Unexpected behaviour of crossing sequences

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Abstract

The n^{th} crossing number of a graph G, denoted $cr_n(G)$, is the minimum number of crossings in a drawing of G on an orientable surface of genus n. We prove that for every $a > b > 0$, there exists a graph G for which $cr_0(G) = a$, $cr_1(G) = b$, and $cr_2(G) = 0$. This provides support for a conjecture of Archdeacon et al. and resolves a problem of Salazar.

1 Introduction

Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept—crossing number, and embeddings in more complicated surfaces—are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [5], and studied further in [1]. Following these authors, we define for every nonnegative

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integer i and every graph G, the i^{th} *crossing number*, $cr_i(G)$, (and also the i^{th} *nonorientable crossing number*, $\tilde{cr}_i(G)$) to be the minimum number of crossings in a drawing of G on the orientable (nonorientable, respectively) surface of genus i. We consider drawings where each vertex x of G is represented by a point $\phi(x)$ of the suface, each edge uv by a curve with ends at points $\phi(u)$ and $\phi(v)$ and with interior avoiding all points $\phi(x)$ for $x \in V(G)$. Moreover, we assume that no three edges are drawn so that they have an interior point in common. Observe that $cr_i(G) = 0$ (respectively, $\tilde{cr}_i(G) = 0$) if and only if i is greater or equal to the genus (resp., nonorientable genus) of G. This gives, for every graph G, two finite sequences of integers, $(cr_0(G), cr_1(G), \ldots, 0)$ and $(\tilde{cr}_0(G), \tilde{cr}_1(G), \ldots, 0)$, both of which terminate with a single zero. The first of these is the *orientable crossing sequence* of G, the second the *nonorientable crossing sequence* of G.

A natural question is to characterize crossing sequences of graphs. This is the focus of both [5] and [1]. If we are given a drawing of a graph in a surface $\mathcal{S}$ with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of G in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0. This necessary condition was conjectured to be sufficient in [1].

Conjecture 1.1 (Archdeacon, Bonnington, and Širáň)
If $(a_1, a_2, \ldots, 0)$ is a sequence of nonnegative integers which strictly decreases until 0, then there is a graph whose crossing sequence (nonorientable crossing sequence) is $(a_1, a_2, \ldots, 0)$.

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form $(a, b, 0)$, Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases. We shall postpone discussion of their examples for the oriented case until later, but let us highlight their result for the nonorientable case here.

Theorem 1.2 (Archdeacon, Bonnington, and Širáň) *If a and b are integers with $a > b > 0$, then there exists a graph G with nonorientable crossing sequence $(a, b, 0)$.*

It has been believed by some that such a result cannot hold for the orientable case. For the most extreme special case $(N, N-1, 0)$, where N is a large integer, Salazar asked [4] if this sequence could really be the crossing sequence of a graph. The following quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

> If G has crossing sequence $(N, N-1, 0)$, then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Our main theorem is an analogue of Theorem 1.2 for the orientable case, and its special case $a = N$, $b = N - 1$ resolves Salazar's question [4].

Theorem 1.3 *If a and b are integers with $a > b > 0$, then there exists a graph G whose orientable crossing sequence is $(a, b, 0)$.*

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence $\mathbf{a} = (a_0, a_1, \ldots, a_g)$ and define the sequence $(d_1, \ldots, d_g)$ by the rule $d_i = a_{i-1} - a_i$. If $\mathbf{a}$ is the crossing sequence of a graph, then, roughly speaking, d_i is the number of crossings which can be saved by adding the i^{th} handle. It seems intuitively clear that sequences for which $d_1 \geq d_2 \geq \cdots \geq d_g$ should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [5] constructed a graph with crossing sequence $\mathbf{a}$ whenever $d_1 \geq d_2 \geq \cdots \geq d_g$.

Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence $(a, b, 0)$ where a/b is arbitrarily close to 1. The most extreme examples are due to Archdeacon, Bonnington and Širáň [1] and have a/b approximately equal to 6/5. Although our main theorem gives us a graph with every possible crossing sequence of the form $(a, b, 0)$, we don't know what happens for longer sequences. In particular, it would be nice to resolve the following problem which asks for graphs where the first s handles save only an epsilon fraction of what is saved by the $s + 1^{st}$ handle.

Problem 1.4 *For every positive integer s and every $\varepsilon > 0$, construct a graph G for which $cr_0(G) - cr_s(G) \leq \varepsilon\,(cr_s(G) - cr_{s+1}(G))$.*

For graph embeddings, the genus of a disconnected graph is the sum of the genera of its connected components. For drawing, this situation is presently unclear. If we have a graph which is a disjoint union of G_1 and

G_2, then we can always "use part of the surface for G_1 and the other part for G_2", leading to

$$cr_i(G_1 \cup G_2) \le \min_j \bigl(cr_j(G_1) + cr_{i-j}(G_2)\bigr) .$$

To the best of our knowledge, this inequality might always be an equality. More generally we shall pose the following problem.

Problem 1.5 *Let G be a disjoint union of the graphs G_1 and G_2, and let $\mathcal{S}$ be a (possibly nonorientable) surface. Is there an optimal drawing of G on $\mathcal{S}$, such that no edge of G_1 crosses an edge of G_2?*

This problem is trivially true when $\mathcal{S}$ is the plane, but it also holds when $\mathcal{S}$ is the projective plane:

Proposition 1.6 *Let G be a disjoint union of the graphs G_1 and G_2. Then*

$$\tilde{cr}_1(G) = \min\{\tilde{cr}_1(G_1) + cr_0(G_2), cr_0(G_1) + \tilde{cr}_1(G_2)\} .$$

In other words, there is an optimal drawing of G where planar drawing of G_2 is put into one of the regions defined by the drawing of G_1; or vice versa.

Proof: To see this, consider an optimal drawing of G on the projective plane, and suppose (for a contradiction) that some edge of G_1 crosses an edge of G_2. If there is a crossing involving two edges in G_1, then by creating a new vertex at this crossing point, we obtain an optimal drawing of this new graph. Continuing in this manner, we may assume that both G_1 and G_2 are individually embedded in the projective plane. For $i = 1, 2$, let a_i be the length of a shortest noncontractible cycle in the dual graph of the embedding of G_i. Note that $a_i \ge 2$ as otherwise G_i embeds in the plane, so G embeds in the projective plane. Assume (without loss) that $a_1 \le a_2$. Now, it follows from a theorem of Lins [2] that there exists a half-integral packing of noncontractible cycles in G_i with total weight a_i for $i = 1, 2$. Since any two noncontractible curves in the projective plane meet, it follows that the total number of crossings in this drawing is at least $a_1 a_2$. However, we can draw G in the projective plane by embedding G_2 and then drawing G_1 in a face of this embedding with a total of $\binom{a_1}{2} = \frac{1}{2}a_1(a_1 - 1) < a_1 a_2$ crossings, a contradiction. $\square$

Our primary family of graphs used in proving Theorem 1.3 can be constructed with relatively little machinery, so we shall introduce them here.

We will however use a couple of gadgets which are common in the study of crossing numbers. Let us pause here to define them precisely. A *special graph* is a graph G together with a distinguished subset $T \subseteq E(G)$ of *thick* edges, a subset $U \subseteq V(G)$ of *rigid* vertices and a family $\{\pi_u\}_{u \in U}$ of prescribed *local rotations* for the rigid vertices. Here, π_u describes the cyclic ordering of the ends of edges incident with u. A *drawing* of a special graph G in a surface Σ is a drawing of the underlying graph G with the added property that for every $u \in U$, the local rotation of the edges incident with u given by this drawing either in the local clockwise or counterclockwise order matches π_u. The *crossing number* of a drawing of the special graph G is ∞ if there is an edge in T which contains a crossing, and otherwise it is the same as the crossing number of the drawing of the underlying graph. We define the *crossing number* of a special graph G in a surface Σ to be the minimum crossing number of a drawing of G in Σ, and $cr_i(G)$ to be the crossing number of G in a surface of genus i. In the next section, we shall prove the following result.

Lemma 1.7 *If G is a special graph with crossing sequence* **a** *consisting of real numbers, then there exists an (ordinary) simple graph with crossing sequence* **a**.

This result permits us to use special graphs in our constructions. Indeed, starting in the third section, we shall consider special graphs on par with ordinary ones, and we shall drop the term special. When defining a (special) graph with a diagram, we shall use the convention that thick edges are drawn thicker, and vertices which are marked with a box instead of a circle have the distinguished rotation scheme as given by the figure. With this terminology, we can now introduce our principal family of graphs.

The n^{th} *hamburger graph* H_n is a special graph with $3n+8$ vertices. Its thick edges form a cycle $C = qv_1 \dots v_n rr's'su_n \dots u_1 tt'q'q$ of length $2n+8$ together with two additional thick edges $\tau_0 = qr$ and $\tau_1 = st$. See Figure 1. In addition to these, H_n has n special vertices u'_i (for odd values of i) and v'_i (for even values of i) with rotation as shown in the figure. These vertices are of degree 4 and they lie on paths $r_1 = q'v'_2v'_4 \dots v'_m r'$ (where $m = n$ if n is even and $m = n-1$ otherwise) and $r_2 = t'u'_1u'_3 \dots u'_l s'$ (where $l = n$ if n is odd and $m = n-1$ otherwise). These two paths will be referred to as the *rows* of H_n. Each u'_i and each v'_i is adjacent to u_i and v_i, and the 2-path $c_i = u_iu'_iv_i$ (or $c_i = u_iv'_iv_i$, depending on the parity of i) is called a *column* of H_n, $i = 1, \dots, n$.

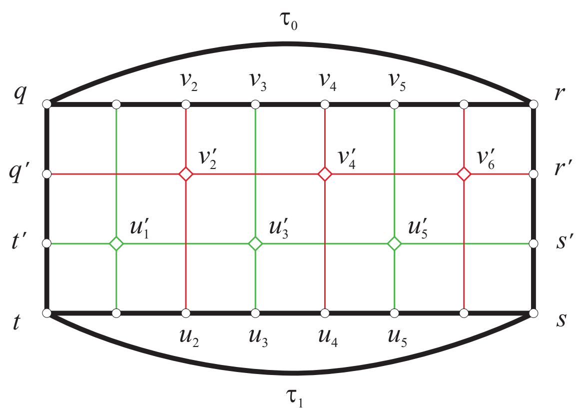

Figure 1: The graph H_n (for $n = 6$).

We claim that the hamburger graph H_n has crossing sequence $(n, n-1, 0)$ whenever $n \geq 5$ (or $n = 3$). Although this does not handle all possible sequences of the form $(a, b, 0)$, as discussed above, these are in some sense the most difficult and counterintuitive cases. Indeed, a rather trivial modification of these will be used to get all possible sequences.

Since it is quite easy to sketch proofs of $cr_0(H_n) = n$ and $cr_2(H_n) = 0$, let us pause to do so here (rigorous arguments will be given later). The first of these equalities follows from the observation that every row must meet every column in any planar drawing in which thick edges are crossing-free. The second equality follows from the observation that H_n minus the thick edges τ_0, τ_1 is a graph which can be embedded in the sphere. Using an extra handle for each of τ_0, τ_1 gives an embedding of the whole graph in a surface of genus 2. Of course, it is possible to draw H_n in the torus with only $n - 1$ crossings by starting with the drawing in the figure and then adding a handle to remove one crossing. In the third section we shall show that these are indeed optimal drawings (for $n = 3$ and $n \geq 5$).

2 Gadgets

The goal of this section is to establish Lemma 1.7 which permits us to use special graphs in our constructions. The first part (dealing with thick edges) appears in [1], we include it for readers convenience.

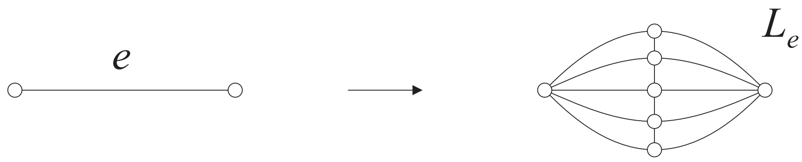

Figure 2: Putting weights on the edges (here $w(e) = 5$).

Thick edges

For every $e \in E(G)$ choose positive integer $w(e)$ and replace e by a copy of $L_{w(e)}$ whenever $w(e) > 1$. Let G' be the resulting graph. We claim, that the crossing number of G' is the same as the "weighted crossing number" of G: each crossing of edges e_1, e_2 is counted $w(e_1)w(e_2)$-times. Obviously, $cr(G')$ is at most that, as we can draw each L_e sufficiently close to where e was drawn. Moreover, there is an optimal drawing of this form (which proves the converse inequality): Given an optimal embedding of G', consider the subgraph L_e and from the $w(e)$ paths of length 2 between its "end-points" pick the one, that is crossed the least number of times. We can draw the whole subgraph L_e close to this path without increasing the number of crossings.

This shows that we can "simulate weighted crossing number" by crossing number of a modified graph. In particular, we can let $w(e) = 1$ for each ordinary edge and $w(e) > cr(G)$ for each thick edge e of G. This proves Lemma 1.7 for graphs with thick edges.

Rigid vertices

Suppose that we are considering drawings in surfaces of Euler genus $\leq g$; put $n = 3g+2$. Let G be a special graph with rigid vertices. We replace each rigid vertex v by a copy of $V_{n,\deg(v)}$. That is, we add n nested thick cycles of length $d = \deg(v)$ around v as shown in Figure 3 for $d = 6$ and $n = 5$. When doing this, the cycles meet the edges incident with v in the same order as requested by the local rotation π_v around v. If an edge incident with v is thick, then all edges in G' arising from it are thick too (as indicated in the figure for one of the edges). Call the resulting graph G'.

We claim that the crossing number of G' (graph with thick edges but no rigid vertices) is the same as that of G. Any drawing of G that respects the rotations at each rigid vertex can be extended to a drawing of G' without any

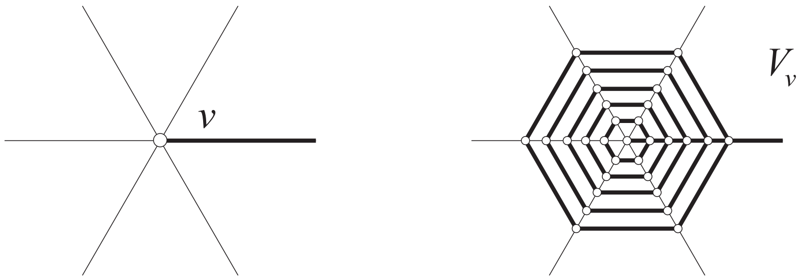

Figure 3: Controlling the prescribed local rotations.

new crossing; in this drawing all n thick cycles in each V_v are contractible and v is contained in the disc that any of them is bounding. We will show, that there is an optimal drawing of G' of this "canonical" type.

Let us consider an optimal drawing (respecting thick edges) of G' in S (of genus $\leq g$). Let v be a rigid vertex of G, and consider the inner $n-1$ out of the n thick cycles in V_v. No edge of these cycles is crossed; so by [3, Proposition 4.2.6], either one of these cycles is contractible in S, or two of them are homotopic.

Suppose first, that one of the cycles, Q, is contractible. Since Q separates the graph into two connected components, either the disk D bounded by Q or its exterior contains no vertex or edge of G' apart from some cycles and edges of V_v. Let us assume that this is the interior of D. Now delete the drawing of all thick cycles in V_v except Q, and delete the drawing of all $\deg(v)$ paths from Q to v. Now think of Q as the outermost cycle of V_v and draw the rest on V_v inside D without crossings.

Suppose next, that two of the cycles, Q_1 and Q_2 are homotopic (and that Q_1 is closer to v in G'). We cut S along Q_1, and patch the two holes with a disc. This simplifies the surface, so if we can draw G' on it without new crossings, we get a contradiction. Such drawing of G' indeed exists, as we may delete the drawing of all of V_v that is "inside" Q_1 and draw it in one of the new discs.

By performing such a change to each rigid vertex, we obtain an optimum drawing of G' which is canonical. Consequently, it gives rise to a legitimate drawing of the special graph G, and which is also optimal for G. This shows that Lemma 1.7 holds also when there are special vertices.

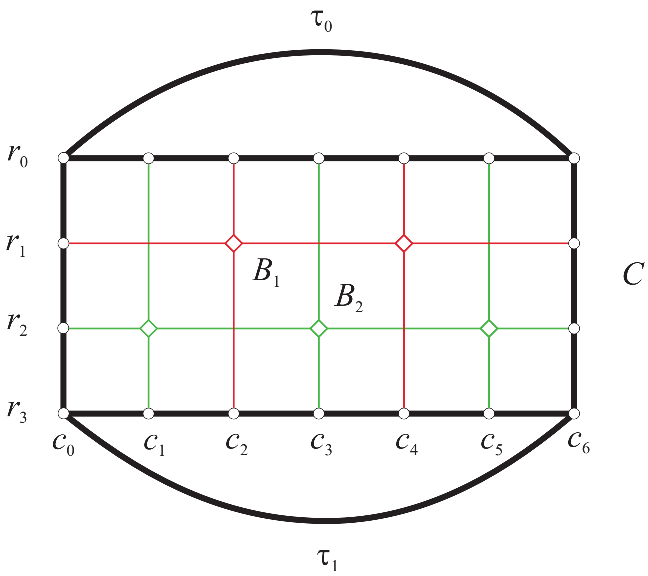

Figure 4: Main constituents of the graph H_n (for $n = 5$).

3 Hamburgers

The goal of this section is to prove Theorem 1.3, showing the existence of a graph with crossing sequence $(a, b, 0)$ for every $a > b > 0$. The hamburger graphs H_n (defined in the introduction) have all of the key features of interest. These are actually special graphs, but thanks to Lemma 1.7 it is enough to consider crossing sequences of special graphs. Indeed, in the remainder of the paper we will omit the term 'special'.

We have redrawn H_n (for $n = 5$) again in Figure 4 where we have given names to numerous subgraphs of it. We have previously defined the rows r_1, r_2 and columns $c_1, \ldots, c_n$. For convenience we add rows r_0 and r_3 and columns c_0 and c_{n+1} (see Figure 4). The cycle C (consisting of c_0, r_0, c_{n+1}, and r_3) has two trivial bridges (the thick edges τ_0 and τ_1) and two other bridges. The first, denoted by B_1, consists of the row r_1 together with all columns c_i with i even (and, of course, $1 \leq i \leq n$). The second one is denoted by B_2 and consists of the row r_2 and columns c_i with i odd (and, again, $1 \leq i \leq n$).

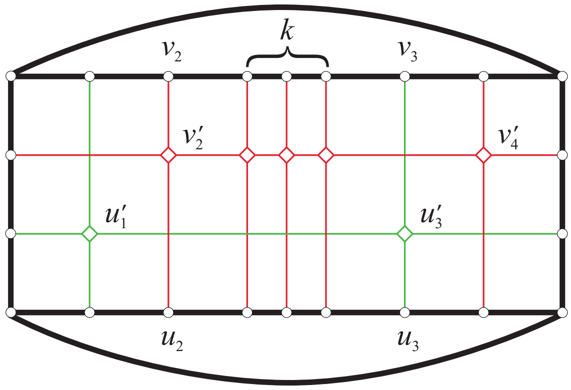

Figure 5: The graph $H_{n,k}$ (for $n = 4$ and $k = 3$).

To get every possible crossing sequence $(a, b, 0)$, we will also require a slightly more general class of graphs. For every $n, k \in \mathbb{N}$ with $n \geq 3$, we define the graph $H_{n,k}$, which is obtained from H_n by adding k duplicates of the second column c_2 as shown in Figure 5 for the case of $n = 4$ and $k = 3$. Note that $H_n \cong H_{n,0}$.

We shall denote by $\mathbb{S}_g$ $(g \geq 0)$ the orientable surface of genus g.

Lemma 3.1 *$cr_2(H_{n,k}) = 0$ for every $n, k \in \mathbb{N}$ with $n \geq 3$.*

Proof: To draw H_n in the double torus $\mathbb{S}_2$, start by embedding $H_n - \tau_0 - \tau_1$ in the sphere $\mathbb{S}_0$. Now, use one handle to route the edge τ_0, and another handle for τ_1. $\square$

Lemma 3.2 *$cr_0(H_{n,k}) = n + k$ for every $n, k \in \mathbb{N}$ with $n \geq 3$.*

Proof: Consider a drawing of $H_{n,k}$ in the sphere. If this drawing has finite crossing number, the cycle C must be embedded as a simple closed curve which separates the surface into two discs D_1, D_2 and is not crossed by any edge. Moreover, both thick edges τ_0 and τ_1 are drawn in the same disc, say D_2. Now every column of B_1 crosses the row r_2 and every column of B_2 crosses the row r_1, so we have at least $n + k$ crossings. Since $H_{n,k}$ is drawn in $\mathbb{S}_0$ with $n + k$ crossings in Figure 5, we conclude that $cr_0(H_{n,k}) = n + k$ as required. $\square$

Not surprisingly, the situation when drawing our graphs H_n on the torus is considerably more complicated to analyze. By drawing H_n in the plane with n crossings and then using a handle to remove one crossing, we see that $cr_1(H_n) \leq n-1$ for all $n \geq 3$ (even $cr_1(H_{n,k}) \leq n-1$ for all $n \geq 3$ and $k \geq 0$). For $n \geq 5$, we shall prove that this is the best which can be achieved. For $n \leq 4$, however, there is some exceptional behavior (cf. Lemma 3.7).

Lemma 3.3 *For every optimal drawing of H_n (in some surface), each column c_i $(1 \leq i \leq n)$ is a simple curve.*

Proof: It is easy to see that in every optimal drawing, every edge is represented by a simple curve. Let us now consider a column $c_i = v_i v_i' u_i$ (or similarly for $v_i u_i' u_i$) and suppose that the edges $e = v_i v_i'$ and $f = u_i v_i'$ cross. Suppose that e is represented by the simple curve $\alpha(t)$, $0 \leq t \leq 1$, where $\alpha(0) = v_i$ and $\alpha(1) = v_i'$. Similarly, let f be represented by the simple curve $\beta(t)$, $0 \leq t \leq 1$, where $\beta(0) = u_i$ and $\beta(1) = v_i'$. Let $\alpha(t') = \beta(t')$ $(0 < t' < 1)$ be where they cross. Now let $\tilde{\alpha}(t) = \alpha(t)$ for $t \leq t'$ and $\tilde{\alpha}(t) = \beta(t)$ for $t \geq t'$. Change similarly β to $\tilde{\beta}$. Then the crossing becomes a touching of the two curves, which can be eliminated yielding a drawing with fewer crossings. Observe that the local rotation at the special vertex v_i' changes from clockwise to anticlockwise but this is still consistent with the requirement for this special vertex. Therefore the new drawing contradicts the optimality of the original one. □

At several occasions in the proof we will use the following well-known fact about closed curves on the torus.

Lemma 3.4 ([3, Proposition 4.2.6]) *Let φ, ψ be two simple closed non-contractible curves on the torus that are not freely homotopic. Then φ and ψ cross each other.*

The following is well-known (cf., e.g., [6]).

Lemma 3.5 *Let φ, ψ be two closed curves on some surface, assume ψ is contractible. The curves may intersect themselves and each other, but we assume that*

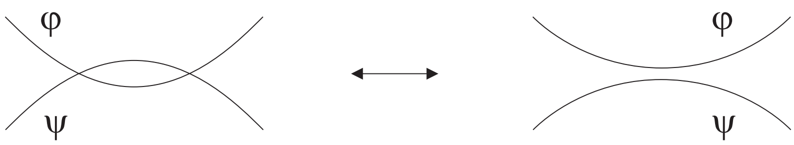

Figure 6: Illustration for the proof of Lemma 3.5.

1. *the total number of intersections is finite, and*
2. *each point of intersection is a crossing (the curves do not touch and there are no more than two arcs that run through the point).*

Then, the number of intersections of φ with ψ is even.

Proof (hint): Let us transform ψ continuously to a trivial curve. The number of intersections of φ with ψ stays the same, or changes by 2 when we modify ψ as in Figure 6.

It will be convenient for us to classify different types of drawings of H_n in the torus depending on the drawing of the thick subgraph $C + \tau_0 + \tau_1$. In Figure 7 we have listed nine possible embeddings of $C + \tau_0 + \tau_1$ in $\mathbb{S}_1$, where τ_0 and τ_1 are drawn with dashed lines. We shall say that a drawing of H_n is of *type* A, B, C, C', D, E, E', E'', or E''' if the induced drawing of $C + \tau_0 + \tau_1$ is as in the corresponding part of Figure 7. Although there are other possible drawings of $C + \tau_0 + \tau_1$ in the torus, our next lemma shows that the only ones which extend to finite crossing number drawings of H_n have one of these types.

Lemma 3.6 *Every drawing of H_n for $n \geq 3$ on a torus $\mathcal{S}$ with crossing number less than n has type A, B, C, C', D, E, E', E'', or E'''.*

Proof: Let $\mathcal{S}'$ be the bordered surface obtained from $\mathcal{S}$ by cutting along the cycle C. First suppose that C is contractible. Then $\mathcal{S}'$ is disconnected, with one component a disc D, and the other component $\mathcal{S}''$ homeomorphic to $\mathbb{S}_1$ minus a disc. If both B_1 and B_2 are drawn in D, then we have at least n crossings (as in Lemma 3.2). If only one of B_1 or B_2, say B_1 is drawn in D, then B_2 and the edges τ_0 and τ_1 are drawn in $\mathcal{S}''$ (else the crossing number is infinite). Consider the curves $\tau_0 \cup r_0$ and $\tau_1 \cup r_3$ in $\mathcal{S}''$. If either of these is contractible, then B_2 must cross it (yielding infinite crossing number). Otherwise (using the Fact stated before this lemma) they must be freely homotopic noncontractible curves in $\mathcal{S}''$, so $\tau_0 \cup c_0 \cup \tau_1 \cup c_{n+1}$ is a contractible curve. Therefore B_2 must cross it, yielding again infinitely many crossings. Thus, we may assume that both τ_0 and τ_1 are drawn in the disc D and B_1 and B_2 are drawn in $\mathcal{S}''$ so our drawing is of type A.

Next suppose that C is not contractible. In this case, the surface $\mathcal{S}'$ is a cylinder bounded by two copies of the cycle C. If both τ_0 and τ_1 have all of their ends on the same copy of C, we must have a drawing of type B, C, or C'. If one has both ends on one copy of C, and the other has both ends on the other copy of C, then there are infinitely many crossings, unless the drawing is of type D. Finally, if one of τ_0, τ_1, has its ends on distinct copies of C, then the crossing number will be infinite unless the other one of τ_0, τ_1, has both ends on the same copy of C giving us a drawing of type E, E', E'', or E'''. $\square$

If G is a graph drawn on a surface and $A, B \subseteq G$, then we shall denote by $Cr(A \mid B)$ the total number of crossings of an edge from A with an edge from B, where crossings of an edge $e \in E(A \cap B)$ with another edge $f \in E(A \cap B)$ are counted only once. In particular, the total number of crossings of graph G is equal to $Cr(G \mid G)$.

Lemma 3.7 *$cr_1(H_n) = n - 1$ if $n = 3$ or $n \geq 5$, while $cr_1(H_4) = 2$. Furthermore, Figure 8(a)–(c') shows the only drawings of H_3 in the torus with two crossings and the added property that $Cr(r_2|G) = 0$. Figure 9 displays the unique drawing of H_4 in the torus with two crossings.*

Proof: We proceed by induction on n. Consider a drawing $\mathcal{D}$ of H_n in a surface $\mathcal{S}$ homeomorphic to the torus, such that $\mathcal{D}$ yields minimum crossing number. We shall frequently use the inductive assumption for $n-1$ and $n-2$, since by deleting the edges of the column c_1, the column c_n, or two consecutive columns c_i and c_{i+1} we obtain a new graph which is a subdivision of H_{n-1} or H_{n-2} (assuming $n \geq 3$). This technique will be used throughout the proof. It is also worth noting that after applying this operation to $\mathcal{D}$, the drawing of the smaller hamburger graph is of the same type as the drawing $\mathcal{D}$.

The cycle C is not crossed in $\mathcal{D}$, so we may cut our surface along this curve. This leaves us with a drawing of H_n in a closed bordered surface—which we shall denote $\mathcal{S}'$—where each edge of C appears twice on the boundary. We shall use C^1 and C^2 to denote these copies.

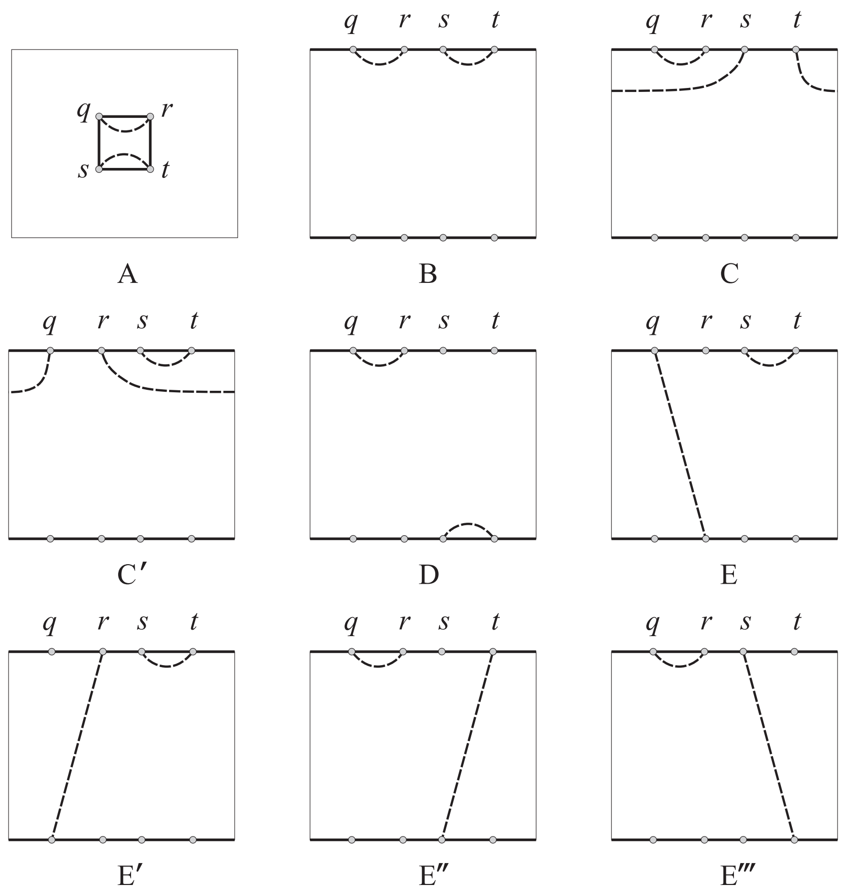

Figure 7: Nine special types of embedding of the thick subgraph $C + \tau_0 + \tau_1$ in the torus. In types B–E''', the cycle C is drawn on the top and bottom sides of the square.

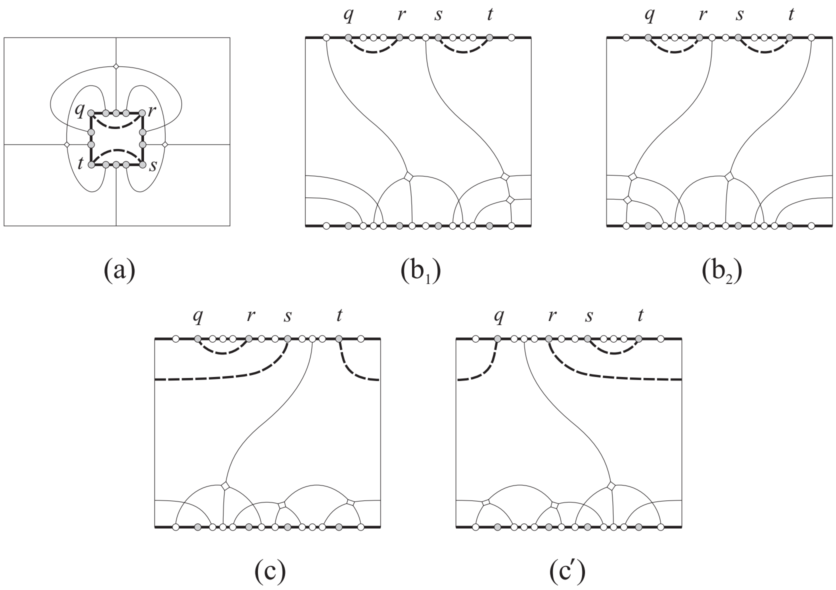

Figure 8: Exceptional drawings of H_3.

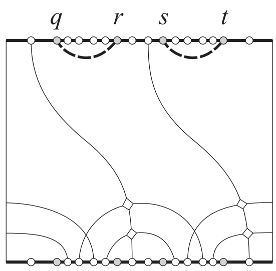

Figure 9: Exceptional type B drawing of H_4.

Essential to our proof is an analysis of the homotopy behavior of the rows and columns. To make this precise, let us now choose a point N in the interior of the row r_0, S in the interior of r_3, W in the interior of c_0 and E in the interior of c_{n+1}. (Actually, for each of these points we have two copies: N^1 and N^2, etc. But we will avoid distinguishing these if there is no danger of confusion). For each column c_i $(0 \le i \le n+1)$ let c_i^+ be a simple curve in $\mathcal{S}'$ obtained by extending c_i along the appropriate copies of the rows r_0 and r_3 so that it has ends N and S. Similarly, for each row r_i $(0 \le i \le 3)$ let r_i^+ be a curve in $\mathcal{S}'$ obtained by extending r_i along the appropriate copies of the columns c_0 and c_{n+1} so that it has ends E and W. We shall focus our attention on the homotopy types in $\mathcal{S}'$ of the curves c_i^+ where N and S are the fixed end points (and similarly r_i^+ where E and W are fixed): we say that c_i^+ and c_j^+ are *homotopic* if c_i^+ may be continuously deformed to c_j^+ in the surface $\mathcal{S}'$, while keeping their endpoints fixed. Note that c_i^+ and c_j^+ can only be homotopic if c_i and c_j are connecting the same copies of N and S—that is they attach on the same side of C in the original surface $\mathcal{S}$. Also note, that for $i = 0$ or $i = n+1$ we actually have two copies of c_i, so we should be speaking of, e.g., c_0^{+1} and c_0^{+2}. We will refrain from this distinction whenever possible to keep the notation clearer—so when saying c_0^+ and c_1^+ are homotopic we will actually mean that c_1^+ is homotopic to c_0^{+s} for some $s \in \{1, 2\}$.

We will use frequently the following fact that connects the homotopy types of columns and their crossing behaviour with respect to the rows (and vice versa). We will refer to this statement as to "the Claim".

> **Claim:** If c_i^+ and c_{i+1}^+ are homotopic $(1 \le i < n)$, then $Cr(r_j \mid c_i \cup c_{i+1}) \ge 1$ for $j = 1, 2$. Similarly, if r_1^+ and r_2^+ are homotopic, then $Cr(r_1 \cup r_2 \mid c_i) \ge 1$ for every $1 \le i \le n$.

To see this, let us observe that the closed curve obtained by following c_i^+ from S to N and then c_{i+1}^+ from N to S is contractible, after deleting part of its intersection with the cycle C, we get a contractible curve ψ that intersects itself only at finitely many points. The row r_j must cross either c_i^+ or c_{i+1}^+ (depending on the parity) in their common vertex (it cannot only touch it as their common vertex has prescribed local rotation). We may extend r_j^+ into a closed curve φ by following closely along the cycle C. This way we are adding two (or zero) intersections with ψ. By Lemma 3.5

curves φ and ψ have an even number of intersection, thus r_j must have another crossing with ψ and we are done. The same argument holds when the rows and columns exchange their roles.

Corollary: If r_1^+ and r_2^+ are homotopic, we are done, as there are at least n intersections.

In light of Lemma 3.6 we may assume that our drawing is of type A, B, C, C', D, E, E', E'', or E''', and we now split our argument into these nine cases.

Case 1: Type A.

Let us first suppose that $n \geq 4$. If there exists $1 \leq i \leq n$ so that c_i^+ is homotopic to c_0^+, then either c_1 crosses c_i, or c_1^+ is homotopic to c_0^+. In the latter case, c_1 crosses r_1. So, in short, $Cr(c_1 \mid H_n) \geq 1$ and by removing this column and applying induction, we deduce that there are at least $n-1$ crossings in our drawing. Note here that the resulting drawing of H_{n-1} is still of type A, so it must have at least $(n-1)-1$ crossings, even if $n = 5$. Thus, we may assume that c_i^+ is not homotopic to c_0^+ for any $1 \leq i \leq n$. By a similar argument, c_i^+ is not homotopic to c_{n+1}^+. If there exist $i, j \in \{1, \ldots, n\}$ with c_i^+ not homotopic to c_j^+, then c_i^+ and c_j^+ cross (Lemma 3.4), and further, $Cr(c_k \mid c_i \cup c_j) \geq 1$ for every $k \in \{1, \ldots, n\}$ with $k \neq i, j$. This implies that we have at least $n-1$ crossings, as desired. The only other possibility is that c_i^+ and c_j^+ are homotopic for every $i, j \in \{1, \ldots, n\}$. In this case, it follows from the Claim (applied to c_1^+ and c_2^+, c_3^+ and c_4^+, $\ldots$) that there are at least $n-1$ crossings.

Suppose now that $n = 3$. If c_2^+ is homotopic to c_1^+ or c_3^+, then it follows from the Claim that each row has at least one crossing, and we are done. Thus, we may assume that c_2^+ has distinct homotopy type from that of c_1^+ and from that of c_3^+. If c_2^+ is homotopic to c_0^+, then $Cr(c_2 \mid r_2) \geq 1$ and $Cr(c_2 \mid c_1) \geq 2$ (since c_1^+ is not homotopic to c_2^+) giving us too many crossings. Thus, c_2^+ is not homotopic to c_0^+, and by a similar argument, we find that c_2^+ is not homotopic to c_4^+. Now, either c_1^+ is homotopic to c_0^+ (in which case $Cr(c_1 \mid r_1) \geq 1$) or c_1^+ is not homotopic to c_0^+ (in which case $Cr(c_1 \mid c_2) \geq 1$). So, in short $Cr(c_1 \mid r_1 \cup c_2) \geq 1$. By a similar argument, $Cr(c_3 \mid r_1 \cup c_2) \geq 1$. Since there are at most two crossings, we must have $Cr(c_1 \cup c_3 \mid r_1 \cup c_2) = 2$ and this accounts for all of our crossings. In particular, this implies that r_1 and r_2 are simple curves. Since

$Cr(r_2 \mid G) = 0$, it follows that r_2^+ is not homotopic to r_0^+ or r_3^+. By the Claim, r_1^+ is not homotopic to r_2^+, and this together with $Cr(r_1 \mid r_2) = 0$ implies that r_1^+ is homotopic to r_0^+. It follows from this that $Cr(r_1 \mid c_i) = 1$ for $i = 1, 3$ and this accounts for all of the crossings. Such a drawing is possible, but must be equivalent with that in Figure 8(a).

In all the remaining cases, we have that $\mathcal{S}'$ is a cylinder, and in our figures we have drawn $\mathcal{S}'$ with the boundary component C^1 on the top and C^2 on the bottom.

Case 2: Type B.

Here all of the column curves c_i^+ have ends N^2 and S^2. Recall that these are copies of N and S drawn at the "bottom copy" C^2 of C. Since all of these curves are simple, it follows that for every $1 \leq i \leq n$, the curve c_i^+ is either homotopic to the simple curve N^2–W^2–S^2 in C^2 (we shall call this homotopy type ℓ), or to the simple curve N^2–E^2–S^2 in C^2 (homotopy type r). Let $\mathbf{a} = a_1 a_2 \ldots a_n$ be the word given by the rule that a_i is the homotopy type of c_i^+. We now have the following simple crossing property.

P1. If $a_i = r$ and $a_j = \ell$ where $1 \leq i < j \leq n$, then $Cr(c_i \mid c_j) \geq 2$.

If there exists an i ($1 \leq i \leq n$) so that $Cr(c_i \mid H_n) \geq 4$, then $n \geq 5$ (otherwise the drawing is not optimal), and by removing c_i and either c_{i-1} or c_{i+1} and applying the theorem inductively to the resulting graph, we deduce that there are at least $4 + cr_1(H_{n-2}) \geq n$ crossings in our drawing, a contradiction. It follows from this and P1, that either $\mathbf{a} = \ell^i r^{n-i}$ or $\mathbf{a} = \ell^i r \ell r^{n-i-2}$. We now split into subcases depending on n.

Suppose first that $n = 3$. If $a_1 = a_2 = \ell$ or $a_2 = a_3 = r$, then it follows from the Claim that $Cr(r_j \mid c_1 \cup c_2 \cup c_3) \geq 1$ for $j = 1, 2$ and we are finished. Otherwise, $\mathbf{a}$ must be $\ell r \ell$ or $r \ell r$ and $Cr(c_2 \mid c_1 \cup c_3) \geq 2$. These configurations are possible, but require that our drawing is equivalent with the one in Figure 8(b)—this comes from $\mathbf{a} = \ell r \ell$, if $\mathbf{a} = r \ell r$ we get a mirror image.

Next we consider the case when $n = 4$ and $\mathbf{a} = \ell^i r^{4-i}$. Applying the Claim for the columns c_1, c_2 and c_3, c_4 resolves the cases when $\mathbf{a}$ is one of ℓ^4, r^4, or $\ell^2 r^2$ (each gives at least four crossings—a contradiction). Suppose that $\mathbf{a} = \ell^3 r$ (or, with the same argument, $\mathbf{a} = \ell r^3$). It follows from the Claim that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$ and $Cr(c_2 \cup c_3 \mid r_1 \cup r_2) \geq 2$, so the only possibility for fewer than three crossings is that our drawing has 2 crossings, both of which are between c_2 and the rows r_1 and r_2. But then c_2 does not cross c_1 or c_3, so c_2 is separated from c_0 by $c_1^+ \cup c_3^+$, so $Cr(r_1 \mid c_1 \cup c_3) > 0$, a contradiction.

Next suppose that $n = 4$ and $\mathbf{a} = \ell^i r \ell r^{2-i}$. If $\mathbf{a} = \ell^2 r \ell$, then it follows from P1 that $Cr(c_3 \mid c_4) \geq 2$ and from the Claim that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$, so we have at least four crossings—a contradiction. Similarly $\mathbf{a} = r \ell r^2$ is impossible. The only remaining possibility is $\mathbf{a} = \ell r \ell r$. In this case, we have $Cr(c_2 \mid c_3) \geq 2$, so the only possibility is that there are exactly two crossings, both between c_2 and c_3. This case can be realized, but requires that our drawing is equivalent to that of Figure 9.

Lastly, suppose that $n \geq 5$. Since $\mathbf{a} \in \{\ell^i r^{n-i}, \ell^i r \ell r^{n-i-2}\}$, either $a_1 = a_2 = \ell$ or $a_{n-1} = a_n = r$. As these arguments are similar, we shall consider only the former case. Now, it follows from the Claim that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$, so removing the first two columns gives us a drawing of H_{n-2} with at least two crossings less than in our present drawing of H_n. By applying our theorem inductively to this new drawing, we find that the only possibility for less than $n-1$ crossings is that $n = 6$ and $\mathbf{a} = \ell^3 r \ell r$. In this case, we have $Cr(c_4 \mid c_5) \geq 2$, so we may eliminate two crossings by removing columns 4 and 5. This leaves us with a drawing of a graph isomorphic to H_4 as above with the pattern $\ell^3 r$. It follows from our earlier analysis, that this drawing has at least three crossings. This completes the proof of this case.

Case 3: Type C.

Now each column curve has one end on the segment of C^2 between q^2 and r^2. As above, every curve c_i^+ with both ends on C^2 must be homotopic with either the simple curve N^2–W^2–S^2 in C^2 (denoted by ℓ), or with the simple curve N^2–E^2–S^2 in C^2 (homotopy type r). Each row has both its ends on C^2.

The homotopy types of the other column curves will be represented by integers. Since $\mathcal{S}'$ is a cylinder, we may choose a continuous deformation Ψ of $\mathcal{S}'$ onto the circle $\mathbb{S}^1$ with the property that C^1 and C^2 map bijectively to $\mathbb{S}^1$, and N^2 and S^1 map to the same point $x \in \mathbb{S}^1$. Now, each curve c_i^+ maps to a closed curve in $\mathbb{S}^1$ from x to x, and for an integer $\alpha \in \mathbb{Z}$, we say that c_i^+ has homotopy type α if the corresponding curve in $\mathbb{S}^1$ has (counterclockwise) winding number α. It follows that c_i^+ and c_j^+ are homotopic if and only if they have the same homotopy type. As before, we let $\mathbf{a} = a_1 a_2 \ldots a_n$ be the word given by the rule that a_i is the homotopy type of c_i^+. We now have the following crossing properties (for the appropriate choice of "clockwise" direction), whenever $1 \le i < j \le n$:

P1. $Cr(c_i \mid c_j) \ge |a_i - a_j - 1|$ if $a_i, a_j \in \mathbb{Z}$.

P2. $Cr(c_i \mid c_j) \ge 2$ if $a_i = r$ and $a_j = \ell$.

P3. $Cr(c_i \mid c_j) \ge 1$ if either $a_i = r$ and $a_j \in \mathbb{Z}$ or $a_i \in \mathbb{Z}$ and $a_j = \ell$.

By choosing Ψ appropriately, we may further assume that the smallest integer $1 \le i \le n$ for which $a_i \in \mathbb{Z}$ (if such i exists) satisfies $a_i = 0$. Again, we split into subcases depending on n.

Suppose first that $n = 3$. Note that every column of type r or ℓ separates the segment q^2t^2 on C^2 from r^2s^2. Consequently, $Cr(r_1 \cup r_2 \mid c_i) \ge 1$ whenever $a_i \in \{\ell, r\}$. Next we shall consider the homotopy types of our rows. If r_1^+ is not homotopic to r_0^+ or r_3^+, then $Cr(r_1 \mid r_1) \ge 1$ and further $Cr(r_1 \mid c_1 \cup c_3) \ge 2$ (as in this case, r_1 separates C^2 from C^1 and also segment q^2r^2 from s^2t^2) which gives us too many crossings. If r_2^+ is not homotopic to r_0^+ or r_3^+, then $Cr(r_2 \mid r_2) \ge 1$ and $Cr(r_2 \mid c_2) \ge 1$, and we have nothing left to prove. Thus, we may assume that r_1^+ (and also r_2^+) is homotopic to one of r_0^+, r_3^+. If r_1^+ and r_2^+ are homotopic, then the Claim implies that there are at least three crossings. Hence, we may assume that r_1^+ is homotopic to r_0^+ and r_2^+ to r_3^+ (the other possibility yields two crossings and each row crossed). It now follows from our assumptions that $Cr(r_1 \mid c_i) \ge 1$ for $i = 1, 3$, so assuming we have at most two crossings, our only crossings are between r_1 and c_1 and between r_1 and c_3. If $a_i \in \mathbb{Z}$ for $i \in \{1, 3\}$, then c_i also crosses r_2 because of the requirements concerning local rotations at the special vertices u_1' and u_3'. It follows that there are at least three crossings unless $\mathbf{a} = \ell 0 \ell$, $\ell 0 r$, $r 0 \ell$, or $r 0 r$. Each of these, except $\ell 0 r$ gives at least three crossings by (P3). The remaining case is possible, but only as it appears in Figure 8(c).

Suppose now that $n \ge 4$. If either c_1 or c_n is crossed, then we delete it and use the induction hypothesis. If neither has a crossing, then both a_1 and a_n are integers (otherwise $Cr(c_1 \cup c_n \mid r_1 \cup r_2) \ge 1$ as above). It follows that $a_1 = 0$, and $a_n = -1$ (otherwise c_1 and c_n cross). Now there is no value for a_2 to avoid crossing with either c_1 or c_n. Hence one of c_1, and c_n is crossed, after all, and we may use induction. This completes the proof of Case 3.

Case 4: Type C'.

This case is nearly identical to the previous one. We may define the homotopy types for the columns to be r, ℓ, or an integer, exactly as before, so that the same homotopy properties are satisfied. Then the analysis for $n \geq 4$ is identical, and the only difference is the case when $n = 3$. As before, if r_1^+ is not homotopic to r_0^+ or r_3^+, then $Cr(r_1 \mid r_1) \geq 1$ and $Cr(r_1 \mid c_1 \cup c_3) \geq 2$ giving us too many crossings. Similarly, if r_2^+ is not homotopic to r_0^+ or r_3^+, then $Cr(r_2 \mid r_2) \geq 1$ and $Cr(r_2 \mid c_2) \geq 1$ and there is nothing left to prove. Now, using the Claim, we deduce that r_1^+ is homotopic to r_0^+ and r_2^+ is homotopic to r_3^+. It follows from this that $Cr(c_2 \mid r_2) \geq 1$. If $a_2 \in \mathbb{Z}$ then, as the vertex v_2' is rigid, it follows that $Cr(c_2 \mid r_1) \geq 1$ and we have nothing left to prove. Thus, we may assume that $a_2 \in \{\ell, r\}$. If $a_i \in \{\ell, r\}$ for $i = 1$ or $i = 3$, then c_i crosses r_1 and we are done. Thus, we may assume that $a_1, a_3 \in \mathbb{Z}$. It now follows that $Cr(c_2 \mid c_1 \cup c_3) \geq 1$. This can be realized with exactly two crossings, but row r_2 must be crossed.

Case 5: Type D.

In this case, every column has one end on r_0^2 and one end on r_3^1. We define the homotopy types of curves c_i^+ using integers as in the previous case. Again, c_i^+ and c_j^+ are homotopic if and only if they have the same homotopy type. As before, we let $\mathbf{a} = a_1 a_2 \dots a_n$ be the word given by the rule that a_i is the homotopy type of c_i^+. And as before, we have the following useful crossing property:

P1. $Cr(c_i \mid c_j) \geq |a_i - a_j - 1|$ if $1 \leq i < j \leq n$.

Suppose first that $n \geq 4$. If the first column c_1 does not cross any other columns, then $\mathbf{a} = 0(-1)^{n-1}$. Similarly, if the last column does not cross any other columns, then $\mathbf{a} = 0^{n-1}(-1)$. Since these cases are mutually exclusive for $n \geq 4$, either the first, or the last column contains a crossing. Then we may remove it and apply induction.

If $n = 3$, we proceed as follows. Using P1 (and the convention $a_1 = 0$) we get that the number of crossings between the columns is at least $|a_2 + 1| + |a_3 + 1| + |a_2 - a_3 - 1| \geq |a_2 + 1| + |a_2|$ (using the triangle inequality). Symmetrically, we get another lower bound for the number of crossings: $|a_3 + 1| + |a_3 + 2|$. If any of these bounds is at least 3, we are done. It follows that $a_2 \in \{0, -1\}$ and $a_3 \in \{-1, -2\}$. Now, if there are two consecutive columns with the same homotopy type, then each row will cross some of these columns, and we are done. Consequently $\mathbf{a} = 0, -1, -2$. It follows that $Cr(c_1 \mid c_3) \geq 1$. If c_2 crossed either c_1 or c_3, then it would

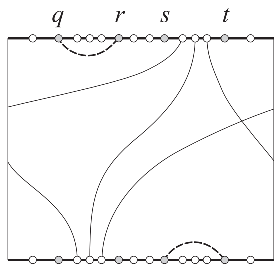

Figure 10: Part of a type D drawing of H_3.

have to cross the column twice—which would yield too many crossings. Similarly, if $Cr(c_1 \mid c_3) > 1$, then $Cr(c_1 \mid c_3) \geq 3$ and we would have too many crossings. It follows that the three columns c_1, c_2, c_3 are drawn as in Figure 10. Now we have that c_1 and c_3 separate c_2 from c_0^1, c_0^2, c_{n+1}^1, and c_{n+1}^2. It follows that $Cr(r_1 \mid c_1 \cup c_3) \geq 2$ giving us too many crossings.

Case 6: Type E.

In this case, every curve c_i^+ must have one end in r_3^2 and the other end in either r_0^1 or r_0^2. In the first case, we say that c_i^+ has homotopy type 0 and in the second we say it has type ℓ. It is immediate that any two such curves of the same type are homotopic. As usual, we let $\mathbf{a} = a_1 a_2 \dots a_n$ be the word given by the rule that a_i is the homotopy type of c_i^+. The following rule indicates some forced crossing behavior.

P1. $Cr(c_i \mid c_j) \geq 1$ if $a_i = 0$ and $1 \leq i < j \leq n$.

Let us first treat the case when $n \geq 4$. If the last column c_n contains at least one crossing, then we may remove it and apply induction. Otherwise, (P1) implies that $\mathbf{a} = \ell^n$ or $\mathbf{a} = \ell^{n-1}0$. It follows from the Claim that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$. Thus, if $n \geq 5$, we may remove the first two columns and apply induction. If $n = 4$ and $\mathbf{a} = \ell^4$, then the Claim gives us at least four crossings—a contradiction with the minimality of our drawing. It remains to check $\mathbf{a} = \ell^3 0$. If there are fewer than three crossings, then (again by applying the Claim twice) there are exactly two, and both occur on c_2. However, in this case $Cr(r_1 \mid c_3) = 0$. As c_3 separates c_2 from both $r^1 s^1$ and $r^2 s^2$ and r_1 has a common vertex with c_2, we get a contradiction.

Finally, suppose that $n = 3$. If there are two consecutive columns with the same homotopy type, then we are finished (by the Claim), so we may

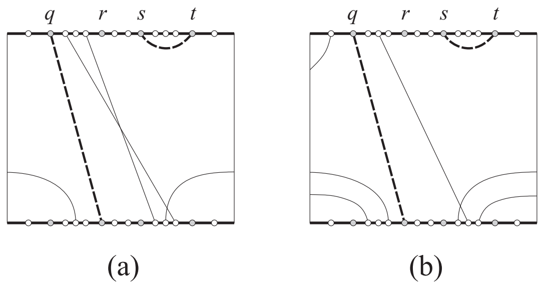

Figure 11: Towards type E drawings of H_3.

assume $\mathbf{a} = 0\ell 0$ or $\mathbf{a} = \ell 0\ell$. In the former case, we have $Cr(c_1 \mid c_2 \cup c_3) \geq 2$, so we may assume that there are exactly two crossings, and the columns must be drawn as in Figure 11(a). However, it is impossible to complete this drawing to a drawing of H_3 with fewer than three crossings.

In the case $\mathbf{a} = \ell 0\ell$ we have $Cr(c_2 \mid c_3) \geq 1$ (see Figure 11(b)) and the total number of crossings is at most two. If r_2 is crossed, then the drawing is not exceptional and we are done. There is a unique way to add r_2 to Figure 11(b) without creating any new crossing. Then there is no way to add r_1 without crossing r_2.

Case 7: Type E'.

This case is very close to the previous one. A similar analysis reduces the problem to the case when $n = 3$. This case is actually identical to the above: By reflecting both the torus pictured in E' and the standard drawing of H_3 (as in Figure 1) about a vertical symmetry axis we find ourselves in this previous case.

Case 8: Type E''.

This case is somewhat similar to that of Type E. We may define the homotopy types for the columns 0, ℓ exactly as before, so that the crossing property (P1) from Type E is satisfied. Then the analysis for $n \geq 4$ is identical, and the only difference is the case when $n = 3$. As before, if there are two consecutive columns with the same homotopy type, we are finished. Thus we may assume that $\mathbf{a} = 0\ell 0$ or $\mathbf{a} = \ell 0\ell$. Then we get another drawing of H_3 with two crossings, but again, in this case r_1 and r_2 cross each other.

Case 9: Type E'''.

This case is essentially the same as the previous one, in the same way as type E' was related to E. This completes the proof of Lemma 3.7. $\square$

Next we bootstrap to the following Lemma.

Lemma 3.8 *The graph $H_{n,k}$ has crossing sequence $(n+k, n-1, 0)$ for every $n \geq 3$ and $k \geq 0$ with the exception of $n = 4$ and $k = 0$.*

Proof: Lemmas 3.1 and 3.2 show that $cr_0(H_{n,k}) = n+k$ and $cr_2(H_{n,k}) = 0$. We can draw $H_{n,k}$ in the torus with $n-1$ crossings by adding a handle to the drawing from Figure 5. It remains to show that $cr_1(H_{n,k}) \geq n-1$ (for $n \geq 3$, unless $n = 4$ and $k = 0$). Take a drawing of $H_{n,k}$ in the torus. By removing the k extra columns we obtain a drawing of $H_{n,0}$ in the torus, which (by Lemma 3.7) has $\geq n-1$ crossings, unless $n = 4$. This completes the proof in all cases except when $n = 4$.

If $n = 4$, the same argument as above shows that $cr_1(H_{4,k}) \geq cr_1(H_{4,1})$; we shall prove now that $cr_1(H_{4,1}) \geq 3$. Suppose this is false, and consider a drawing of $H_{4,1}$ in the torus with at most two crossings. By removing the added column, we obtain a drawing of H_4 in the torus with at most two crossings. It follows from Lemma 3.7 that this drawing is equivalent to that in Figure 9. Since this drawing does not extend to a drawing of $H_{4,1}$ with ≤ 2 crossings, this gives us a contradiction.

Thus $H_{n,k}$ (for $(n,k) \neq (4,0)$), has crossing sequence $(n+k, n-1, 0)$ as claimed. $\square$

Next we introduce one additional graph to get the crossing sequence $(4,3,0)$. We define the graph H_3^+ in the same way as H_3 except that we have three rows instead of two. See Figure 12.

Lemma 3.9 *The graph H_3^+ has crossing sequence* $(4,3,0)$

Proof: It follows from an argument as in Lemma 3.2 that $cr_0(H_3^+) = 4$. Since $H_3^+ - \tau_0 - \tau_1$ is planar, it follows that $cr_2(H_3^+) = 0$. It remains to show that $cr_1(H_3^+) = 3$. Since $cr_1(H_3^+) \leq 3$, we need only to show the reverse inequality. Consider an optimal drawing of H_3^+ in the torus, and suppose (for a contradiction) that it has fewer than three crossings. If the first row contains a crossing, then by removing its edges, we obtain a drawing of a subdivision of H_3 in the torus with at most one crossing—a contradiction.

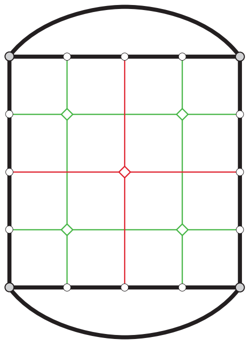

Figure 12: The special graph H_3^+.

Thus, the first row must not have a crossing, and by a similar argument, the third row must not have a crossing. Now, we again remove the first row. This leaves us with a drawing of a subdivision of H_3 in the torus with at most two crossings, and with the added property that one row (r_2 in this H_3) has no crossings. By Lemma 3.7 this must be a drawing as in Figure 8. A routine check of these drawings shows that none of them can be extended to a drawing of H_3^+ with fewer than 3 crossings. $\square$

We require one added Lemma for some simple crossing sequences.

Lemma 3.10 *For every $a > 1$ there is a graph with crossing sequence $(a, 1, 0)$.*

Proof: Let G_1 be a copy of K_5, let G_2 be the graph obtained from a copy of K_5 by replacing each edge, except for one of them, with $a - 1$ parallel edges joining the same pair of vertices. Let G be the disjoint union of G_1 and G_2. It is immediate that $cr_0(G) = a$, $cr_2(G) = 0$, and $cr_1(G) \geq 1$. A drawing of G in $\mathbb{S}_1$ with this crossing number is easy to obtain by embedding G_2 in the torus, and then drawing G_1 disjoint from G_2 with one crossing. Thus, G has crossing sequence $(a, 1, 0)$ as required. $\square$

Proof of Theorem 1.3: Let $(a, b, 0)$ be given with integers $a > b > 0$. If $b = 1$, then the previous lemma shows that there is a graph with crossing sequence $(a, b, 0)$. If $(a, b, 0) = (4, 3, 0)$ then Lemma 3.9 provides such a graph. Otherwise, Lemma 3.8 shows that the graph $H_{b+1,a-b-1}$ has crossing sequence $(a, b, 0)$. $\square$

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