Non-representability of finite projective planes by convex sets

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Abstract

We prove that there is no d such that all finite projective planes can be represented by convex sets in $\mathbb{R}^d$, answering a question of Alon, Kalai, Matoušek, and Meshulam. Here, if $\mathbb{P}$ is a projective plane with lines $\ell_1, \dots, \ell_n$, a *representation of* $\mathbb{P}$ *by convex sets* in $\mathbb{R}^d$ is a collection of convex sets $C_1, \dots, C_n \subseteq \mathbb{R}^d$ such that $C_{i_1}, C_{i_2} \dots, C_{i_k}$ have a common point if and only if the corresponding lines $\ell_{i_1}, \dots, \ell_{i_k}$ have a common point in $\mathbb{P}$. The proof combines a positive-fraction selection lemma of Pach with a result of Alon on "expansion" of finite projective planes. As a corollary, we show that for every d there are 2-collapsible simplicial complexes that are not d-representable, strengthening a result of Matoušek and the author.

1 Introduction

Intersection patterns of convex sets. One of the important areas in discrete geometry regards understanding to "intersection patterns" of convex sets. A pioneering result in this area is Helly's theorem [Hel23] which asserts that if $C_1, C_2, \dots, C_n$ are convex sets in $\mathbb{R}^d$, $n \geq d+1$ and every $d+1$ of the C_i have a common point, then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Consequently, intersection patterns of convex sets have been studied intensively. This study led to the introduction of d-representable and d-collapsible simplicial complexes. We recall that the *nerve* of a family $\mathcal{S} =$

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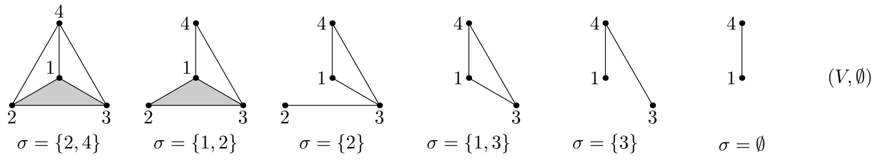

Figure 1: An example of 2-collapsing.

$\{S_1, S_2, \ldots, S_n\}$ is the simplicial complex with vertex set $[n] := \{1, 2, \ldots, n\}$ and with a set $\sigma \subseteq [n]$ forming a simplex if $\bigcap_{i \in \sigma} S_i \neq \emptyset$. A simplicial complex K is *d-representable* if it is isomorphic to the nerve of a family of convex sets in $\mathbb{R}^d$. In this language Helly's theorem implies that a d-representable complex is determined by its d-skeleton. We refer to [DGK63, Eck93, Mat02, MT08] for more examples and background.

Wegner in his seminal 1975 paper [Weg75] introduced d-collapsible simplicial complexes. To define this notion, we first introduce an *elementary d-collapse*. Let K be a simplicial complex and let $\sigma, \tau \in \mathsf{K}$ be faces (simplices) such that

(i) $\dim \sigma \leq d - 1$,
(ii) τ is an inclusion-maximal face of K,
(iii) $\sigma \subseteq \tau$, and
(iv) τ is the *only* face of K satisfying (ii) and (iii).

Then we say that σ is a *d-collapsible face* of K and that the simplicial complex $\mathsf{K}' := \mathsf{K} \setminus \{\eta \in \mathsf{K} : \sigma \subseteq \eta \subseteq \tau\}$ arises from K by an elementary d-collapse. A simplicial complex K is *d-collapsible* if there exists a sequence of elementary d-collapses that reduces K to the empty complex $\emptyset$. Fig. 1 shows an example of 2-collapsing.

Wegner [Weg75] showed that every d-representable simplicial complex is d-collapsible and also gave an example of 2-collapsible simplicial complex that is not 2-representable.

Alon et al. [AKMM02, Problem 17] asked whether there is a function $f = f(d)$ such that every d-collapsible simplicial complex is $f(d)$-representable. Matoušek and Tancer [MT08] showed that $f(d) \geq 2d - 1$ if exists. We improve this result by showing that no such f exists.

Finite projective planes. A *finite projective plane of order* $q \geq 2$ is a pair $(P, \mathcal{L})$ where P is a finite set of *points*, and $\mathcal{L} \subseteq 2^P$ is a set of subsets of P (called *lines*) such that (i) every two points are contained in a unique line, (ii) every two lines intersect in a unique point, and (iii) every line contains

$q+1$ points. It follows that every point is contained in $q+1$ lines and $|P| = |\mathcal{L}| = q^2+q+1$, see e.g. [MN98] for more details.

It is well known that a projective plane of order q exists whenever q is a power of a prime. We remark that it is a well known open problem to decide whether there are projective planes of other orders.

It is also known that a finite projective plane cannot be represented in $\mathbb{R}^d$ so that the points of the projective plane are points in $\mathbb{R}^d$ and the lines of the projective plane are the inclusionwise maximal collections of points lying on a common Euclidean line. This fact follows for example from Sylvester-Gallai theorem: if $p_1, \dots, p_n$ are points in the plane not all of them lying on a common line, then there is a line in the plane which intersects exactly two of these points (see [Gal44] for original solution and [Kel86] for an elegant proof). Our task is to obtain a similar result where the Euclidean lines are replaced by convex sets.

Let $(X, \mathcal{F})$ be a set system where X is a finite set and $\mathcal{F}$ is a set of some subsets of X. We say that $(X, \mathcal{F})$ is *representable by convex sets* in $\mathbb{R}^d$ if there are convex sets $C_F \subset \mathbb{R}^d$ for $F \in \mathcal{F}$ such that for any $F_1, \dots, F_k \in \mathcal{F}$ the convex sets $C_{F_1}, \dots, C_{F_k}$ intersect if and only if the sets $F_1, \dots, F_k$ have a common point in X.[1]

Theorem 1.1. *For every $d \in \mathbb{N}$ there is a $q_0 = q_0(d) \in \mathbb{N}$ such that any projective plane $(P, \mathcal{L})$ of order $q \geq q_0$ is not representable by convex sets in $\mathbb{R}^d$.*

We remark that the assumption $q \geq q_0$ cannot be left out since every projective plane $(P, \mathcal{L})$ of order q can be easily represented by convex sets in $\mathbb{R}^{q^2+q}$. For consider P as the set of vertices of a (q^2+q)-simplex in $\mathbb{R}^{q^2+q}$ and set $C_\ell := \mathrm{conv}\{p : p \in \ell\}$ for a line $\ell \in \mathcal{L}$.[2]

Our main tools are the positive-fraction selection lemma and the fact that the projective planes (considered as bipartite graphs) are expanders.

Theorem 1.1 is proved in Section 2.

We also have the following consequence of Theorem 1.1 which answers the question of Alon at al. [AKMM02] (as announced above) and which is proved in Section 3.

[1]We strongly distinguish the terms d-representable simplicial complex and a set system representable by convex sets in $\mathbb{R}^d$; they have a different meaning. In fact, they are dual in a certain sense. A set system $(X, \mathcal{F})$ is representable by convex sets in $\mathbb{R}^d$ if and only if the nerve of $\mathcal{F}$ is d-representable.

[2]With a bit more effort can be shown that a projective plane of order q can be represented by convex sets in $\mathbb{R}^{2q+1}$. This is based on the fact that a simplicial complex of dimension d is $(2d+1)$-representable.

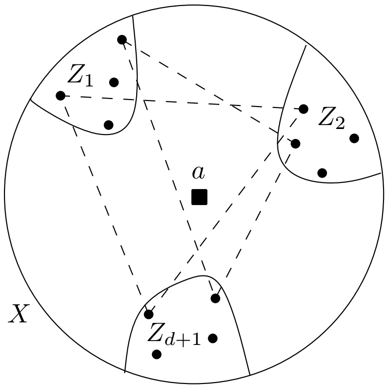

Figure 2: Positive-fraction selection lemma: every triangle formed by the sets Z_i contains a.

Corollary 1.2. *Let $d > 1$ be an integer and let $q_0 = q_0(d)$ be the integer from Theorem 1.1. Let $(P, \mathcal{L})$ be the projective plane of order $q \geq q_0$. Let K_q be a simplicial complex whose vertices are points in P and whose faces are subsets of lines in $\mathcal{L}$. Then K_q is 2-collapsible and is not d-representable.*

2 Proof of the main result

In this section we prove Theorem 1.1. We need few preliminaries. Let $(Z_1, \ldots, Z_k)$ be a k-tuple of sets. By a *transversal* of this k-tuple we mean any set $T = \{t_1, \ldots, t_k\}$ such that $t_i \in Z_i$ for every $i \in [k]$. We need the following result due to Pach [Pac98]; see also [Mat02, Theorem 9.5.1]. See Figure 2.

Theorem 2.1 (Positive-fraction selection lemma; a special case). *For every natural number d, there exists $c = c(d) > 0$ with the following property. Let $X \subset \mathbb{R}^d$ be a finite set of points in general position (i.e., there are no $d+1$ points lying in a common hyperplane). Then there is a point $a \in \mathbb{R}^d$ and disjoint subsets $Z_1, \ldots, Z_{d+1}$, with $|Z_i| \geq c|X|$ such that the convex hull of every transversal of $(Z_1, \ldots, Z_{d+1})$ contains a.*

We remark that the proof of Theorem 2.1 uses several involved tools such as weak hypergraph regularity lemma or same-type lemma (therefore we do not reproduce any details of the proof here). We should also remark that this is only a special case of Pach's theorem (but general enough); Pach moreover assumes that $Z_i \subseteq X_i$, where $X_1 \cup \cdots \cup X_{d+1}$ is a partition of X, and in this seting $|Z_i| \geq c|X_i|$.

We also need the following expansion property of the projective plane [Alo85, Theorem 2.1], [Alo86].

Theorem 2.2. *Let $(P, \mathcal{L})$ be a projective plane of order q. Let $A \subseteq P$. Then $|\{\ell \in \mathcal{L} : \ell \cap A = \emptyset\}| \leq n^{3/2}/|A|$, where $n = q^2 + q + 1$.*

Alon, Haussler and Welzl [AHW87] used this expansion property in the context of range-search problems. They showed that the points of a projective plane (of high enough order) cannot be partitioned into a small number of sets $P_1, \ldots, P_m$ so that for every projective line ℓ the set $\bigcup_{\ell \cap P_i \neq \emptyset} P_i$ contains only a given fraction of all the points. Known results on range search problems imply that a projective plane of a high order cannot be represented by half-planes or simplices in $\mathbb{R}^d$. However, the author is not aware that this approach would imply the result for convex sets.

For completeness, we also reproduce a short proof of Theorem 2.2.

Proof of Theorem 2.2. Let $M = (m_{p\ell})$ be an $n \times n$ matrix with rows indexed by the points of P and columns indexed by the lines of $\mathcal{L}$. We set $m_{p\ell} := 1$ if $p \in \ell$ and $m_{p\ell} := 0$ otherwise. The matrix MM^T has real nonnegative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

By a theorem of Tanner [Tan84]

$$|N(A)| \geq \frac{(q+1)^2|A|}{((q+1)^2 - \lambda_2)|A|/n + \lambda_2}$$

where $N(A)$ denotes $\{\ell \in \mathcal{L} : \ell \cap A \neq \emptyset\}$, the neighborhood of A.

It is not hard to compute that $\lambda_1 = (q+1)^2$ and $\lambda_2 = \cdots = \lambda_n = q$. Consequently,

$$|N(A)| \geq \frac{(q+1)^2|A|}{|A| + q} = n - \frac{q(n - |A|)}{|A| + q} \geq n - \frac{n^{3/2}}{|A|}.$$

□

Proof of Theorem 1.1. For contradiction, we assume that $(P, \mathcal{L})$ is representable by convex sets in $\mathbb{R}^d$; i.e., there are convex sets C_ℓ for $\ell \in \mathcal{L}$ such that $C_{\ell_1}, \ldots, C_{\ell_k}$ intersect if and only if $\ell_1, \ldots, \ell_k$ contain a common point.

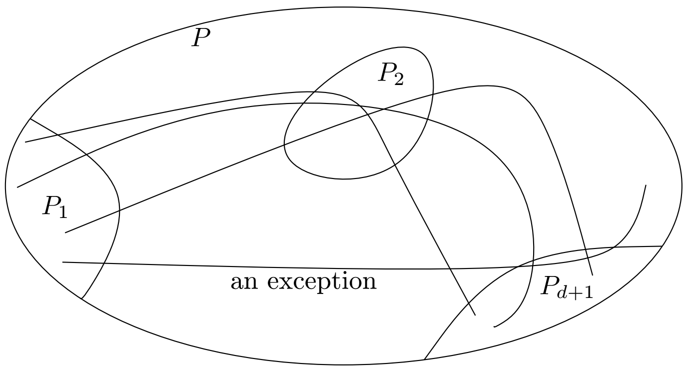

Figure 3: Almost all lines intersect all of $P_1, \ldots, P_{d+1}$.

By standard tricks, we can assume that these sets are open. We explain this step at the end of the proof.

Let $p \in P$. We know that $\bigcap_{p \in \ell} C_\ell$ is nonempty (and open). Let x_p be a point of this intersection. We define $X := \{x_p : p \in P\}$. Because of the openness of the intersections we can assume that X is in general position.

Let $c = c(d) > 0$, $a \in \mathbb{R}^d$, and $Z_1, \ldots, Z_{d+1}$ be the (output) data from Theorem 2.1 (when applied to X). We know that $|Z_i| \geq c|X|$. Let us set $P_i := \{p \in P : x_p \in Z_i\}$ and $\mathcal{M}_i := \{\ell \in \mathcal{L} : \ell \cap P_i = \emptyset\}$. By Theorem 2.2 with $A = P_i$ we get

$$|\mathcal{M}_i| \leq \frac{n^{3/2}}{|P_i|} = \frac{n^{3/2}}{|Z_i|} \leq \frac{n^{3/2}}{c|X|} = \frac{n^{3/2}}{cn} \leq \frac{n}{2(d+1)} = \frac{|\mathcal{L}|}{2(d+1)}$$

provided that q (and hence n as well) is sufficiently large (depending on d and c).

Hence the set $\mathcal{L}' := \mathcal{L} \setminus (\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{d+1})$ of lines that intersect each of $P_1, \ldots, P_{d+1}$ contains at least half of the lines of $\mathcal{L}$. Now let $\ell \in \mathcal{L}'$ and let $p_i \in \ell \cap P_i$. Then $(x_{p_1}, \ldots, x_{p_{d+1}})$ is a transversal of $(Z_1, \ldots, Z_{d+1})$. Thus $a \in \operatorname{conv}\{x_{p_1}, \ldots, x_{p_{d+1}}\} \subseteq C_\ell$, and so a is contained in at least $\frac{|\mathcal{L}|}{2}$ of the C_ℓ. This is a contradiction since at most $q+1$ sets among the C_ℓ can have a nonempty intersection.

It remains to show that we can assume the C_ℓ open. Let us have a set of lines $\mathcal{S} = \{\ell_1, \ldots, \ell_k\}$ such that $C(\mathcal{S}) := \bigcap_{i=1}^k C_{\ell_i}$ is nonempty. In such a case we pick a point $y_{\mathcal{S}} \in C(\mathcal{S})$. For $\ell \in \mathcal{L}$ we set

$$C'_\ell := \operatorname{conv}\{y_{\mathcal{S}} : \mathcal{S} \subseteq \mathcal{L}, \ell \in \mathcal{S}, C(\mathcal{S}) \neq \emptyset\}.$$

In the definition of C'_ℓ we have considered all the possible intersection, thus the sets $\{C'_\ell : \ell \in \mathcal{L}\}$ have the same "intersection pattern" as the sets $\{C_\ell : \ell \in \mathcal{L}\}$. It means that the sets $\{C'_\ell : \ell \in \mathcal{L}\}$ are a counterexample for the theorem if and only if the sets $\{C_\ell : \ell \in \mathcal{L}\}$ are a counterexample. The sets C'_ℓ are compact sets it means that since now we can assume that all the considered sets are compact.

Finally we blow up this sets by a small $\varepsilon > 0$. Thus we get open sets instead of compact sets.

□

3 Proof of the gap between d-representability and d-collapsibility

Proof of Theorem 1.2. The fact that the complex K_q is 2-collapsible is essentially mentioned in [AKMM02, discussion below Problem 15] (without a proof).

All the inclusionwise maximal faces of K_q are of the form $\sigma_\ell = \{p : p \in \ell\}$ for $\ell \in \mathcal{L}$. Two such faces intersect only in a vertex, thus it is possible to 2-collapse these faces gradually to the vertices; the details are given in Lemma 3.1 below. After these collapsings, it is sufficient to remove the vertices (which are already inclusionwise maximal).

It remains to show that K_q is not d-representable. We consider the dual projective plane $(\mathcal{L}, \bar{P})$, where $\bar{P} := \{\{\ell \in \mathcal{L} : p \in \ell\} : p \in P\}$. In particular we can identify a point $p \in P$ with a dual line $\{\ell \in \mathcal{L} : p \in \ell\} \in \bar{P}$. Theorem 1.1 applied for this dual plane $(\mathcal{L}, \bar{P})$ essentially states that K_q is not d-representable (convex sets in the statement now correspond to the lines in $\bar{P}$, which we have identified with P—the set of vertices of K_q).

□

Lemma 3.1. *Let Δ be a d-simplex, i.e., a simplicial complex with $[d+1]$ as the set of vertices and with all the possible faces. Then there is a sequence of elementary 2-collapses that starts with Δ and ends with the simplicial complex that contains all the vertices of Δ and no faces of higher dimension.*

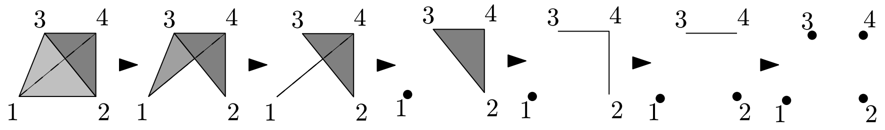

Figure 4: Collapsing a simplex to its vertices.

Proof. In every elementary 2-collapse we only mention the smaller face σ (here we adopt the notation from the definition of an elementary d-collapse), since a 2-collapse is uniquely determined by σ.

The following sequence of choices of σ provides the required 2-collapsing (the faces are ordered in the lexicographical order, see Figure 4).

$$\{1,2\},\{1,3\},\dots,\{1,d+1\},\{2,3\},\dots,\{2,d+1\},\{3,1\},\dots,\{d,d+1\}.$$

□

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