Combinatorial Characterizations of K-matrices[a]

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We present a number of combinatorial characterizations of K-matrices. This extends a theorem of Fiedler and Pták on linear-algebraic characterizations of K-matrices to the setting of oriented matroids. Our proof is elementary and simplifies the original proof substantially by exploiting the duality of oriented matroids. As an application, we show that a simple principal pivot method applied to the linear complementarity problems with K-matrices converges very quickly, by a purely combinatorial argument.

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1 Introduction

A matrix $M \in \mathbb{R}^{n\times n}$ is a *P-matrix* if all its principal minors (determinants of principal submatrices) are positive; it is a *Z-matrix* if all its off-diagonal elements are non-positive; and it is a *K-matrix* if it is both a P-matrix and a Z-matrix.

Z- and K-matrices often occur in a wide variety of areas such as input–output production and growth models in economics, finite difference methods for partial differential equations, Markov processes in probability and statistics, and linear complementarity problems in operations research [2].

In 1962, Fiedler and Pták [8] listed thirteen equivalent conditions for a Z-matrix to be a K-matrix. Some of them concern the sign structure of vectors:

Theorem 1.1 (Fiedler–Pták [8])**.** *Let M be a Z-matrix. Then the following conditions are equivalent:*

(a) *There exists a vector $x \geq 0$ such that $Mx > 0$;*
(b) *there exists a vector $x > 0$ such that $Mx > 0$;*
(c) *the inverse M^{-1} exists and $M^{-1} \geq 0$;*
(d) *for each vector $x \neq 0$ there exists an index k such that $x_k y_k > 0$ for $y = Mx$;*
(e) *all principal minors of M are positive (that is, M is a P-matrix, and thus a K-matrix).*

Our interest for K-matrices originates in the *linear complementarity problem (LCP)*, which is for a given matrix $M \in \mathbb{R}^{n\times n}$ and a given vector $q \in \mathbb{R}^n$ to find two vectors w and z in $\mathbb{R}^n$ so that

$$\begin{aligned} w - Mz &= q, \\ w, z &\geq 0, \\ w^{\mathrm{T}} z &= 0. \end{aligned} \tag{1}$$

In general, the problem to decide whether a LCP has a solution is NP-complete [6, 14]. If the matrix M is a P-matrix, however, a unique solution to the LCP always exists [22]. Nevertheless, no polynomial-time algorithm to find it is known, nor are hardness results for this intriguing class of LCPs. It is unlikely to be NP-hard, because that would imply that NP = co-NP [17]. For some recent results, see also [19].

If the matrix M is a Z-matrix, a polynomial-time (pivoting) algorithm exists [5] (see also [21, sect. 8.1]) that finds the solution or concludes that no solution exists. Interestingly, LCPs over this simple class of matrices have many practical applications (pricing of American options, portfolio selection problems, resource allocation problems).

A frequently considered class of algorithms to solve LCPs is the class of *simple principal pivoting methods* (see Sect. 6 or [7, Sect. 4.2]). We speak about a *class* of algorithms because the concrete algorithm depends on a chosen *pivot rule*. It has recently been proved in [10] that a simple principal pivoting method with *any* pivot rule takes at most a number of pivot steps linear in n to solve a LCP with a K-matrix M.

The study of pivoting methods and pivot rules has led to the devising of *combinatorial abstractions* of LCPs. One such abstraction is unique-sink orientations of cubes [23]; the one we are concerned with here is one of oriented matroids.

Oriented matroids were pioneered by Bland and Las Vergnas [4] and Folkman and Lawrence [9]. Todd [24] and Morris [18] gave a combinatorial generalization of LCPs by formulating the complementarity problem of oriented matroids (OMCP). Todd [24] proposed a generalization of Lemke's method [15] to solve the OMCP. Later Klafszky and Terlaky [13] and Fukuda and Terlaky [11] proposed a generalized criss-cross method; in [11] it is used for a constructive proof of a duality theorem for OMCPs in sufficient matroids (and hence also for LCPs with sufficient matrices). Hereby we revive their approach.

In this paper, we present a combinatorial generalization (Theorem 5.5) of the Fiedler–Pták Theorem 1.1. To this end, we devise oriented-matroid counterparts of the conditions (a)–(d). If the oriented matroid in question is realizable as the sign pattern of the null space of a matrix, then our conditions are equivalent to the conditions on the realizing matrix. In general, however, our theorem is stronger because it applies also to nonrealizable oriented matroids.

As a by-product, our proof yields a new, purely combinatorial proof of Theorem 1.1. Rather than on algebraic properties, it relies heavily on oriented matroid duality.

We then use our characterization theorem to show that an OMCP on an n-dimensional K-matroid (that is, a matroid satisfying the equivalent conditions of Theorem 5.5) is solved by any pivoting method in at most $2n$ pivot steps. This implies the result of [10] mentioned above that any simple principal pivoting method is fast for K-matrix LCPs.

2 Oriented matroids

The theory of oriented matroids provides a natural concept which not only generalizes combinatorial properties of many geometric configurations but presents itself in many other areas as well, such as topology and theoretical chemistry.

2.1 Definitions and basic properties

Here we state the definitions and basic properties of oriented matroids that we need in our exposition. For more on oriented matroids consult, for instance, the book [3].

Let E be a finite set of size n. A *sign vector* on E is a vector X in $\{+1, 0, -1\}^E$. Instead of $+1$, we write just $+$; instead of -1, we write just $-$. We define $X^- = \{e \in E : X_e = -\}$, $X^\ominus = \{e \in E : X_e = -$ or $X_e = 0\}$, and the sets X^0, $X^\oplus$ and X^+ analogously. For any subset F of E we write $X_F \geq 0$ if $F \subseteq X^\oplus$, and $X_F \leq 0$ if $F \subseteq X^\ominus$; furthermore $X \geq 0$ if $X_E \geq 0$ and $X \leq 0$ if $X_E \leq 0$. The *support* of a sign vector X is $\underline{X} = X^+ \cup X^-$. The *opposite* of X is the sign vector $-X$ with $(-X)^+ = X^-$, $(-X)^- = X^+$ and $(-X)^0 = X^0$. The *composition* of two sign vectors X and Y is given by

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise.} \end{cases}$$

The *product* $X \cdot Y$ is of two sign vectors is the sign vector given by

$$(X \cdot Y)_e = \begin{cases} 0 & \text{if } X_e = 0 \text{ or } Y_e = 0, \\ + & \text{if } X_e = Y_e \text{ and } X_e \neq 0, \\ - & \text{otherwise.} \end{cases}$$

Definition 2.1. An *oriented matroid* on E is a pair $\mathcal{M} = (E, \mathcal{V})$, where $\mathcal{V}$ is a set of sign vectors on E satisfying the following axioms:
(V1) $0 \in \mathcal{V}$.
(V2) If $X \in \mathcal{V}$, then $-X \in \mathcal{V}$.
(V3) If $X, Y \in \mathcal{V}$, then $X \circ Y \in \mathcal{V}$.
(V4) If $X, Y \in \mathcal{V}$ and $e \in X^+ \cap Y^-$, then there exists $Z \in \mathcal{V}$ with $Z^+ \subseteq X^+ \cup Y^+$, $Z^- \subseteq X^- \cup Y^-$, $Z_e = 0$, and $(\underline{X} \setminus \underline{Y}) \cup (\underline{Y} \setminus \underline{X}) \cup (X^+ \cap Y^+) \cup (X^- \cup Y^-) \subseteq \underline{Z}$.

The axioms (V1) up to (V4) are called *vector axioms*; (V4) is the *vector elimination axiom.* We say that the sign vector Z is the result of a vector elimination of X and Y at element e.

An important example is a matroid whose vectors are the sign vectors of elements of a vector subspace of $\mathbb{R}^n$. If A is an $r \times n$ real matrix, define

$$\mathcal{V} = \{\operatorname{sgn} x : x \in \mathbb{R}^n \text{ and } Ax = 0\}, \tag{2}$$

where $\operatorname{sgn} x = (\operatorname{sgn} x_1, \ldots, \operatorname{sgn} x_n)$. Then $\mathcal{V}$ is the vector set of an oriented matroid. In this case we speak of *realizable* oriented matroids.

A *circuit* of $\mathcal{M}$ is a nonzero vector $\mathcal{C} \in \mathcal{V}$ such that there is no nonzero vector $X \in \mathcal{V}$ satisfying $\underline{X} \subset \underline{C}$.

Proposition 2.2. *Let $\mathcal{M} = (E, \mathcal{V})$ be a matroid and let $\mathcal{C}$ be the collection of all its circuits. Then:*

(C1) $0 \notin \mathcal{C}$.
(C2) *If* $C \in \mathcal{C}$, *then* $-C \in \mathcal{C}$.
(C3) *For all* $C, D \in \mathcal{C}$, *if* $\underline{C} \subseteq \underline{D}$, *then* $C = D$ *or* $C = -D$.
(C4) *If* $C, D \in \mathcal{C}$, $C \neq -D$ *and* $e \in X^+ \cap Y^-$, *then there is a* $Z \in \mathcal{C}$ *with* $Z^+ \subseteq (C^+ \cup D^+) \setminus \{e\}$ *and* $Z^- \subseteq (C^- \cup D^-) \setminus \{e\}$.
(C5) *If* $C, D \in \mathcal{C}$, $e \in X^+ \cap Y^-$ *and* $f \in (C^+ \setminus D^-) \cup (C^- \setminus D^+)$, *then there is a* $Z \in \mathcal{C}$ *with* $Z^+ \subseteq (C^+ \cup D^+) \setminus \{e\}$, $Z^- \subseteq (C^- \cup D^-) \setminus \{e\}$ *and* $Z_f \neq 0$.
(C6) *For every vector* $X \in \mathcal{V}$ *there exist circuits* $C^1, C^2, \ldots, C^k \in \mathcal{C}$ *such that* $X = C^1 \circ C^2 \circ \cdots \circ C^k$ *and* $C^i_e \cdot C^j_e \geq 0$ *for all indices* i, j *and all* $e \in \underline{X}$.

Moreover, if a set $\mathcal{C}$ *of sign vectors on* E *satisfies (C1)–(C4), then it is the set of all circuits of a unique matroid; this matroid's vectors are then all finite compositions of circuits from* $\mathcal{C}$.

The property (C4) is called *weak circuit elimination*; (C5) is called *strong circuit elimination*. In (C6) we speak about a *conformal decomposition* of a vector into circuits.

A *basis* of an oriented matroid $\mathcal{M}$ is an inclusion-maximal set $B \subseteq E$ for which there is no circuit C with $\underline{C} \subseteq B$. Every basis B has the same size, called the *rank* of $\mathcal{M}$.

Proposition 2.3. *Let* B *be a basis of an oriented matroid* $\mathcal{M}$. *For every* e *in* $E \setminus B$ *there is a unique circuit* $C(B, e)$ *such that* $\underline{C(B, e)} \subseteq B \cup \{e\}$ *and* $C(B, e)_e = +$.

Such a circuit $C(B, e)$ is called the *fundamental circuit* of e with respect to B.

Two sign vectors X and Y are *orthogonal* if the set $\{X_e \cdot Y_e : e \in E\}$ either equals $\{0\}$ or contains both $+$ and $-$. We then write $X \perp Y$.

Proposition 2.4. *For every oriented matroid* $\mathcal{M} = (E, \mathcal{V})$ *of rank* n *there is a unique oriented matroid* $\mathcal{M}^* = (E, \mathcal{V}^*)$ *of rank* $|E| - n$ *given by*

$$\mathcal{V}^* = \left\{ Y \subseteq \{-, 0. +\}^E : X \perp Y \text{ for every } X \in \mathcal{V} \right\}.$$

This $\mathcal{M}^*$ is called the *dual* of M. Note that $(\mathcal{M}^*)^* = \mathcal{M}$. The circuits of $\mathcal{M}^*$ are called the *cocircuits* of $\mathcal{M}$ and the vectors of $\mathcal{M}^*$ are called the *covectors* of $\mathcal{M}$. The covectors of a realizable matroid given by (2) are sign vectors of the elements of the row space of the matrix M.

We conclude this short overview by introducing the concept of matroid minors and extensions. For any $F \subseteq E$, the vector $X \backslash F$ denotes the subvector $(X_e : e \in E \backslash F)$ of X. Then let

$$\mathcal{V} \backslash F = \{X \backslash F : X \in \mathcal{V} \text{ and } X_f = 0 \text{ for all } f \in F\}$$

be the *deletion* and

$$\mathcal{V}/F = \{X \backslash F : X \in \mathcal{V}\}$$

the *contraction* of the circuits $\mathcal{V}$ by the elements F. It is easy to check that the pairs $\mathcal{M} \backslash F = (E \backslash F, \mathcal{V} \backslash F)$ and $\mathcal{M}/F = (E \backslash F, \mathcal{V}/F)$ are oriented matroids. For any disjoint $F, G \subseteq E$ we call the oriented matroid $(\mathcal{M} \backslash F)/G$ a *minor* of $\mathcal{M}$.

Definition 2.5. A matroid $\hat{\mathcal{M}} = (E \cup \{q\}, \hat{\mathcal{C}})$ with $q \notin E$ is a *one-point extension* of $\mathcal{M}$ if $\hat{\mathcal{M}} \backslash \{q\} = \mathcal{M}$ and there is a vector X of $\hat{\mathcal{M}}$ with $X_q \neq 0$.

2.2 Complementarity in oriented matroids

In the rest of the paper, we are considering oriented matroids endowed with a special structure. The set of elements E_{2n} is a $2n$-element set with a fixed partition $E_{2n} = S \cup T$ into two n-element sets and a mapping $e \mapsto \overline{e}$ from E_{2n} to E_{2n} which is an involution (that is, $\overline{\overline{e}} = e$ for every $e \in E_{2n}$) and for every $e \in S$ we have $\overline{e} \in T$. Note that this mapping constitutes a bijection between S and T.

The element $\overline{e}$ is called the *complement* of e. For a subset F of E_{2n} let $\overline{F} = \{\overline{e} : e \in F\}$. A subset F of E_{2n} is called *complementary* if $F \cap \overline{F} = \emptyset$.

The matroids we are working with are of the kind $\mathcal{M} = (E_{2n}, \mathcal{V})$, where the set $S \subseteq E_{2n}$ is a basis of $\mathcal{M}$. In addition, we study their one-point extensions $\hat{\mathcal{M}} = (\hat{E}_{2n}, \hat{\mathcal{V}})$, where $\hat{E}_{2n} = E_{2n} \cup \{q\}$ for some element $q \notin E_{2n}$. Sometimes we make the canonical choice $E_{2n} = \{1, \ldots, 2n\}$ with $S = \{1, \ldots, n\}$ where the complement of an $i \in S$ is the element $i + n$.

Definition 2.6. The *oriented matroid complementarity problem* (OMCP) is to find a vector X of an oriented matroid $\hat{\mathcal{M}}$ so that

$$X \in \hat{\mathcal{V}}, \tag{3a}$$

$$X \geq 0, \ X_q = +, \tag{3b}$$

$$X_e \cdot X_{\overline{e}} = 0 \quad \text{for every } e \in E_{2n}, \tag{3c}$$

or to report that no such vector exists.

A vector X which satisfies (3b) is called *feasible*, one which satisfies (3c) is called *complementary*. Note that a vector is complementary if and only if its support is a complementary set. If an $X \in \hat{\mathcal{V}}$ satisfies (3b) and (3c), then X is a *solution* to the OMCP($\hat{\mathcal{M}}$).

Now we show how LCPs are special cases of OMCPs. Finding a solution to the LCP (1) is equivalent to finding an element x of

$$V = \left\{ x \in \mathbb{R}^{2n+1} : \begin{bmatrix} I_n & -M & -q \end{bmatrix} x = 0 \right\}$$

such that

$$\begin{aligned} x \geq 0, \ x_{2n+1} = 1, \\ x_i \cdot x_{i+n} = 0 \quad \text{for every } i \in [n]. \end{aligned} \tag{4}$$

We set $\hat{\mathcal{V}} = \{\operatorname{sgn} x : x \in V\}$ and consider the OMCP for the matroid $\hat{\mathcal{M}} = (\hat{E}_{2n}, \hat{\mathcal{V}})$. Clearly, if the OMCP has no solution, then V contains no vector x satisfying (4). If on the other hand there is a solution X satisfying (3a)–(3c), then the solution to the LCP can be obtained by solving the system of linear equations

$$\begin{aligned} \begin{bmatrix} I_n & -M & -q \end{bmatrix} x &= 0, \\ x_i &= 0 \quad \text{whenever } X_i = 0, \\ x_{2n+1} &= 1. \end{aligned}$$

This correspondence motivates the following definition.

Definition 2.7. An oriented matroid $\mathcal{M} = (E_{2n}, \mathcal{V})$ is *LCP-realizable* if there is a matrix $M \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{V} = \Big\{ \operatorname{sgn} x : x \in \mathbb{R}^{2n} \text{ and } \begin{bmatrix} I_n & -M \end{bmatrix} x = 0 \Big\}.$$

The matrix M is then a *realization matrix* of $\mathcal{M}$. This is a little non-standard, because usually the matrix A from (2) is called a realization matrix.

The extension $\hat{\mathcal{M}} = (\hat{E}_{2n}, \hat{\mathcal{V}})$ is *LCP-realizable* if there is a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ such that

$$\hat{\mathcal{V}} = \Big\{ \operatorname{sgn} x : x \in \mathbb{R}^{2n+1} \text{ and } \begin{bmatrix} I_n & -M & -q \end{bmatrix} x = 0 \Big\}.$$

To study the algorithmic complexity of OMCPs, we must specify how the matroid $\hat{\mathcal{M}}$ is made available to the algorithm. We will assume that it is given by an oracle which, for a basis B of $\hat{\mathcal{M}}$ and a nonbasic element $e \in \hat{E}_{2n} \setminus B$, outputs the unique (fundamental) circuit C of $\hat{\mathcal{M}}$ with support $\underline{C} \subseteq B \cup \{e\}$ such that $X_e = +$.

In the LCP-realizable case such an oracle can be implemented in polynomial time; in fact, it consists in solving a system of $O(n)$ linear equations in $2n + 1$ variables. Thus, in the RAM model, the oracle can be implemented so that it performs arithmetic operations whose number is bounded by a polynomial in n. Hence our goal is to develop an algorithm that solves an OMCP using a number of queries to the oracle that is polynomial in n.

Such an algorithm for the OMCP would obviously provide a strongly polynomial algorithm for the LCP. Since the LCP is NP-hard in general, the existence of such an algorithm is unlikely. In Section 6 we do, nevertheless, prove the existence of such an algorithm for a special class of oriented matroids: K-matroids.

3 P-matroids

In this and the following sections, we investigate what properties of oriented matroids characterize oriented matroids realizable by special classes of matrices. We start with P-matrices; recall that a P-matrix is a matrix whose principal minors are all positive.

Several conditions are equivalent to the positivity of principal minors:

Theorem 3.1. *For a matrix $M \in \mathbb{R}^{n\times n}$, the following are equivalent:*

(a) *All principal minors of M are positive (i.e., M is a P-matrix);*
(b) *there is no nonzero vector x such that $x_k y_k \leq 0$ for all $i = 1, 2, \ldots, n$, where $y = Mx$;*
(c) *the LCP* (1) *with the matrix M and any right-hand side q has exactly one solution.*

The equivalence of (a) and (b) is due to Fiedler and Pták [8]. The equivalence of (a) and (c) was proved independently by Samelson, Thrall and Wesler [22], Ingleton [12], and Murty [20].

The following notions and our definition of a P-matroid are motivated by the condition (b) in Theorem 3.1. It is much easier to express in the oriented-matroid language than (a).

A sign vector $X \in \{-, 0, +\}^{E_{2n}}$ is *sign-reversing (s.r.)* if $X_e \cdot X_{\overline{e}} \leq 0$ for every $e \in S$. If in addition $\underline{X} = E_{2n}$, the vector is *totally sign-reversing (t.s.r.)*. Analogously, an X is *sign-preserving (s.p.)* if $X_e \cdot X_{\overline{e}} \geq 0$ for every e, and *totally sign-preserving (t.s.p.)* if $\underline{X} = E_{2n}$ as well.

Definition 3.2 (Todd [24])**.** An oriented matroid $\mathcal{M}$ on E_{2n} is a *P-matroid* if it has no sign-reversing circuit.

Note that a P-matroid contains no sign-reversing vectors, because every vector is the composition of some circuits and composing non-s.r. circuits yields non-s.r. vectors. Hence, using Theorem 3.1, we immediately get:

Proposition 3.3.

(i) *If $\mathcal{M}$ is LCP-realizable and there exists a realization matrix M that is a P-matrix, then $\mathcal{M}$ is a P-matroid.*
(ii) *If $\mathcal{M}$ is an LCP-realizable P-matroid, then every realization matrix M is a P-matrix.*

P-matroids were extensively studied by Todd [24]. He lists eight equivalent conditions for a matroid to be a P-matroid. We recall three of them (conditions (a), (a*) and (c) below) and add two new ones.

Theorem 3.4. *For an oriented matroid $\mathcal{M}$ on E_{2n}, the following conditions are equivalent:*

(a) *$\mathcal{M}$ has no s.r. circuit;*
(a*) *$\mathcal{M}$ has no s.p. cocircuit;*
(b) *every t.s.p. X is a vector of $\mathcal{M}$;*
(b*) *every t.s.r. Y is a covector of $\mathcal{M}$;*
(c) *every one-point extension $\hat{\mathcal{M}}$ of $\mathcal{M}$ to $\hat{E}_{2n}$ contains exactly one complementary circuit C such that $C \geq 0$ and $C_q = +$.*

Proof. The equivalence of the conditions (a), (a*) and (c) was shown by Todd [24]. Morris [18] proved that (a) implies (b). We show the equivalence of (a) with (b*). The equivalence of (a*) with (b) is proved analogously.

First we prove that (a) implies (b*). Since no circuit of $\mathcal{M}$ is s.r., there is for every circuit C an element e such that $C_e \cdot C_{\overline{e}} = +$. It follows that every t.s.r. sign vector Y is orthogonal to every circuit, hence Y is a covector.

For the opposite direction, suppose that there is a s.r. circuit C. If so, then any t.s.r. vector Y for which $Y^+ \subseteq C^+$ and $Y^- \subseteq C^-$ is *not* orthogonal to C, which is a contradiction with (b*). $\square$

The condition (b) of this theorem has a translation for realization matrices of P-matroids, that is, for P-matrices:

Corollary 3.5. *A matrix $M \in \mathbb{R}^{n\times n}$ is a P-matrix if and only if for every $\sigma \in \{-1,+1\}^n$ there exists a vector $x \in \mathbb{R}^n$ such that for $y = Mx$ and for each $i \in \{1,2,\ldots,n\}$ we have*

$$\sigma_i x_i > 0,$$
$$\sigma_i y_i > 0.$$

Todd [24] also gives an oriented-matroid analogue of the "positive principal minors" condition. Stating it would require some more explanations; later in this article we need a weaker property of P-matroids, though, which corresponds to the fact that all principal minors of a P-matrix are nonzero.

Lemma 3.6 (Todd [24])**.** *For a P-matroid $\mathcal{M}$ every complementary subset $B \subseteq E_{2n}$ of cardinality n is a basis.*

Remark. In addition, every such complementary B is also a cobasis, i.e., it is a basis of the dual matroid $\mathcal{M}^*$.

Next we consider principal pivot transforms (see [25, 26]) of P-matrices. The fact that every principal pivot transform of a P-matrix is again a P-matrix [27] is well-known. The proof is not very difficult but it uses involved properties of the Schur complement. In the setting of oriented matroids the equivalent is much simpler. First let us define principal pivot transforms of oriented matroids.

Definition 3.7. Let $F \subseteq E_{2n}$ be a complementary set. The *principal pivot transform* of a sign vector X with respect to F is the sign vector $\tilde{C}$ given by

$$\tilde{C}_e = \begin{cases} C_e & \text{if } e \notin F, \\ C_{\overline{e}} & \text{if } e \in F. \end{cases}$$

The *principal pivot transform* of a matroid $\mathcal{M}$ with respect to F is the matroid whose circuits (vectors) are the principal pivot transforms of the circuits (vectors) of $\mathcal{M}$.

It is easy to see that, in the LCP-realizable case, principal pivot transforms of a matroid correspond to matroids realized by corresponding principal pivot transforms of the realization matrix. Thus the following proposition implies the analogous theorem for P-matrices.

Proposition 3.8. *Every principal pivot transform of a P-matroid is a P-matroid.*

Proof. The principal pivot transform of a circuit C is sign-reversing if and only if C is sign-reversing. □

4 Z-matroids

The second class of matrices we examine are Z-matrices; the corresponding matroid generalizations are Z-matroids. Recall that a *Z-matrix* is a matrix whose every off-diagonal element is non-positive. The definition of Z-matroids was first proposed in [16].

Definition 4.1. A matroid $\mathcal{M}$ on E_{2n} is a *Z-matroid* if for every circuit C of $\mathcal{M}$ we have:

$$\begin{aligned} &\text{If } C_T \geq 0, \text{ then} \\ &\qquad C_{\overline{e}} = + \text{ for all } e \in S \text{ with } C_e = +. \end{aligned} \tag{5}$$

Remark. In the definition of Z-matroid we could replace all occurrences of the word "circuit" with the word "vector". Indeed, in a conformal decomposition of a vector violating (5), there would always be a circuit violating (5) as well.

It makes perfect sense to define Z-matroids in this way. We show that in LCP-realizable cases, any realization matrix M is a Z-matrix.

Proposition 4.2.
(i) *If $\mathcal{M}$ is LCP-realizable and there exists a realization matrix M that is a Z-matrix, then $\mathcal{M}$ is a Z-matroid.*
(ii) *If $\mathcal{M}$ is an LCP-realizable Z-matroid, then every realization matrix M is a Z-matrix.*

Proof. We fix $E_{2n} = \{1, \ldots, 2n\}$ with $S = \{1, \ldots, n\}$ where the complement of an $i \in S$ is the element $i + n$.

(i) Let e_i denote the ith unit vector and m_j the jth column of the matrix M. The sign pattern of the Z-matrix M implies that there is no linear combination of the form

$$e_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j e_j - \sum_{j=n+1}^{2n} x_j m_j = 0,$$

where $x_j \geq 0$ for every $j > n$ and $x_{i+n} = 0$, because the ith row of the left-hand side is strictly positive. Hence there is no vector $X \in \mathcal{V}$ for which $X_T \geq 0$, $X_i = +$ but $X_{i+n} = 0$ for some $i \in S$.

(ii) Proof by contradiction. Assume that for an LCP-realizable Z-matroid $\mathcal{M}$ (where S is a basis), there is a realization matrix M that is not a Z-matrix, that is, there is an off-diagonal $m_{ij} > 0$. If so, there is a vector X with $X_{j+n} = +$ and $X_{T\setminus\{j+n\}} = 0$, but $X_i = +$. This X violates the Z-matroid property (5) since also $X_{i+n} = 0$, a contradiction. Thus no positive m_{ij} can exist and M has to be a Z-matrix. □

Another option is to characterize a Z-matroid with respect to the dual matroid $\mathcal{M}^*$.

Proposition 4.3. *An oriented matroid $\mathcal{M}$ on E_{2n} is a Z-matroid if and only if for every cocircuit D of $\mathcal{M}$ we have:*

$$\begin{aligned} &\textit{If } D_S \leq 0, \textit{ then} \\ &\qquad\qquad D_{\overline{e}} = - \textit{ for all } e \in T \textit{ with } D_e = +. \end{aligned} \tag{6}$$

Proof. First we prove the "only if" direction. Suppose that there is a cocircuit D which does not satisfy (6). Accordingly $D_S \leq 0$ and there is $e \in T$ such that $D_e = +$, but $D_{\overline{e}} = 0$. But note that the fundamental circuit $C(S,e)$ and D are not orthogonal because the Z-matroid property (5) implies that $C_{S\setminus\{e\}} \leq 0$. Hence no such D can exist.

For the "if" direction suppose that there is a circuit C for which $C_T \geq 0$ and $C_e = +$, but $C_{\overline{e}} = 0$ for some $e \in S$. This circuit C and the fundamental cocircuit $D(T,e)$ are not orthogonal since by assumption (6) holds for D and of course $-D$, hence $D_{T\setminus\{\overline{e}\}} \geq 0$. □

5 K-matroids

Definition 5.1. A matroid $\mathcal{M}$ on E_{2n} is a *K-matroid* if it is a P-matroid and a Z-matroid.

Combining Proposition 3.3 and Proposition 4.2 we immediately get:

Proposition 5.2.

(i) *If $\mathcal{M}$ is LCP-realizable and there exists a realization matrix M that is a K-matrix, then $\mathcal{M}$ is a K-matroid.*

(ii) *If $\mathcal{M}$ is an LCP-realizable K-matroid, then every realization matrix M is a K-matrix.*

In the proofs in this section we often make use of fundamental circuits. We first observe that all fundamental circuits with respect to the basis S follow the same sign pattern.

Lemma 5.3. *Let $\mathcal{M}$ be a K-matroid. Let $e \in T$ and let $C = C(S, e)$ be the fundamental circuit of e with respect to the basis S. Then*

$$\begin{aligned} C_e &= +, \\ C_{\overline{e}} &= +, \\ C_{T \setminus \{e\}} &= 0, \\ C_{S \setminus \{\overline{e}\}} &\leq 0. \end{aligned}$$

Proof. The first and the third equality follow directly from the definition of a fundamental circuit. Thus $C_T \geq 0$. Hence the fourth property follows from the Z-matroid property (5). The third equality follows from the fact that $\mathcal{M}$, as a P-matroid, contains no sign-reversing circuit. □

An oriented matroid minor $\mathcal{M} \setminus F / \overline{F}$ where F is a complementary subset of E_{2n} is called a *principal minor* of $\mathcal{M}$.

Lemma 5.4. *Let $\mathcal{M}$ be a K-matroid. Then every principal minor $\mathcal{M} \setminus F / \overline{F}$ is a K-matroid.*

Proof. It was shown by Todd [24] that every principal minor of a P-matroid is a P-matroid. Thus, it is enough to show that such a minor is a Z-matroid.

First, we prove that if $e \in T$, then $\mathcal{M} \setminus \{e\} / \{\overline{e}\}$ is a Z-matroid. Such a principal minor consists of all circuits $C \setminus \{e, \overline{e}\}$, where C is a circuit of $\mathcal{M}$ and $C_e = 0$. Since every circuit of $\mathcal{M}$ satisfies the Z-matroid characterization (5), such a circuit $C \setminus \{e, \overline{e}\}$ trivially satisfies it too.

Secondly, let $e \in S$. Here we apply a case distinction. If $C_{\overline{e}} = +$, then $(C \setminus \{e, \overline{e}\})_T \geq 0$ if and only if $C_T \geq 0$. As a direct consequence, $C \setminus \{e, \overline{e}\}$ satisfies (5) because C does. If $C_{\overline{e}} = -$, we can show that there is another element $f \in T$ such that $C_f = -$ too, that is, $(C \setminus \{e, \overline{e}\})_T \ngeq 0$ and thus the Z-matroid property (5) is obviously satisfied. Assume for the sake of contradiction that there is no such $f \in T$. The strong circuit elimination (C5) of C and the fundamental circuit $C(S, \overline{e})$ at $\overline{e}$ then yields a circuit C' with $C'_T \geq 0$, $C'_{\overline{e}} = 0$ and $C'_e = +$. Since $e \in S$, such a C' would violate Z-matroid definition, a contradiction. $\square$

Our main result, the combinatorial generalization of the Fiedler–Pták Theorem 1.1 is the following.

Theorem 5.5. *For a Z-matroid $\mathcal{M}$ (with vectors $\mathcal{V}$, covectors $\mathcal{V}^*$, circuits $\mathcal{C}$ and cocircuits $\mathcal{D}$), the following statements are equivalent:*

(a) $\exists X \in \mathcal{V} : X_T \geq 0$ *and* $X_S > 0$;

(b) $\exists X \in \mathcal{V} : X > 0$;

(c) $\forall C \in \mathcal{C}$: $C_S \geq 0 \implies C_T \geq 0$;

(d) *there is no s.r. circuit $C \in \mathcal{C}$ (that is, $\mathcal{M}$ is a P-matroid);*

(a*) $\exists Y \in \mathcal{V}^* : Y_S \leq 0$ *and* $Y_T > 0$;

(b*) $\exists Y \in \mathcal{V}^* : Y_S < 0$ *and* $Y_T > 0$;

(c*) $\forall D \in \mathcal{D}$: $D_T \geq 0 \implies D_S \leq 0$;

(d*) *there is no s.p. cocircuit $D \in \mathcal{D}$.*

In order to use duality in the proof of this theorem, let us first define the *reflection* of a matroid $\mathcal{M} = (E_{2n}, \mathcal{V})$ to be the matroid $\Re(\mathcal{M}) = (E_{2n}, \Re(\mathcal{V}))$, where $\Re(\mathcal{V}) = \{\Re(X) : X \in \mathcal{V}\}$ with

$$\big(\Re(X)\big)_e = \begin{cases} X_{\overline{e}} & \text{if } e \in S, \\ -X_{\overline{e}} & \text{if } e \in T. \end{cases}$$

Observe that $\Re\big(\Re(\mathcal{M})\big) = \mathcal{M}$ because of (V2), and that $\Re(\mathcal{M}^*) = \Re(\mathcal{M})^*$; thus

$$\Re\big(\Re(\mathcal{M}^*)^*\big) = \mathcal{M}. \tag{7}$$

Proof of Theorem 5.5.

(a) $\implies$ (b): Let X be as in (a). Since $X_T \geq 0$, the Z-matroid property (5) implies that if $X_e = +$ for an $e \in S$, then $X_{\overline{e}} = +$. Thus $X_T > 0$.

(b) $\implies$ (c): Let X be the all-plus vector as in (b). Suppose that there is a circuit $C \in \mathcal{C}$ not satisfying (c), that is, $C_S \geq 0$ but $C_e = -$ for some element e in T. Starting with $Y^0 = C$, we apply a sequence of vector eliminations (V4) to get vectors Y^i. We eliminate Y^{i-1} and X at any element e where $Y^{i-1}_e = -$. For a resulting vector Y^i it holds that $(Y^i)^- \subset (Y^{i-1})^-$. Thus, at some point $(Y^k)^- = \emptyset$ while $Y^k_e = 0$ and $Y^k_{\overline{e}} = +$ where $e \in T$ is the element eliminated in step $k-1$. This vector Y^k does not satisfy the Z-matroid property (5), which is a contradiction.

(b) $\implies$ (a): Suppose that there is a s.r. circuit $C \in \mathcal{C}$, that is, $C_e \cdot C_{\overline{e}} \leq 0$ for every $e \in S$. Let $C^0 = C$. We apply consecutive circuit eliminations (C4). To get C^i, we eliminate C^i with any fundamental circuit $C(S, e)$ at position $e \in T$ where $C^i_e = -$. After finitely many steps we end up with a circuit C^k for which $C^k_T \geq 0$. In addition, by the Z-matroid property (5) and the fact that the initial C^0 is s.r., we have $C^k_S \leq 0$. Note that $-C^k$ violates the property (b). Hence there cannot exist a s.r. circuit C.

(d) $\implies$ (a*): Because of (d), for every circuit C there is an $e \in S$ such that $C_e \cdot C_{\overline{e}} = +$. The sign vector Y where $Y_S < 0$ and $Y_T > 0$ is orthogonal to every circuit C, because the sign of $Y_e \cdot C_e$ is opposite to the sign of $Y_{\overline{e}} \cdot C_{\overline{e}}$. Hence such a Y is a covector.

To finish the proof, notice that a matroid $\mathcal{M}$ satisfies (a*) if and only if the reflection of its dual $\Re(\mathcal{M}^*)$ satisfies (a); analogously for (b*) and (b), (c*) and (c), and (d*) and (d). Thus if $\mathcal{M}$ satisfies (a*), then $\Re(\mathcal{M}^*)$ satisfies (a), hence also (b), and so (using (7)) $\mathcal{M}$ satisfies (b*). The missing implications (b*) $\Longrightarrow$ (c*), (c*) $\Longrightarrow$ (d*), and (d*) $\Longrightarrow$ (a) are proved analogously. $\square$

6 Algorithmic aspects

Let an OMCP($\hat{\mathcal{M}}$) be given, where $\hat{\mathcal{M}}$ is any one-element extension of an n-dimensional matroid $\mathcal{M}$ on E_{2n}. We present *simple principal pivot algorithms* to find a solution. This kind of algorithms is a well-established solving method for LCPs. Sometimes called *Bard-type methods*, they were first studied by Zoutendijk [28] and Bard [1].

Here we extend a recent result of [10] to the generalizing setting of OMCP. We prove below that the unique solution to every OMCP($\hat{\mathcal{M}}$) where the underlying matroid $\mathcal{M}$ is a K-matroid, is found by every simple principal pivot algorithm in a linear number of pivot steps.

Let $\hat{\mathcal{M}}$ be given by an oracle which, for a basis B of $\hat{\mathcal{M}}$ and a non-basic element $e \in \hat{E}_{2n} \backslash B$, outputs the unique fundamental circuit $C(B, e)$. A simple principal pivot algorithm starts with a fundamental circuit $C^0 = C(B^0, q)$ where B^0 is any complementary basis. For instance, $B^0 = S$. It then proceeds in *pivot steps*. Assume that the ith step leads to a fundamental circuit $C^i = C(B^i, q)$. We require the complementary condition (3c) to be an invariant, that is, B^i is supposed to be complementary. To get C^{i+1} we proceed as follows: If C^i is feasible, that is, the condition (3b) is satisfied, then the C^i is the solution and the algorithm terminates. Otherwise one chooses an $e^i \in B^i$ for which $C^i_{e^i} = -$ according to a *pivot rule*. Then the element e^i is replaced in the basis with its complement $\overline{e^i}$, that is, $B^{i+1} = B^i \backslash \{e^i\} \cup \{\overline{e^i}\}$. Lemma 3.6 asserts that B^{i+1} is indeed a basis. Then $C^{i+1} = C(B^{i+1}, q)$ is computed by feeding the oracle with basis B^{i+1} and the non-basic element q. The algorithm then proceeds with pivot step $i + 2$.

SIMPLEPRINCIPALPIVOT($\hat{\mathcal{M}}, B^0$)
$i := 0$
$C^0 := C(B^0, q)$
while $(C^i)^- \neq \emptyset$ **do**
  choose $e^i \in (C^i)^-$ according to a pivot rule R
  $B^{i+1} := B^i \backslash \{e^i\} \cup \{\overline{e^i}\}$
  $C^{i+1} := C(B^{i+1}, q)$
  $i := i + 1$
end while
return C^i

If the number of pivots is polynomial in n, then the whole algorithm runs in polynomial time too, provided that the oracle computes the fundamental circuit in polynomial time. This is the case if the LCP is given by a matrix M and a right-hand side q as in (1).

The number of pivots depends on the applied pivot rule and some rules may even enter a loop on some inputs $\hat{\mathcal{M}}$. If the input is a K-matroid extension, though, then the SIMPLEPRINCIPALPIVOT method is fast. We claim that no matter which pivot rule is applied, SIMPLEPRINCIPALPIVOT runs in a linear number of pivot steps on every K-matroid extension. The following two lemmas are required to prove this fact. The first one holds for every P-matroid extension, the second is restricted to K-matroid extensions.

Lemma 6.1. *If $\hat{\mathcal{M}}$ is a P-matroid extension, then $C^{i+1}_{\overline{e^i}} \geq 0$ for every $i \geq 0$.*

Proof. Suppose that $C^{i+1}_{\overline{e^i}} = -$ in some pivot step $i+1$. Let C' be the result of a weak circuit elimination of C^i and $-C^{i+1}$ at p. Then C' is an almost-complementary circuit of the P-matroid $\mathcal{M}$ with $C'_{e^i} \leq 0$ and $C'_{\overline{e^i}} \geq 0$, in other words it is a s.r. vector. According to the Definition 3.2 of a P-matroid, no s.r. circuit can exist. This contradiction finishes the proof. $\square$

Lemma 6.2. *If $\hat{\mathcal{M}}$ is a K-matroid extension, then for every $f \in T$:*

$$\text{If } C_f^h \geq 0 \text{ for some } h \geq 0, \text{ then } C_f^k \geq 0 \text{ for every } k \geq h.$$

Proof. For the sake of contradiction suppose that the statement does not hold and let $l \geq h$ be the smallest value such that $C_f^l \geq 0$, but $C_f^{l+1} = -$. Let X be an elimination vector of C^l and $-C^{l+1}$ at q. Note that $X_{e^l} = -$, $X_f = +$ and $X_{\overline{f}} = 0$. In addition by Lemma 6.1 it holds that $X_{\overline{e^l}} \leq 0$. Since $X_q = 0$, the sign vector $X \backslash \{q\}$ is a vector of the K-matroid $\mathcal{M}$. Now have a look at the principal minor $\mathcal{M} \backslash \left(X^0 \backslash \{\overline{f}, \overline{e^l}\}\right) / \overline{\left(X^0 \backslash \{\overline{f}, \overline{e^l}\}\right)}$. This minor is a matroid on the element set $\{f, \overline{f}, e^l, \overline{e^l}\}$. Moreover, by Lemma 5.4 it is also a K-matroid. Further it contains the vector $X' = X \backslash \left(\hat{E_{2n}} \backslash \{f, \overline{f}, e^l, \overline{e^l}\}\right)$ with $X'_{e^l} = -$, $X'_f = +$, $X'_{\overline{f}} = 0$ and $X'_{\overline{e^l}} \leq 0$. The contradiction follows from the fact that $-X'$ violates the K-matroid property (c) in Theorem 5.5. □

Theorem 6.3. *Every simple principal pivot algorithm runs in at most $2n$ pivot steps on every K-matroid extension.*

Proof. We prove that, no matter which pivot rule R one applies, every element $e \in E_{2n}$ is chosen at most once as the pivot element. Consider any pivot step h in the SimplePrincipalPivot algorithm. First suppose that the pivot element e^h is in S. According to Lemma 6.1 $C_{\overline{e^h}}^{h+1} \geq 0$. Moreover, for every $k \geq h$ we have $C_{\overline{e^h}}^k \geq 0$ (Lemma 6.2) and $C_{e^h}^k = 0$. In other words, the elements e^h and $\overline{e^h}$ cannot become pivot elements in later steps. Secondly, if the pivot e^h is in T, the argumentation from above fails. Even if $C_{\overline{e^h}}^{h+1} \geq 0$ (Lemma 6.1), we cannot conclude that $C_{\overline{e^h}}^k \geq 0$ for every $k \geq h$, because Lemma 6.2 does not apply. It may eventually happen for some k that $\overline{e^h}$ is chosen as pivot e^k. However if so, our first argument applies for pivot step k and neither $\overline{e^h}$ nor e^h can become pivot elements again. □

7 Extension to principal pivot closures

So far, we have considered a matroid $\mathcal{M}$ on a complementary set E_{2n} where the maximal complementary set S is fixed from the beginning. In the following, S' is an arbitrary complementary subset of size n and $T' = \{\overline{e} : e \in S'\}$.

Definition 7.1. A matroid $\mathcal{M}$ on E_{2n} is a *Z*-matroid* if there is a complementary set $S' \subseteq E_{2n}$ of cardinality n such that for $T' = \{\overline{e} : e \in S'\}$ and every circuit C of $\mathcal{M}$ we have:

$$\text{If } C_{T'} \geq 0, \text{ then } \quad C_{\overline{e}} = + \text{ for all } e \in S' \text{ with } C_e = +.$$

Analogously $\mathcal{M}$ is a *K*-matroid* if it is a P-matroid and a Z*-matroid. Note that the class of Z*-, K*-matroids is the closure under the principal pivot transform applied to Z-matroids, K-matroids respectively. Moreover, Proposition 4.3, Lemma 5.4 up to Theorem 6.3 have equivalent counterparts for these closure classes, obtained by substituting S by S' and accordingly T by T' in the original statements. Hence we get the following.

Corollary 7.2. *Every simple principal pivot algorithm finds the solution to OMCP($\hat{\mathcal{M}}$), where $\hat{\mathcal{M}}$ is a K*-matroid extension, in at most $2n$ pivot steps.*

The reader might wonder why we introduced Z-matroids and K-matroids at all and did not start off with their principal pivot closures. One good reason for our approach is to point out the correspondence of LCP-realizable Z-matroids and their matrix counterparts, see Proposition 4.2. With respect to this, the main problem is that a principal pivot transform of a Z-matroid or a K-matroid is in general not a Z-matroid, a K-matroid respectively. However every principal pivot algorithm is still able to solve an LCP(M, q) where M is a principal pivot transform of a K-matrix in a linear number of pivot steps.

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