Support set invariancy for interval bimatrix games

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Abstract

Traditionally, game theory problems were considered for exact data and the decisions were based on known payoffs. However, this assumption is rarely true in practice. Uncertainty in measurement and imprecise information must be taken into account. Interval-based approach for handling such uncertainties assumes that one has lower and upper bounds on payoffs. Herein, interval bimatrix games are studied.

Especially, we focus on three kinds of support set invariancy. Support of a mixed strategy consists of that pure strategies having positive probabilities. Given an interval-valued bimatrix game and supports for both players, the question states as follows: Does every bimatrix game instance have an equilibrium with the prescribed support? The other two kinds of invariancies are slight modifications: Has every bimatrix game instance an equilibrium being a subset/superset of the prescribed support? It is difficult to answer these questions: the first case costs solving a large number of linear programs or mixed integer programs. For the remaining two cases a sufficient condition and a necessary condition is proposed, respectively.

Keywords: *Bimatrix game, interval matrix, interval analysis.*

Notation

e	a vector of all ones (with convenient dimension)
e_k	the k-th basis vector (with convenient dimension), i.e., the k-th column of the identity matrix
$x \leq y$, $A \leq B$, $A < B$, ...	vector and matrix relations are understood componentwise
$A_{i,\bullet}$	the i-th row of a matrix A
$A_{\bullet,i}$	the i-th column of a matrix A

1 Introduction

Competitive situations arise in many part of real life and game theory gives a mathematical background for dealing with such conflicting events. There are diverse kinds of mathematical games; we focus on matrix games.

A bimatrix game [18, 19] is a two player game, each of them has finite number of strategies. The payoff function is determined by two matrices $A, B \in \mathbb{R}^{m \times n}$. When player I chooses the i-th strategy and player II his j-th strategy then a_{ij} and b_{ij} are payoffs of the player I and II, respectively. Thus, a bimatrix game is determined by a pair of matrices (A, B). Rational behavior of players is assumed, that is, each of them attempts to maximize his/her reward.

Without loss of generality assume that all payoffs are positive, i.e., $A > 0$ and $B > 0$. This can be always achieved by adding a sufficiently large constant to all the payoffs. Such a transformation does not affect strategies, and the players' rewards are just shifted by a constant.

An equilibrium point in pure strategies needn't exist, so we have to consider mixed strategies. *Mixed strategy* is a probability vector over a set of pure strategies. That is, a mixed strategy for player I is a vector $x \in \mathbb{R}^m$, $x \geq 0$, $e^T x = 1$. Likewise, a mixed strategy for player II is a vector $y \in \mathbb{R}^n$, $y \geq 0$, $e^T y = 1$. Now, the expected reward for player I and II is $x^T Ay$ and $x^T By$, respectively. A pair of mixed strategies $(\hat{x}, \hat{y})$ is called *(Nash) equilibrium* if

$$\hat{x}^T A\hat{y} \geq x^T A\hat{y}, \tag{1}$$

$$\hat{x}^T B\hat{y} \geq \hat{x}^T By \tag{2}$$

for any mixed strategy x and y. In other words, no player has a motivation to change his/her strategy. In mixed strategies, at least one equilibrium always exists [14].

Equilibria can be computed by solving a linear complementarity problem [20], however, for our purposes it is more convenient to use the following result by Audet et al. [3]. It says that equilibria corresponds one-to-one to solutions of a particular mixed integer programming problem. Note that variables α and β are respectively payoffs of the player I and II.

Theorem 1. *Let*

$$L_1 := \max_{i,j} a_{ij} - \min_{i,j} a_{ij},$$
$$L_2 := \max_{i,j} b_{ij} - \min_{i,j} b_{ij}.$$

The set of equilibria is the set of mixed strategies (x, y) *for which there are* $\alpha, \beta \in \mathbb{R}$ *and vectors* $u \in \{0,1\}^m$ *and* $v \in \{0,1\}^n$ *satisfying*

$$e^T x = 1, \ x \geq 0, \tag{3}$$
$$e^T y = 1, \ y \geq 0, \tag{4}$$
$$\alpha e - L_1 u \leq Ay \leq \alpha e, \tag{5}$$
$$\beta e - L_2 v \leq B^T x \leq \beta e, \tag{6}$$
$$x + u \leq e, \tag{7}$$
$$y + v \leq e. \tag{8}$$

In many economical situations the payoffs are not known precisely. Uncertainty in mathematical problems can be dealt by various ways. A popular approach uses fuzzy values and arithmetic [4]. Another, interval-based approach, is considered in this papers. Herein, we assume that we have lower and upper bounds on the imprecise data. Such interval-valued games have already been studied in the recent years. Yager & Kreinovich [21] showed how fair division under interval uncertainty can be performed. Zero-sum interval matrix games were considered by Liu & Kao [13], Collins & Hu [5, 6, 7], Levin [12], and by Shashikhin [17]. While the former authors Liu & Kao discussed the range of possible payoffs, the others solved the interval problem by imposing a binary relation on intervals. Cooperation under interval uncertainty was dealt with by Alparslan-Gök et al. [1, 2].

Let us introduce some interval analysis notation, cf. [15]. *An interval matrix* is a family of matrices

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}\ \forall i,j\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ are given matrices. The midpoint and the radius matrix associated with $\boldsymbol{A}$ are respectively defined as

$$A^c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

An interval matrix $\boldsymbol{B}$ is defined by analogy. By *an instance* we mean a bimatrix game (A, B) with certain $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$. Bimatrix games with interval-values were introduced by Hladík [10], where equilibrium preservation and the set of all possible equilibria were investigated.

2 Support set invariancy

When we perturb entries of matrices A and B a common equilibrium needn't exist, however, equilibria with the same support still may exist. By support [20] of a vector x we understood a set of non-zero components and it is denoted by $\sigma(x) := \{i \mid x_i \neq 0\}$. Support of a mixed strategy says which pure strategies have positive probability, while the others are not involved in that mixed strategy.

Let $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$ be given supports. Usually, these index sets are not given ad hoc, but they are supports of some equilibria of a certain bimatrix game instance instead. We ask whether the support sets preserve (in some sense) when perturbing the payoffs in the intervals. Generally, we will consider the following three cases:

(SSI1) Every instance (A, B), $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ has an equilibrium (x, y) such that $\sigma(x) = S_1$ and $\sigma(y) = S_2$.

(SSI2) Every instance (A, B), $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ has an equilibrium (x, y) such that $\sigma(x) \subseteq S_1$ and $\sigma(y) \subseteq S_2$.

(SSI3) Every instance (A, B), $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ has an equilibrium (x, y) such that $\sigma(x) \supseteq S_1$ and $\sigma(y) \supseteq S_2$.

This categorization was motivated by papers on support set invariancy in linear programming [8, 9, 11].

2.1 The first kind of support set invariancy

Here we consider the first kind of support set invariancy. Let $S_1 \subseteq \{1,\dots,m\}$ and $S_2 \subseteq \{1,\dots,n\}$ be given. Theorem 2 reduced the problem of testing (SSI1) to the problem of checking solvability of $2^{|S_1|}+2^{|S_2|}$ linear systems. Solvability of linear systems can be checked efficiently in a polynomial time, so the tractability of the problem depends on cardinalities of S_1 and S_2.

Theorem 2. *Remove from* $\boldsymbol{A}$ *the columns indexed by* $\{1,\dots,n\}\setminus S_2$ *and from* $\boldsymbol{B}$ *the rows indexed by* $\{1,\dots,m\}\setminus S_1$*. Then* (SSI1) *holds true if and only if for every* $z^1 \in \{\pm 1\}^{|S_1|}$ *and* $z^2 \in \{\pm 1\}^{|S_2|}$ *there exist* $x \in \mathbb{R}^{|S_1|}$*,* $y \in \mathbb{R}^{|S_2|}$*, and* $\alpha, \beta \in \mathbb{R}$ *satisfying the linear system*

$$e^T x = 1,\ \ x \geq \varepsilon_1 e, \tag{9}$$

$$e^T y = 1,\ \ y \geq \varepsilon_2 e, \tag{10}$$

$$\alpha = (A^c_{i,\bullet} - z^1_i A^{\Delta}_{i,\bullet})y, \quad \forall i \in S_1, \tag{11}$$

$$\alpha \geq (\overline{A}_{i,\bullet})y, \quad \forall i \notin S_1, \tag{12}$$

$$\beta = (B^c_{\bullet,i} - z^2_i B^{\Delta}_{\bullet,i})^T x, \quad \forall i \in S_2, \tag{13}$$

$$\beta \geq (\overline{B}_{\bullet,i})^T x, \quad \forall i \notin S_2, \tag{14}$$

where $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are sufficiently small.*

Note that the linear system (9)–(14) is decomposable into two subsystems. The first one consists of (9), (13) and (14) with variables β and x, while the second one covers (10)–(12) with variables α and y. We have to check $2^{|S_2|}$ instances of the former and $2^{|S_1|}$ instances of the latter. That is why the total number of linear programs to solve run comes up to $2^{|S_1|}+2^{|S_2|}$.

Proof. First we note that removing the mentioned columns from $\boldsymbol{A}$ and rows from $\boldsymbol{B}$ makes no loss of generality, since the corresponding indices of that columns and rows have zero probability to be chosen as strategies.

If (SSI1) holds true then (3)–(8) is solvable for every $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$, and the solution satisfies $\sigma(x) = S_1$ and $\sigma(y) = S_2$. Let $z^1 \in \{\pm 1\}^{|S_1|}$, $z^2 \in \{\pm 1\}^{|S_2|}$ and define $A \in \boldsymbol{A}$ in this way: $A_{i,\bullet} := A^c_{i,\bullet} - z^1_i A^{\Delta}_{i,\bullet}$ provided $i \in S_1$ and $A_{i,\bullet} := \overline{A}_{i,\bullet}$ otherwise. Define $B \in \boldsymbol{B}$ accordingly. By Theorem 1, (3)–(8) has a solution x^{z^2}, y^{z^1}, and this solution satisfies (11)–(14) as well. The remaining inequalities $x^{z^2} \geq \varepsilon_1 e$ and $y^{z^1} \geq \varepsilon_2 e$ holds true provided $\varepsilon_1 > 0$

and $\varepsilon_2 > 0$ are small enough, e.g. take ε_1 the minimal component of x^{z^2} over all $z^2 \in \{\pm 1\}^{|S_2|}$.

Suppose now that (9)–(14) is solvable for every $z^1 \in \{\pm 1\}^{|S_1|}$ and $z^2 \in \{\pm 1\}^{|S_2|}$. Particularly, for every $z^1 \in \{\pm 1\}^{|S_1|}$ the system (11) has a nonnegative solution α^{z^1}, y^{z^1}. According to Rohn [15, Theorem 2.17]s, the system

$$\alpha = (A_{i,\bullet})y, \quad \forall i \in S_1$$

is solvable for every $A \in \boldsymbol{A}$. Moreover, its solution α, y (at least one) belongs to the convex hull of the points α^{z^1}, y^{z^1} over $z^1 \in \{\pm 1\}^{|S_1|}$. This solution also fulfills (10) and (12). The latter implies that

$$\alpha \geq (\overline{A}_{i,\bullet})y \geq Ay, \quad \forall i \notin S_1.$$

Now, we construct a solution $\hat{\alpha}, \hat{y}, \hat{u}$ to (4)–(5) as follows. Put $\hat{\alpha} := \alpha$ and define $\hat{y}$ such that $\hat{y}_i = 0$ for $\imath \notin S_2$ and a restriction of $\hat{y}$ to the components indexed by S_2 is equal to y. Eventually, , $\hat{u} \in \{0,1\}^m$ defined as $u_i = 0$ if $i \in S_1$ and $u_i = 1$ otherwise. Similarly we proceed for the second part of (9)–(14) and we obtain that for every $B \in \boldsymbol{B}$ there is solution $\hat{\beta}, \hat{x}, \hat{v}$ of (3) and (6). Moreover, x and y have the right support, so they satisfies (7) and (8) as well. □

Remark 1. Theorem 2 exhibits some sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, but do not say how small values really suffices to use. By means of Schrijver [16] we determine some values depending only on $\underline{A}$, $\overline{A}$, $\underline{B}$ and $\overline{B}$. First we assume that their entries are rational, which is not restricting from practical standpoint. Let $r = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ be a rational number, $w \in \mathbb{Q}^n$ rational vector and $A \in \mathbb{Q}^{m \times n}$ a rational matrix. We define their *sizes* respectively as

$$\begin{aligned}
\operatorname{size}(r) &:= 1 + \lceil \log_2(|p|+1) \rceil + \lceil \log_2(q+1) \rceil, \\
\operatorname{size}(w) &:= n + \operatorname{size}(w_1) + \cdots + \operatorname{size}(w_n), \\
\operatorname{size}(A) &:= mn + \sum_{i,j} \operatorname{size}(a_{ij}),
\end{aligned}$$

where $\lceil a \rceil$ stands for the upper integer part of a. When sizes of rationals are defined in this way, one can see that $\operatorname{size}(r/s) \leq \operatorname{size}(r) + \operatorname{size}(s)$ for any $r \in \mathbb{Q}$ and non-zero $s \in \mathbb{Q}$. By Schrijver [16, Theorem 3.2], one has also $\operatorname{size}(\det A) < 2 \operatorname{size}(A)$ for any $A \in \mathbb{Q}^{n \times n}$.

Let y be a solution to (10)–(12) such that $e^T y = 1$ and $y > 0$. Then it is a solution of the equational sub-system taking some inequalities as equations. Cramer's rule gives that every components of y is a fraction of two determinants composed of a part of the constraint matrix and the right-hand side. Using the formulae for sizes of fractions and determinants we obtain that $\mathrm{size}(y_i)$ is bounded from above by $4\,\mathrm{size}(M)^{z^1}$, where M^{z^1} is a matrix consisting of the constraint matrix and the right-hand side. To get an upper bound for the size of M^{z^1} over $z^1 \in \{\pm 1\}^{|S_1|}$ we define $A \in \boldsymbol{A}$ such that a_{ij} is that of $\underline{a}_{ij}$ and $\overline{a}_{ij}$ which has greater size. Introduce

$$M := \begin{pmatrix} A & e \\ e^T & 1 \end{pmatrix}.$$

We have $\mathrm{size}(M^{z^1}) \leq \mathrm{size}(M)$ for every $z^1 \in \{\pm 1\}^{|S_1|}$, and therefore $\mathrm{size}(y_i) \leq 4\,\mathrm{size}(M)$. Eventually, we can put $\varepsilon_2 := 2^{-4\,\mathrm{size}(M)}$ and similarly for ε_1.

Another problems arise provided the index sets S_1 and S_2 are not known. Do there exist S_1 and S_2 satisfying (SSI1)? And how does one find them?

A direct approach to answer this questions is to check all combinations of $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$. For each $S_1 \subseteq \{1, \dots, m\}$ check solvability of (10)–(12), and for each $S_2 \subseteq \{1, \dots, n\}$ check solvability of the system consisting of (9), (13) and (14). How much computation does it need? For any $S_1 \subseteq \{1, \dots, m\}$ we must solve $|S_1|$ linear programs. Thus the number of solving (10)–(12) is as follows

$$\sum_{S_1 \subseteq \{1,\dots,m\}} 2^{|S_1|} = \sum_{i=1}^{m} \binom{m}{i} 2^i = (2+1)^m = 3^m,$$

using binomial expansion. Similarly, we have to solve 3^n linear programs for the second issue. In total, the direct approach needs solving $3^m + 3^n$ linear programs.

2.2 The second kind of support set invariancy

In this section we discuss (SSI2). Let $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$ be given. If S_1 and S_2 are unknown then we are finished: Put $S_1 := \{1, \dots, m\}$, $S_2 := \{1, \dots, n\}$ and (SSI2) is true, since every bimatrix game has an equilibrium.

Proposition 1. *Remove from $\boldsymbol{A}$ the columns indexed by $\{1,\ldots,n\}\setminus S_2$ and from $\boldsymbol{B}$ the rows indexed by $\{1,\ldots,m\}\setminus S_1$. Then* (SSI2) *holds true if for every $k\in\{1,\ldots,m\}\setminus S_1$ the linear system*

$$\sum_{i\in S_1}\lambda_i\underline{A}_{i,\bullet}\geq\overline{A}_{k,\bullet},\tag{15}$$

$$\sum_{i\in S_1}\lambda_i=1,\quad \lambda_i\geq 0\quad \forall i\in S_1\tag{16}$$

is solvable and for every $k\in\{1,\ldots,n\}\setminus S_2$ the linear system

$$\sum_{i\in S_2}\lambda_i\underline{B}_{\bullet,i}\geq\overline{B}_{\bullet,k},$$

$$\sum_{i\in S_2}\lambda_i=1,\quad \lambda_i\geq 0\quad \forall i\in S_2$$

is solvable.

Proof. For each $A\in\boldsymbol{A}$ and $B\in\boldsymbol{B}$ we construct an equilibrium (x,y) satisfying $\sigma(x)\subseteq S_1$ and $\sigma(y)\subseteq S_2$ as follows. Let $\boldsymbol{A}^r$ be an interval matrix which arise from $\boldsymbol{A}$ by restricting to rows indexed by S_1 and columns indexed by S_2, and similarly for $\boldsymbol{B}^r$. The bimatrix game (A^r,B^r) has an equilibrium (x^r,y^r) for every $A^r\in\boldsymbol{A}^r$ and $B^r\in\boldsymbol{B}^r$. The strategies x^r and y^r can be extended to strategies x and y of the original game by canonical embedding. To show that (x,y) is an equilibrium of (A,B), $A\in\boldsymbol{A}$ and $B\in\boldsymbol{B}$, we must prove that (1)–(2) is true.

Let $k\in\{1,\ldots,m\}\setminus S_1$ and λ be a solution to (15)–(16). Then

$$(x^r)^TA^ry^r\geq\lambda^TA^ry^r\geq\lambda^T\underline{A}^ry^r\geq\overline{A}^r_{k,\bullet}y^r\geq A^r_{k,\bullet}y^r.$$

Extension to full size gives

$$x^TAy\geq A_{k,\bullet}y=e_k^TAy.$$

It means that no one pure strategy e_k is better than x for some strategy of player II, and the same is true when considering mixed strategies instead of pure ones. Therefore (1) holds and (2) can be proven in a similar manner. $\square$

The proposed sufficient condition does not seem to be very strong, but it need to solve merely $m + n - |S_1| - |S_2|$ linear programs.

Another sufficient condition for (SSI2) is given below. It relax the positivity conditions (9)–(10) to non-negative ones (17)–(18), but in general it requires needlessly many equations in (19) and (21). For this reason it is not a necessary condition for (SSI2).

Proposition 2. *Remove from $\boldsymbol{A}$ the columns indexed by $\{1,\dots,m\} \setminus S_2$ and from $\boldsymbol{B}$ the rows indexed by $\{1,\dots,n\} \setminus S_1$. Then* (SSI2) *holds true if for every $z^1 \in \{\pm 1\}^{|S_1|}$ and $z^2 \in \{\pm 1\}^{|S_2|}$ there exist $x \in \mathbb{R}^{|S_1|}$, $y \in \mathbb{R}^{|S_2|}$, and $\alpha, \beta \in \mathbb{R}$ satisfying the linear system*

$$e^T x = 1,\ x \geq 0, \tag{17}$$

$$e^T y = 1,\ y \geq 0, \tag{18}$$

$$\alpha = (A^c_{i,\bullet} - z^1_i A^\Delta_{i,\bullet})y, \quad \forall i \in S_1, \tag{19}$$

$$\alpha \geq (\overline{A}_{i,\bullet})y, \quad \forall i \notin S_1, \tag{20}$$

$$\beta = (B^c_{\bullet,i} - z^2_i B^\Delta_{\bullet,i)})^T x, \quad \forall i \in S_2, \tag{21}$$

$$\beta \geq (\overline{B}_{\bullet,i)})^T x, \quad \forall i \notin S_2. \tag{22}$$

Proof. Let (17)–(22) be solvable for every $z^1 \in \{\pm 1\}^{|S_1|}$ and $z^2 \in \{\pm 1\}^{|S_2|}$. Particularly, for every $z^1 \in \{\pm 1\}^{|S_1|}$ the system (19) has a nonnegative solution α^{z^1}, y^{z^1}. According to Rohn [15, Theorem 2.17], the system

$$\alpha = (A_{i,\bullet})y, \quad \forall i \in S_1$$

has a solution for every $A \in \boldsymbol{A}$. Moreover, at least one of its solution α, y belongs to the convex hull of the points α^{z^1}, y^{z^1} over $z^1 \in \{\pm 1\}^{|S_1|}$. Hence α, y also fulfills (18) and (20). The latter implies that

$$\alpha \geq (\overline{A}_{i,\bullet})y \geq Ay, \quad \forall i \notin S_1.$$

Now, we construct a solution $\hat{\alpha}, \hat{y}, \hat{u}$ to (4)–(5) as follows. Put $\hat{\alpha} := \alpha$ and define $\hat{y}$ such that $\hat{y}_i = 0$ for $\imath \notin S_2$ and a restriction of $\hat{y}$ to the components indexed by S_2 is equal to y. Eventually, $\hat{u} \in \{0,1\}^m$ defined as $u_i = 0$ if $i \in S_1$ and $u_i = 1$ otherwise. Similarly we proceed for the second part of (17)–(22) and we obtain that for every $B \in \boldsymbol{B}$ there is solution $\hat{\beta}, \hat{x}, \hat{v}$ of (3) and (6). Moreover, (7) and (8) are satisfied, too, since at least one of x_i and u_i is zero and likewise for y_i and v_i. $\square$

Again, the linear system (17)–(22) is decomposable into two sub-systems (18)–(20), and (17), (21) and (22). The computational cost is to solve $2^{|S_1|} + 2^{|S_2|}$ linear programs.

2.3 The third kind of support set invariancy

Let $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$ be given. The third kind of support set invariancy is, roughly speaking, like a dual to (SSI2). Not surprisingly, a direct adaptation of Theorem 2 to this case yields necessary condition (contrary to the sufficient condition for (SSI2) proposed in Proposition 2). Note that the constants ε_1 and ε_2 can be adjusted according to Remark 1.

Theorem 3. *If* (SSI3) *holds true then for every* $z^1 \in \{\pm 1\}^m$ *and* $z^2 \in \{\pm 1\}^n$ *there exist* $x \in \mathbb{R}^m$*,* $y \in \mathbb{R}^n$*, and* $\alpha, \beta \in \mathbb{R}$ *satisfying the linear system*

$$e^T x = 1, \tag{23}$$

$$x_i \geq \varepsilon_1, \quad \forall i \in S_1, \tag{24}$$

$$e^T y = 1, \tag{25}$$

$$y_i \geq \varepsilon_2, \quad \forall i \in S_2, \tag{26}$$

$$\alpha = (A^c_{i,\bullet} - z^1_i A^\Delta_{i,\bullet})y, \quad \forall i \in S_1, \tag{27}$$

$$\alpha \geq (\overline{A}_{i,\bullet})y, \quad \forall i \notin S_1, \tag{28}$$

$$\beta = (B^c_{\bullet,i} - z^2_i B^\Delta_{\bullet,i})^T x, \quad \forall i \in S_2, \tag{29}$$

$$\beta \geq (\overline{B}_{\bullet,i})^T x, \quad \forall i \notin S_2, \tag{30}$$

where $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are sufficiently small.*

Proof. Suppose that (SSI3) holds and let $z^1 \in \{\pm 1\}^m$ and $z^2 \in \{\pm 1\}^n$. Define $A \in \boldsymbol{A}$ in this way: For $i \in S_1$ put $A_{i,\bullet} := A^c_{i,\bullet} - z^1_i A^\Delta_{i,\bullet}$, otherwise $A_{i,\bullet} := \overline{A}_{i,\bullet}$. Analogously we define $B \in \boldsymbol{B}$. Condition (SSI3) implies that the bimatrix game (A, B) has an equilibrium (x, y) satisfying $x_i > 0$ for $i \in S_1$ and $y_i > 0$ for $i \in S_2$. Hence x, y fulfill (23)–(26). Vectors x, y solve also (3)–(8), which shows that x, y fulfill (27)–(30). $\square$

Also in this case, the linear system (23)–(30) is simply decomposable into two sub-systems (25)–(28), and (23)–(24) with (29)–(30). It is needed to solve $2^m + 2^n$ linear programs.

2.4 Illustrative example

Example 1. Consider an interval bimatrix game $(\boldsymbol{A}, \boldsymbol{B})$, where

$$\underline{A} := \begin{pmatrix} 0 & 25 & 5 & 5 \\ 40 & 0 & 5 & 10 \\ 10 & 15 & 20 & 0 \\ 20 & 5 & 10 & 15 \end{pmatrix}, \quad \overline{A} := \begin{pmatrix} 5 & 28 & 8 & 8 \\ 43 & 5 & 8 & 13 \\ 13 & 18 & 25 & 3 \\ 23 & 8 & 13 & 20 \end{pmatrix},$$

$$\underline{B} := \underline{A}^T, \quad \overline{B} := \overline{A}^T.$$

We discuss several support sets. The bimatrix game $(\underline{A}, \underline{B})$ has equilibria

1. (x^1, y^1), $x^1 = y^2 = (\frac{1}{3}, \frac{8}{15}, \frac{2}{15}, 0)^T$,
2. (x^2, y^2), $x^2 = y^2 = (0, 0, 1, 0)^T$,
3. (x^3, y^3), $x^3 = y^3 = (\frac{1}{5}, \frac{2}{5}, 0, \frac{2}{5})^T$,

among others. We investigate support set invariancies for their respective supports.

1. Do all instances of the interval game have an equilibrium with the same support as (x^1, y^1)? Put $S_1 := \{1, 2, 3\}$ and $S_2 := \{1, 2, 3\}$. Utilizing Theorem 2 we obtain negative answer.

   What about the second kind of support set invariancy? The simple sufficient condition formulated in Proposition 1 reveals that (SSI2) holds true, even though the sufficient condition in Proposition 2 is not satisfied.

   The necessary condition for (SS3) in Theorem 3 is not fulfilled, meaning that not all instance of the interval game have equilibria with supports covering S_1 and S_2.

2. Let $S_1 := \{3\}$ and $S_2 := \{3\}$. By Theorem 2 we observe that (SSI1) holds true. Both sufficient conditions for (SSI2) are valid, too. Moreover, the necessary condition for (SSI3) is satisfied. This doesn't imply that (SSI3) is really true, but validity of (SSI1) do so.

3. Let $S_1 := \{1, 2, 4\}$ and $S_2 := \{1, 2, 4\}$. In this case, (SSI1) does not hold. The second support set invariancy may not hold as neither the sufficient condition holds true. However, by the result of calling Theorem 3, we can claim certainly that (SSI3) is not valid.

3 Conclusion

We investigated three support set invariancies (SSI1)–(SSI3) for interval bimatrix games. We proposed a characterization of (SSI1) by means of a series of linear systems. For (SSI2) and (SSI3) sufficient and necessary conditions were formulated, respectively. Their complete description remains as an open problem.

There were stated two sufficient conditions for (SSI2), in Proposition 1 and 2. The former costs solving a linear number of linear programs while the latter requires an exponential number. Surprisingly, several performed examples indicate that the former condition is stronger in many cases.

Theorem 3 gives the mentioned necessary condition for (SSI3) and it seems to be quite strong one. In interval bimatrix games, it may be useful in proving non-existence of equilibria with large supports.

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