible reasons why T is not full and total independent in detail. Consider a boundary edge $e_0 = e^i$ and its end $u_0 = u_0^i$ in P^i, and suppose that e^i is also incident with a path P^j. Let u' denote the last vertex of P^j (thus, $u' = u_0^-$) and write $u'' = (u')^-$ and $e' = u''u'$. Recall that if u is a vertex of G, then u^* denotes its mate.

A *conflict* at e^i is any of the situations listed in the middle column of Table 1; the right hand column shows how to modify T in order to resolve the conflict. Note that all the cases are mutually exclusive and that the resolution rules are deterministic. The conflict types are shown in Figure 1.

The resolution of a conflict at e^i only affects vertices and edges in $N_2(e^i)$. Since B is 4-distant, each vertex and edge is in at most one set $N_2(e^i)$. Thus, the order in which the conflicts are resolved is irrelevant. It is easy to see that after the resolution of all the conflicts, the resulting set $\tilde{T}$ is total independent and full.

We need to show that the conflicts occur in a uniform manner throughout G, i.e., that if $e, f \in B$, then the probability of a conflict of any given type is the same at e and f. As an example, we consider the conflict type IIIb and sketch how to prove this claim.

Observe that, under the assumption $\lambda(e_0) = t$, the conflict of type IIIb

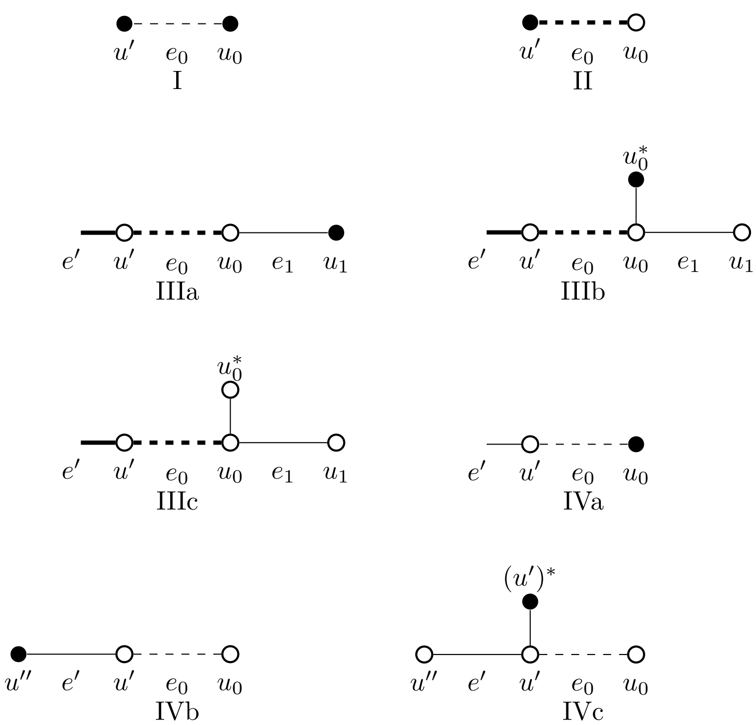

Figure 1: Possible conflict types. The boundary edge is shown dashed, thick edges and black vertices are those included in T.

occurs if and only if $X_1 \wedge X_2 \wedge X_3 \wedge X_4$ occurs, where:

$$\begin{aligned}
&X_1 \equiv e' \in T,\\
&X_2 \equiv e_0 \in T,\\
&X_3 \equiv (u_0^*)^- \notin T \wedge (u_0^*)^- u_0^* \notin T \wedge \lambda(u_0^*) \neq t,\\
&X_4 \equiv (\lambda(u_1^*) < t \wedge u_1^* \in T) \vee \lambda(u_1^*) = t.
\end{aligned}$$

The conditional probability of each of these events (with respect to $\lambda(e_0) = t$) is easy to compute using Lemma 11 and Observation 12. Reasoning similarly as in the proof of Lemma 9, one can show that each set of events $\{X_1, X_2, Y_3, Y_4\}$ is conditionally mutually independent given $\lambda(e_0) = t$, where Y_i $(i = 3, 4)$ ranges over the 'summands' in the union X_i. From this, it is a simple exercise in the use of Lemma 10 to conclude that the events X_i are conditionally mutually independent given that $\lambda(e_0) = t$. In particular, the probability P_{IIIb} of the conflict of type IIIb is

$$\begin{aligned}
P_{\mathrm{IIIb}} &= \sum_{t=1}^{k} \mathbf{P}\left(X_1 \mid \lambda(e_0) = t\right) \cdot \mathbf{P}\left(X_2 \mid \lambda(e_0) = t\right) \cdot \\
&\qquad \mathbf{P}\left(X_3 \mid \lambda(e_0) = t\right) \cdot \mathbf{P}\left(X_4 \mid \lambda(e_0) = t\right) \\
&= (p^*)^2 \cdot \left(1 - p^* + \frac{p(t)}{k}\right) \cdot \frac{1}{k} \left(1 + \sum_{i=1}^{t-1} p(i)\right),
\end{aligned}$$

which does not depend on e_0. For all of the other conflict types, a similar computation applies.

The way we resolve conflicts of type IIIb decreases the probability that $e_0 \in \tilde{T}$ by P_{IIIb} when comparing to $\mathbf{P}\left(e_0 \in T\right)$. The final probability $\mathbf{P}\left(e_0 \in \tilde{T}\right)$ can be determined by considering all the conflict types whose resolution involves e_0, namely types I, IIIa, IIIb, IIIc and IVa. It is important that $\mathbf{P}\left(x \in \tilde{T}\right)$ only depends on the position of $x \in V(T(G))$ relative to B. To formalize this notion, let us define the (F, B)-*type* (or just *type*) of x as follows.

Assume that $x \in N_2(e)$, where $e \in B$. Consider the graph H in Figure 2 and the unique isomorphism between $N_2(e)$ (viewed as a subgraph of G) and H, taking the edges of F to the bold edges in such a way that their orientations match. We define the (F, B)-type of x as the label associated to the image of x in H. Thus, the type is an integer from $\{1, \ldots, 11\}$. Note that it is only defined for the vertices and edges of $N_2(B)$.

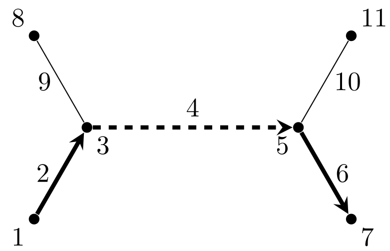

Figure 2: The definition of an (F, B)-type.

Observation 13. *For $x, y \in V(T(G))$, the following holds:*

(i) if $x, y \in V(G) - N_2(B)$, then

$$\mathbf{P}\left(x \in \tilde{T}\right) = \mathbf{P}\left(y \in \tilde{T}\right) = \mathbf{P}\left(x \in T\right),$$

(ii) if $x, y \in E(F) - N_2(B)$, then

$$\mathbf{P}\left(x \in \tilde{T}\right) = \mathbf{P}\left(y \in \tilde{T}\right) = \mathbf{P}\left(x \in T\right),$$

(iii) if $x, y \in N_2(B)$ and the (F, B)-type of x and y is the same, then

$$\mathbf{P}\left(x \in \tilde{T}\right) = \mathbf{P}\left(y \in \tilde{T}\right).$$

5 Cubic bridgeless graphs

In this section, we prove Theorem 6 under the assumption that G is a cubic bridgeless graph, deferring the general case to Section 6. Recall our assumption that the girth of G is at least $15k\ell$, where k and ℓ are appropriately chosen constants to be determined in Lemma 17. Let F be a 2-factor in G and let $B \subseteq E(F)$ be an (F, ℓ)-sparse set of edges.

At this point, we need to introduce the following concept and result. A graph H is *strongly r-colourable* if for any partition of $V(H)$ into $\lceil |V(H)| / r \rceil$ parts, each of size at most r, H admits a proper r-colouring with each colour class intersecting each part of the partition in at most one vertex.

The *strong chromatic number* of H is the smallest r such that H is strongly r-colourable. Haxell [6] proved the following upper bound on the strong chromatic number, improving an earlier result of Alon [1] (see also [7]).

Theorem 14. *The strong chromatic number of H is at most $3\Delta(H) - 1$.*

We will use Theorem 14 to show that under certain conditions, $E(F)$ can be decomposed into (F, ℓ)-sparse sets. In the following result, all that we need in this section is the special case $E(Q) = \emptyset$; the general statement will be used in Section 6.

Lemma 15. *Let F be a 2-factor of G and let Q be a graph with vertex set $E(F)$. If $\ell \geq 83 + 3\Delta(Q)$, then the set $E(F)$ can be decomposed into 3ℓ sets, each of which is (F, ℓ)-sparse and none of which contains a pair of edges that forms an edge of Q.*

Proof. Consider an auxiliary graph H with vertex set $E(G)$ and an edge ef for each pair $e, f \in E(G)$ such that either the distance between e and f in G is at most 3, or $ef \in E(Q)$. It is easy to see that the maximum degree of H is at most $28 + \Delta(Q)$.

Let the cycles of F be $C_1, \dots, C_n$. For $1 \leq i \leq n$, set $m(i) = \lceil |C_i| / \ell \rceil$. Split C_i into edge-disjoint paths $P_{i,1}, \dots, P_{i,m(i)}$ such that each $P_{i,j}$ with $j \geq 2$ has length ℓ, while $P_{i,1}$ has length at least one. Let $\mathcal{P}$ be a partition of $E(F)$ such that the edge set of each path $P_{i,j}$, where $1 \leq i \leq s$ and $j \geq 2$, forms a class of $\mathcal{P}$, and moreover all but at most one class of $\mathcal{P}$ are of size ℓ.

Since $\ell \geq 83 + 3\Delta(Q) \geq 3\Delta(H) - 1$, Theorem 14 (applied to $\mathcal{P}$) implies that there is a colouring (say, c) of the edges of F by ℓ colours such that each colour class $B_1, \dots, B_\ell$ intersects each set in $\mathcal{P}$ in at most one edge. It follows that each B_r contains exactly one edge from each $P \in \mathcal{P}$ with $|P| = \ell$. Furthermore, by the construction of H, each B_r is 4-distant and no B_r contains edges e, f with $ef \in E(Q)$.

We now transform the colouring to the desired partition into (F, ℓ)-sparse sets $B_{r,t}$, where $r \in \{1, \dots, \ell\}$ and $t \in \{0, 1, 2\}$. Since we will choose $B_{r,t}$ as a subset of B_i, all we need to ensure is that each component of $F - B_{r,t}$ is a path of length between ℓ and 7ℓ.

For $i = 1, \dots, n$, we construct a sequence $s_{i,1}, \dots, s_{i,m(i)}$ of symbols $0, 1, 2$ starting with 01, ending with 2, and such that every two consecutive occurences of the same symbol are separated by one, two or three other symbols (when considering the first and last symbol as adjacent). Since the girth of G is at least $15k\ell > 6\ell$, it suffices to construct the sequence for each length starting with 6. We start with one of the following sequences depending on the residue class mod 3 of $m(i)$:

$$\begin{aligned} 012012 \qquad & \text{for } m(i) \equiv 0 \pmod 3, \\ 0102012 \qquad & \text{for } m(i) \equiv 1 \pmod 3, \\ 01021012 \qquad & \text{for } m(i) \equiv 2 \pmod 3 \end{aligned}$$

and insert a suitable number of blocks of 201 before the last symbol 2 to make the length equal to $m(i)$.

The set $B_{r,t}$ ($t \in \{0,1,2\}$) is defined as the intersection of B_r with the edge sets of all paths $P_{i,j}$ such that $s_{i,j} = t$, where $i = 1, \dots, n$ and $j = 1, \dots, m(i)$. For $i = 1, \dots, n$, each symbol of the sequence $s_{i,1} \dots s_{i,m(i)}$, except possibly for $s_{i,1}$, represents ℓ consecutive edges. Furthermore, any two neighbouring symbols in this sequence as well as the second and last symbol are different. It follows that the distance on C_i between any two edges in $B_{r,t} \cap E(C_i)$ is at least ℓ.

The upper bound follows from the fact that neighbouring occurences of any symbol $t \in \{0,1,2\}$ are separated by at most three other symbols. For $t = 0$, this can be improved: neighboring occurences of 0 are separated by at most two symbols. At the same time, all but at most one symbol in the sequence correspond to paths containing edges of all ℓ colours. An easy case analysis implies that the components of $C_i - B_{r,t}$ ($t \in \{0,1,2\}$) are paths of length at most 7ℓ. Thus, the sets $B_{r,t}$ are indeed (F,ℓ)-sparse. □

Recall the values p^* and q^*, defined in Observation 12. The following lemma summarizes the findings of Section 4:

Lemma 16. *If F is an oriented 2-factor of G and B is a 4-distant set of edges, then there exists a function $w_{F,B} : \Phi(G) \to [0,1]$ satisfying, for all $x \in V(T(G))$, the following conditions:*

(i) if $x \notin N_2(B)$, then

$$w_{F,B}[x] = \begin{cases} q^* & \textit{for } x \in V(G), \\ p^* & \textit{for } x \in E(F), \\ 0 & \textit{for } x \in E(G) - E(F), \end{cases}$$

(ii) if $x_1, x_2 \in N_2(B)$ have the same (F,B)-type, then $w_{F,B}[x_1] = w_{F,B}[x_2]$.

Proof. Consider the algorithm, described in Section 4, that produces a total independent set $\tilde{T}(F,B)$. For any full total independent set X in G, let $w_{F,B}(X)$ be the probability that $\tilde{T}(F,B) = X$.

If $v \in V(G) - N_2(B)$, then

$$w_{F,B}[v] = \sum_{v \in X \in \Phi(G)} w_{F,B}(X) = \mathbf{P}\left(v \in \tilde{T}(F,B)\right) = \mathbf{P}\left(v \in T(F,B)\right) = q^*$$

by Observations 12 and 13(i). The rest of part (i) is derived similarly. Part (ii) follows from Observation 13(iii). $\square$

By combining Lemmas 15 and 16, we obtain the following corollary:

Lemma 17. *There are positive rational constants α, β and γ such that $\alpha + \beta + 2\gamma = 1$, $\beta > 1/4$, $\alpha \leq 4/3\ell$ and the following holds: If F is a 2-factor of G, then there exists a function $w : \Phi(G) \to [0,1]$ such that:*

$$w[x] = \begin{cases} \beta & \text{if } x \in V(G), \\ \gamma & \text{if } x \in E(F), \\ \alpha & \text{otherwise} \end{cases}$$

for all $x \in V(T(G))$.

Proof. In this proof, we determine the requirements on the constants k and ℓ. We use Lemma 15 (with $E(Q) = \emptyset$) to find a decomposition $\mathcal{B}$ of $E(F)$ into 3ℓ (F,ℓ)-sparse sets; this can be done whenever $\ell \geq 83$ but we will require $\ell \geq 89$ to be consistent with Section 6. For each $B \in \mathcal{B}$, consider the function $w_{F,B}$ of Lemma 16, and define

$$w = \sum_{B \in \mathcal{B}} \frac{w_{F,B}}{3\ell}.$$

Since each edge of F is contained in exactly one $B \in \mathcal{B}_i$, the number of times that $x \in V(T(G))$ acquires a particular (F_i, B)-type as B ranges over $\mathcal{B}$ is independent of x. It follows that the change in probabilities associated with the resolution of conflicts is the same for all $x \in V(G)$, for all $x \in E(F)$ and all $x \in E(G) - E(F)$. In this way, the values $0, q^*$ and p^* of Lemma 16 change into α, β and γ, respectively.

We claim that for large ℓ, β is close to q^*. To see this, observe that every vertex $v \in V(G)$ is contained in exactly six of the sets $N_2(e)$, where $e \in E(F)$. By Lemma 16,

$$|\beta - q^*| \leq \frac{6}{3\ell}.$$

Furthermore, as k grows large, q^* tends to $1 - 2(3 - \sqrt{7}) \approx 0.2915$ by Lemma 8. Thus, for large enough k and ℓ, we will have $\beta > 1/4$. In fact, it is routine to check that, for instance, the values $k = 11$ and $\ell = 96$ are sufficient.

It remains to prove that $\alpha \leq 4/3\ell$. For any particular choice of B, an edge e of $E(G) - E(F)$ may only be included in $\tilde{T}$ if it is incident with an edge of B. Since this will happen for 4 out of the 3ℓ choices for B, the inequality follows. $\square$

We can now prove Theorem 6 for cubic bridgeless graphs. By Theorem 2, such a graph G has a fractional 3-edge-colouring c. This is equivalent to the existence of perfect matchings $M_1, \ldots, M_{3N}$ such that each edge is contained in exactly N of them. For $i \leq 3N$, let F_i be the 2-factor complementary to M_i.

For $1 \leq i \leq 3N$, we apply Lemma 17 to the 2-factor F_i and call the resulting function w_i. For a total independent set $X \in \Phi(G)$, put

$$w'(X) = \sum_{i=1}^{3N} \frac{w_i(X)}{3N}.$$

Since each edge of G is contained in $2N$ of the factors F_i, each edge gets the same weight $w'[e] = (\alpha + 2\gamma)/3$. Similarly, each vertex gets weight $w'[v] = \beta$. Observe that $w'[v] > w'[e]$ as $4\beta > 1 = \alpha + \beta + 2\gamma$. Thus, we may use the fractional 3-edge-colouring c to make the weight on edges equal to that on vertices. Specifically, extend c by setting $c(Y) = 0$ for any $Y \in \Phi(G)$ that is not a perfect matching, and define

$$w(X) = \frac{1}{\beta} \cdot w'(X) + \left(1 - \frac{\alpha + 2\gamma}{3\beta}\right) \cdot c(X).$$

It is easy to see that $w[x] = 1$ for all $x \in V(T(G))$, so w is a fractional total colouring. Moreover, we claim that $|w| = 4$. To see this, consider the set $\{x_1, x_2, x_3, x_4\}$ consisting of a vertex of G and the three adjacent edges, and note that since each set from $\Phi(G)$ contains exactly one x_i, we have $|w| = \sum_i w[x_i] = 4$. This proves Theorem 6 for $g \geq 15k\ell$, where the required values of k and ℓ have been identified in the proof of Lemma 17 as $k = 11$ and $\ell = 96$. Thus, $g \geq 15\,840$ is sufficient.

6 Subcubic graphs

We are now ready to prove Theorem 6. We show, by induction on the order of the graph G, that if G is a graph with maximum degree at most 3 and girth at least g, then $\chi''_f(G) \leq 4$. The assertion is true for graphs with $\Delta(G) \leq 2$ by Theorem 4 and for bridgeless cubic graphs by the above.

Suppose first that G contains a bridge e with endvertices x_1 and x_2. For $i = 1, 2$, let G_i be the component of $G - e$ containing x_i. By induction, each G_i has a fractional total colouring w_i with $|w_i| \leq 4$. We may assume without loss of generality that $|w_i| = 4$.

In view of Lemma 1, there is a multiset $\mathcal{W}_i$ of $4N$ total independent sets in G_i, such that each $x \in V(T(G_i))$ is contained in N of the sets in $\mathcal{W}_i$ (for a suitable integer N). Let us enumerate the members of each $\mathcal{W}_i$ as $W_{i,1}, \ldots, W_{i,4N}$ in such a way that:

- x_1 is contained in $W_{1,1}, \ldots, W_{1,N}$,
- x_2 is contained in $W_{2,N+1}, \ldots, W_{2,2N}$,
- neither x_i nor any edge incident to it are contained in $W_{i,j}$ for $j > 3N$.

We construct a multiset $\mathcal{W} = \{W_1, \ldots, W_{4N}\}$ of total independent sets in G by setting

$$W_j = \begin{cases} W_{1,j} \cup W_{2,j} & \text{if } j \leq 3N, \text{ and} \\ W_{1,j} \cup W_{2,j} \cup \{e\} & \text{otherwise.} \end{cases}$$

It is easy to see that each set W_j is total independent and each $x \in V(T(G))$, including e, is contained in N of these sets. Hence, G has a fractional total colouring of size 4.

Having dealt with bridges, we may assume that G is a bridgeless subcubic graph. Let $D = \sum_{v \in V(G)} (3 - d(v))$. We know that $D > 0$; assume now that $D \geq 2$. It is well-known that there exists a D-regular graph H with girth at least g; the construction given in [10, Solution to Problem 10.12] moreover ensures that H is 2-connected. Replace each vertex w of H with a copy of G, and for each vertex v of this copy, choose $3 - d(v)$ edges of H formerly incident with w and redirect them to v. The result is a cubic bridgeless graph of girth at least g. Since any fractional total 4-colouring of this graph yields a fractional total 4-colouring of its subgraph G, this case is resolved.

It remains to consider the case that $D = 1$, i.e., all the vertices of G have degree 3 except for one vertex z of degree 2. Let the neighbors of z be denoted by x and y. The graph G_z, obtained by suppressing z (i.e., contracting one of the two edges adjacent to z), is cubic and bridgeless. Let $F_1, \dots, F_{3N}$ be a multiset of 2-factors of G_z such that each edge of G_z is contained in exactly $2N$ of them. We may assume that the edge $e = xy$ is contained in $F_1, \dots, F_{2N}$.

We follow the approach of Sections 4 and 5, with modifications we describe next.

Step I: We first discuss the case $i \le 2N$. We embed G in a graph G' obtained as follows. Let H' be a hamiltonian cubic graph of girth at least g (see [3] for a construction) and let S' be a Hamilton cycle of H'. Subdivide an edge of S', creating a vertex z^*. The graph G' is the disjoint union of G and H' with an added edge zz^*. Note that G' is cubic. It will not pose any problem that G' contains a bridge.

For $i \le 2N$, we define F'_i as the 2-factor of G' corresponding to F_i with the cycle S' added. Using Lemma 15 (with $E(Q) = \emptyset$), we find a decomposition $\mathcal{B}_i$ of $E(G')$ into (F'_i, ℓ)-sparse sets. For $B \in \mathcal{B}^*_i$, we run the algorithm of Section 4 that constructs the sets $T = T(F'_i, B)$ and $\tilde{T}(F'_i, B)$ without modifications. Following the proof of Lemma 17, we find a function w'_i defined on $\Phi(G')$, satisfying the conclusion of that Lemma with respect to the graph G' and 2-factor F'_i. Restricting to G, we obtain a function w_i defined on $\Phi(G)$, each of which assigns weight β to the vertices of G, γ to edges of G in F'_i and α to the other edges of G, where α, β, γ are the constants from Lemma 17.

Altogether, Step I provides us with $2N$ functions $w_1, \dots, w_{2N}$ on $\Phi(G)$ with the above property.

Step II: To process the 2-factors $F_{2N+1}, \dots, F_{3N}$, we first construct a cubic graph H. For some $s \ge g/2$, where g is the girth of G, take s copies $H_1, \dots, H_s$ of $G - z$. For $j = 1, \dots, s$, let the copies of x and y in H_j be denoted by x_j and y_j, and let x'_j and y'_j be new vertices. The graph H is obtained by taking the disjoint union of all the copies H_j and the cycle $S = x'_1 y'_1 x'_2 y'_2 \dots x'_s y'_s$, and adding the edges $x_j x'_j$ and $y_j y'_j$ for all $j = 1, \dots, s$. It is easy to see that H is cubic bridgeless and its girth is at least g.

For each 2-factor F_i of G_z ($2N + 1 \le i \le 3N$) there is a corresponding 2-factor F''_i of H obtained by taking a copy of F_i in each graph H_j ($1 \le j \le s$) and adding the cycle S. We aim to use Lemma 15 in H to find a decomposition of each $E(F''_i)$, $2N + 1 \le i \le 3N$, into (F''_i, ℓ)-sparse sets.

As we will see, we need to ensure additionally that none of the sets contains an edge incident with x_j and another edge incident with y_j for any $j = 1, \dots, s$. To this end, we apply Lemma 15 to a graph Q on $E(F_i'')$ constructed as follows. The edge set of Q contains, for each $j = 1, \dots, s$, all four edges $e_x e_y$ where e_x is an edge of F_i'' incident with x_j and e_y is an edge of F_i'' incident with y_j. Clearly, $\Delta(Q) = 2$. Since $\ell \geq 89 = 83 + 3\Delta(Q)$, Q may indeed be used in Lemma 15.

The graph H is cubic, so we can run the algorithm of Section 4 on it without modifications. For each choice of a set of boundary edges B (an (F_i'', ℓ)-sparse set obtained from Lemma 15) and each total independent set $\tilde{T}$ the algorithm produces, we consider the total independent set $\tilde{T}''$ in G obtained by the following rules:

- each vertex and edge of $G - z$ is in $\tilde{T}''$ if and only if the corresponding vertex or edge in H_1 is in $\tilde{T}$,
- if $x_1 x_1' \in \tilde{T}$, then we add xz to $\tilde{T}''$,
- if $y_1 y_1' \in \tilde{T}$, then we add yz to $\tilde{T}''$,
- if none of $x_1, y_1, x_1 x_1'$ and y_1' is in $\tilde{T}$, we add z to $\tilde{T}''$.

Each set $\tilde{T}''$ is total independent in G. To verify this, we have to check that $\tilde{T}''$ does not contain both xz and yz, i.e., that $\tilde{T}$ does not contain both $x_1 x_1'$ and $y_1 y_1'$. Our algorithm may add an edge of $E(H) - E(F_i'')$ to $\tilde{T}$ only if the edge is incident with an edge of B. Since B is chosen using the above graph Q, this cannot happen for $x_1 x_1'$ and $y_1 y_1'$ at the same time.

Based on the sets $\tilde{T}''$, we define the associated functions $w_i : \Phi(G) \to [0, 1]$ (where $2N+1 \leq i \leq 3N$), obtained as in the proof of Lemma 17. Each w_i assigns weight β to all vertices except z, γ to all edges of F_i and α to all edges of $E(G) - E(F_i)$.

We need to ensure that $w_i[z] \geq \beta$. By the construction, $w_i[z]$ equals $\sum_X w_i(X)$, where X ranges over total independent sets in G containing none of x, y, xz and yz. We thus have:

$$w_i[z] \geq 1 - w_i[x] - w_i[y] - w_i[xz] - w_i[yz] = 1 - 2\beta - 2\alpha. \qquad (5)$$

Note that the inequality $1 - 2\beta - 2\alpha \geq \beta$ is equivalent to $\gamma \geq \beta + \alpha/2$. As ℓ grows large, γ is close to p^*, which in turn is close to $3-\sqrt{7} \approx 0.3542$ for large k (cf. Lemma 8). Similarly, β tends to $1 - 2(3 - \sqrt{7}) = 2\sqrt{7} - 5 \approx 0.2915$. Furthermore, Lemma 17 asserts that $\alpha \leq 4/3\ell$, so for large ℓ and k we will indeed have $\gamma \geq \beta + \alpha/2$. In particular, the values $k = 11$ and $\ell = 96$, used in Section 5, are sufficient.

Thus, $w_i[z] \geq \beta$. Since we may remove z from any total independent set as required, it may be assumed that $w_i[z] = \beta$.

Following the argument at the end of Section 5, we can define

$$w' = \sum_{i=1}^{3N} \frac{w_i}{3N}$$

and note that w' assigns weight β to each vertex and weight $(\alpha + 2\gamma)/3$ to each edge. Unfortunately, we are no longer able to augment it to a fractional total 4-colouring using a fractional 3-edge-colouring, since the latter need not exist in G. We need to modify the proof in yet another way.

In the recurrence of Section 3, let us replace the equation (1) by

$$q_k(i) = \xi \cdot p_k(i) \left(1 - \frac{1}{k} - \frac{1}{k} \sum_{j=1}^{i-1} p_k(j) \right),$$

where ξ is a real number from the interval $[0, 1]$. In the algorithm of Section 4, we adjust the rule for the inclusion of a vertex accordingly: whenever a vertex u_j is to be included by the original algorithm (that is, the events (3) and (4) occur), we decide with probability $1 - \xi$ not to include it. With this modification, Observation 12 analyses the algorithm correctly if we interpret p^* and q^* as functions of ξ. Similarly, let us regard α, β and γ as functions of ξ, so we can write, e.g., $\beta = \beta(\xi)$. Likewise, for a function such as $q_k(i)$ we may write $q_k(i) = q_k(i, \xi)$. Lemma 17 remains valid, except for the assertion that $\beta > 1/4$. Indeed, $\beta(0)$ will be small, since for $\xi = 0$, the only way that a vertex will be included in the set $\tilde{T}$ is through the resolution of a conflict of type (IIIc). An argument similar to the one used to bound α in Lemma 17 shows that $\beta(0) \leq 1/3\ell < 1/4$.

Each of the functions $p_k(i, \xi)$ and $q_k(i, \xi)$ is easily seen to be continuous in ξ. As we have observed in Section 4, the probability of a particular type of conflict at a given edge can be expressed in terms of these functions, and as a function of ξ it will be continuous. From this it follows that $\beta(\xi)$ is continuous, so there is a value η for which $\beta(\eta) = 1/4$. If we use this value in our algorithm and construct the functions w_i and w' as above, each vertex v will get weight $w'[v] = 1/4$. Similarly, each edge e will get weight $(\alpha(\eta) + 2\gamma(\eta))/3 = (1 - \beta(\eta))/3 = 1/4$. Thus, the function $4w$ is a fractional total colouring of weight 4. The proof of Theorem 6 is complete.

7 Graphs with even maximum degree

In this section, we show that with minor modifications, the method used to prove Theorem 6 yields a proof of Theorem 7.

Let G be a graph of maximum degree Δ, where $\Delta \geq 4$ is even. Using the method described in Section 6, we construct a Δ-regular graph H such that H contains G as a subgraph and the girth of H is greater than or equal to that of G. A well-known result of Petersen (see, e.g., [4, Corollary 2.1.5]) implies that H can be decomposed into edge-disjoint 2-factors $F_1, \dots, F_{\Delta/2}$ of H.

For each $i = 1, \dots, \Delta/2$ and suitable constants k, ℓ, we use an analogue of Lemma 15 to find a decomposition $\mathcal{B}_i$ of $E(F_i)$ into (F_i, ℓ)-sparse sets. For $B \in \mathcal{B}_i$, we then run the algorithm of Section 4 with a single modification: each vertex u will now have $\Delta - 2$ 'mates' (rather than just one), and will only be included in the set $T(F_i, B)$ if this set contains none of the mates whose level is lower than that of u; if a mate of u has the same level as u, then neither of them will be included in $T(F_i, B)$. Although we can no longer use the analysis from Section 3, the following variant of Lemma 17 holds:

Lemma 18. *Let $\Delta \geq 4$ be an even integer. There are positive rational constants α, β and γ such that $(\Delta - 2)\alpha + \beta + 2\gamma = 1$ and the following holds: If F is a 2-factor of G, then there exists a function $w : \Phi(G) \to [0, 1]$ such that:*

$$w[x] = \begin{cases} \beta & \text{if } x \in V(G), \\ \gamma & \text{if } x \in E(F), \\ \alpha & \text{otherwise} \end{cases}$$

for all $x \in V(T(G))$.

As i ranges over $1, \dots, \Delta/2$, the average of the weights $w[x]$ given to $x \in V(T(H))$ is

$$\begin{array}{cl} \beta & \text{if } x \text{ is a vertex,} \\ \dfrac{(\Delta - 2)\alpha + 2\gamma}{\Delta} & \text{if } x \text{ is an edge.} \end{array}$$

A simple computation shows that if $\beta \geq 1/(\Delta+1)$, then the average value for a vertex is greater or equal to that for an edge. In this case, we can use the argument described at the end of Section 6, modifying the equivalent of the equation (1) by introducing a parameter ξ and using a value of ξ for which both of the above averages are equal to $1/(\Delta+1)$. The associated probability distribution on the full total independent sets then clearly determines a fractional total $(\Delta+1)$-colouring of H and hence of G.

It remains to derive the lower bound on the constant β:

Proposition 19. *Let $\Delta \geq 4$. In Lemma 18, we can choose β in such a way that $\beta > 1/(\Delta+1)$.*

Proof. For the present setting, the recurrence of Section 3 changes to

$$\begin{aligned} 2p_k(i) + q_k(i) &= 1, \\ q_k(i) &= p_k(i)\left(1 - \frac{1}{k} - \frac{1}{k}\sum_{j=1}^{i-1} \tilde{q}_k(j)\right)^{\Delta-2}, \\ \tilde{q}_k(i) &= p_k(i)\left(1 - \frac{1}{k} - \frac{1}{k}\sum_{j=1}^{i-1} \tilde{q}_k(j)\right)^{\Delta-3}, \end{aligned} \tag{6}$$

where the term $\tilde{q}_k(i)$ represents the probability that a vertex u is included in the total independent set T assuming that the level of u equals i and the level of a given mate of u exceeds i (for the number of levels being k). Following the method of the proof of Lemma 8, we define piecewise linear functions h_k on the interval $[0,1]$ by the equations

$$h_k\left(\frac{i-1}{k-1}\right) = \tilde{q}_k(i),$$

where $i = 1, \dots, k$, and by the requirement that h_k be linear on each $[\frac{i-1}{k-1}, \frac{i}{k-1}]$ for $i \leq k-1$. We let $\tilde{q} : [0,1] \to [0,1]$ be the limit of h_k as $k \to \infty$. Thus, $\tilde{q}$ can be viewed as an asymptotic version of $\tilde{q}_k$. Note that in the limit, $\sum_{j=1}^{i-1} \tilde{q}_k(j)/k$ becomes $\int_0^x \tilde{q}(t)\,dt$ (for a suitable x). In accordance with (6), we set

$$q(x) = \tilde{q}(x) \cdot \left(1 - \int_0^x \tilde{q}(t)\,dt\right). \tag{7}$$

If we define, for $x \in [0,1]$,

$$\tilde{Q}(x) = \int_0^x \tilde{q}(t)\,dt,$$
$$Q(x) = \int_0^x q(t)\,dt,$$

then $Q(1)$ is the limit value of $\sum_{i=1}^{k} q_k(i)/k$, i.e., the asymptotic probability of the inclusion of a vertex in the set constructed by Phase I of our algorithm. It follows that to prove the assertion of the proposition, it suffices to prove

$$Q(1) > \frac{1}{\Delta+1}. \tag{8}$$

This is what we do in the rest of this proof.

Using the definition of q_k in (6) and passing to the asymptotic form, we find that

$$\tilde{Q}'(x) = \frac{1-Q'(x)}{2} \cdot \left(1-\tilde{Q}(x)\right)^{\Delta-3}. \tag{9}$$

In this equation, $Q'(x)$ can be expressed in terms of $\tilde{Q}(x)$ and its derivative using (7):

$$Q'(x) = \tilde{Q}'(x)(1-\tilde{Q}(x)). \tag{10}$$

Substituting into (9) and setting $F(x) = 1-\tilde{Q}(x)$, we obtain the differential equation

$$F'(x) = -\frac{F(x)^{\Delta-3}}{F(x)^{\Delta-2}+2}. \tag{11}$$

One can check that $F''(x)$ is positive on $[0,1]$, and that $F'(0) = -1/3$. Hence, $F(x) \geq 1-x/3$. This implies an upper bound on $F'(x)$: since the function $h(t) = -t^{\Delta-3}/(t^{\Delta-2}+2)$ is decreasing on $[0,1]$, we obtain from (11) that

$$F'(x) \leq -\frac{(1-\frac{x}{3})^{\Delta-3}}{(1-\frac{x}{3})^{\Delta-2}+2}.$$

Integrating the right hand side, we obtain

$$\begin{aligned}
F(1) &= F(0) + \int_0^1 F'(x)\,dx \\
&\le 1 + \int_0^1 -\frac{(1-\frac{x}{3})^{\Delta-3}}{(1-\frac{x}{3})^{\Delta-2}+2}\,dx \\
&= 1 + \left[\frac{3\log\left((1-\frac{x}{3})^{\Delta-2}+2\right)}{\Delta-2}\right]_0^1 \\
&= 1 + \frac{3}{\Delta-2}\cdot\log\frac{\left(\frac{2}{3}\right)^{\Delta-2}+2}{3}.
\end{aligned}$$

We claim that the result does not exceed $\sqrt{(\Delta-1)/(\Delta+1)}$. This can be checked directly for $4 \le \Delta < 7$. For $\Delta \ge 7$, the argument of the logarithm is easily seen to be at most $e^{-1/3}$, which yields

$$F(1) \le 1 - \frac{1}{\Delta-2} < \sqrt{\frac{\Delta-1}{\Delta+1}}.$$

By (10) and the fact that $Q(0) = \tilde{Q}(0) = 0$, we have

$$Q(x) = \tilde{Q}(x) - \frac{\tilde{Q}(x)^2}{2} = \frac{1-F(x)^2}{2},$$

so the above upper bound on $F(1)$ implies

$$Q(1) > \frac{1-\frac{\Delta-1}{\Delta+1}}{2} = \frac{1}{\Delta+1},$$

proving the desired inequality (8). $\square$

Note that (despite the technicalities in the proof of Proposition 19) the argument for graphs with even maximum degree is simpler than that for subcubic graphs in that it works with just a decomposition of the graph into 2-factors, without the need to use a uniform cover by 1-factors as in Theorem 2. For graphs with odd maximum degree r, however, it is not clear how to proceed without a suitable analogue of Theorem 2. Furthermore, the natural analogue of Theorem 2 for r-regular graphs (r odd) does not hold in general. Still, it seems plausible that the following is true:

Conjecture 20. *The conclusion of Theorem 7 holds for graphs with odd maximum degree as well.*

So far, we have only been able to verify Conjecture 20 for the case of r*-graphs* (r-regular graphs with no odd edge-cuts of size smaller than r).

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