Covering line graphs with equivalence relations*

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Abstract

An equivalence graph is a disjoint union of cliques, and the equivalence number $eq(G)$ of a graph G is the minimum number of equivalence subgraphs needed to cover the edges of G. We consider the equivalence number of a line graph, giving improved upper and lower bounds: $\frac{1}{3}\log_2\log_2\chi(G) < eq(L(G)) \leq 2\log_2\log_2\chi(G)+2$. This disproves a recent conjecture that $eq(L(G))$ is at most three for triangle-free G; indeed it can be arbitrarily large.

To bound $eq(L(G))$ we bound the closely-related invariant $\sigma(G)$, which is the minimum number of orientations of G such that for any two edges e, f incident to some vertex v, both e and f are oriented out of v in some orientation. When G is triangle-free, $\sigma(G) = eq(L(G))$. We prove that even when G is triangle-free, it is NP-complete to decide whether or not $\sigma(G) \leq 3$.

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1 Introduction

Given a binary relation $\sim$ over a set A, it is natural to consider expressing $\sim$ as a union of k transitive subrelations for the smallest possible value of k. If $\sim$ is reflexive and symmetric, each subrelation is an equivalence relation and we can restate the problem as a graph covering problem: We seek to cover the edges of a graph G with k *equivalence subgraphs*, i.e. subgraphs each of which is a disjoint union of cliques. This is an *equivalence covering* of G. The minimum k for which this is possible is the *equivalence number* of G, denoted $eq(G)$.

The equivalence covering number was introduced by Duchet in 1979 [3]. Not surprisingly, it is NP-complete to compute, even for split graphs [2]. In [1], Alon proved upper and lower bounds for general graphs:

Theorem 1 *Let G be a graph on n vertices with minimum degree δ, and let $cc(G)$ be the minimum number of cliques needed to cover the edges of G. Then*

$$\log_2 n - \log_2(n-\delta-1) \leq eq(G) \leq cc(G) \leq 2e^2(n-\delta)^2 \log_e n.$$

Observe that if G is triangle-free, then every equivalence subgraph of G is a matching. It follows that in this case an equivalence covering of G is actually an edge coloring, and that $eq(G)$ is equal to the chromatic index $\chi'(G)$. Thus equivalence coverings can also be thought of as a generalization of edge colorings. In fact, McClain [8] formulated them seemingly independently of earlier work in precisely this context, calling $eq(G)$ the *clique chromatic index* of G.

In this paper we address the problem, first studied by McClain, of bounding the equivalence number of line graphs. For a graph G, the line graph $L(G)$ of G has a vertex corresponding to each edge of G, and two vertices of $L(G)$ are adjacent precisely if the two corresponding edges of G share an endpoint (i.e. are *incident*)[1]. McClain proved that for a graph G on n vertices, $eq(L(G)) \leq 4\left\lceil \frac{\log_e n}{\log_e 12} \right\rceil$, and asked if this bound could be improved [7]. We will prove that

$$\tfrac{1}{3}\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right) \leq eq(L(G)) \leq 2\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right),$$

[1] We need only consider line graphs of simple graphs: If two vertices u and v of G have the same closed neighborhood, it is easy to see that $eq(G) = eq(G-v)$. Thus we can easily reduce the problem for line graphs of multigraphs.

where $\chi(G)$ is the chromatic number of G. We will actually prove a slightly better (but more unwieldy) lower bound. Since triangle-free graphs can have arbitrarily high chromatic number, our lower bound disproves a recent conjecture of McClain [8] stating that any triangle-free graph G has $eq(L(G)) \leq 3$.

In order to bound $eq(L(G))$ we consider a closely-related invariant of G, namely $\sigma(G)$. In the next section we introduce $\sigma(G)$ and prove that it is close to $eq(L(G))$. In Section 3 we relate $\sigma(G)$ to two other interesting invariants arising from orientations. In Section 4 we will briefly discuss tightness and complexity concerns, in particular proving that it is NP-complete to decide whether or not $eq(L(G)) \leq 3$, even if G is triangle-free.

2 Covering incidence pairs with orientations

Equivalence subgraphs of a line graph $L(G)$ are intimately related to orientations of G. We begin the section by explaining why this is so.

For every vertex v of G, there is a clique C_v of $L(G)$ corresponding to those edges of G incident to v. Every vertex of $L(G)$ is in exactly two of these cliques, since every edge of G has two endpoints. This fact invites a natural mapping from the set of orientations of G to the set of equivalence subgraphs of $L(G)$. Given an orientation $\overrightarrow{G}$ we define the clique $\overrightarrow{C_v}$ of $L(G)$ as the set of vertices of $L(G)$ corresponding to the out-edges of v. For $u, v \in V(G)$ the cliques $\overrightarrow{C_u}$ and $\overrightarrow{C_v}$ are disjoint, so the disjoint union of $\overrightarrow{C_v}$ for all $v \in V(G)$ is an equivalence subgraph of $L(G)$ corresponding to the orientation $\overrightarrow{G}$. We call this equivalence subgraph of $L(G)$ the *analogue* of $\overrightarrow{G}$.

Using this idea, we can construct an equivalence covering of $L(G)$ using orientations of G. Let $\overrightarrow{G_1}, \overrightarrow{G_2}, \ldots, \overrightarrow{G_k}$ be a set of orientations of G with the following property: For every vertex u of G with neighbors v and w, there is an i such that $\overrightarrow{uv}, \overrightarrow{uw} \in \overrightarrow{E}(\overrightarrow{G_i})$. In other words, for every $e, f \in E(G)$ sharing an endpoint v, some orientation $\overrightarrow{G_i}$ directs both e and f out of v. We call such a set of orientations an *orientation covering* of G, and accordingly define the *orientation covering number* of G, denoted $\sigma(G)$, as the size of a minimum orientation covering. Figure 1 shows an orientation covering of size three for K_4, along with a corresponding equivalence covering of size three for $L(K_4)$.

Noting the discussion above, we can make an easy observation:

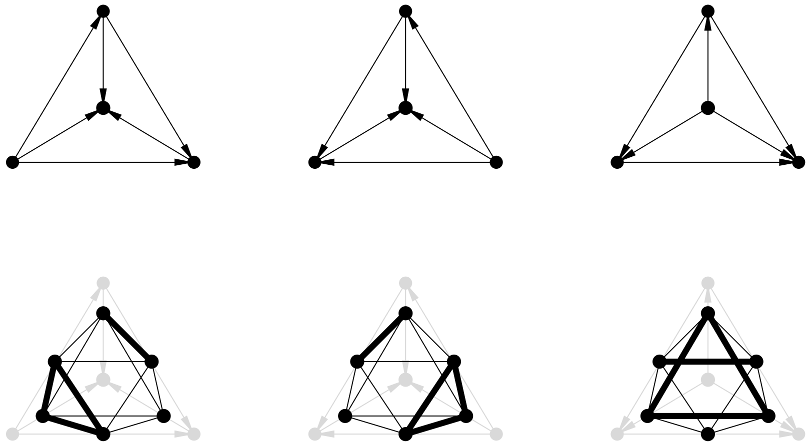

Figure 1: An orientation covering of size three for K_4, and the corresponding equivalence subgraphs in $L(K_4)$.

Observation 2 *For any graph G, we have $eq(L(G)) \leq \sigma(G)$.*

The two invariants are actually equal for triangle-free G:

Proposition 3 *For any triangle-free graph G, we have $eq(L(G)) = \sigma(G)$.*

Proof. Let H be an equivalence subgraph of $L(G)$, and consider a vertex w of $L(G)$ corresponding to the edge $uv \in E(G)$. Since G is triangle-free, the neighborhood of w in H is contained in either C_u or C_v. We construct an orientation of G from H as follows: For every such w, u, and v, orient uv towards v if $N_H(w) \subseteq C_u$, and orient it towards u otherwise. For w having no neighbor in H, orient uv arbitrarily.

If we construct an orientation of G in this way for every equivalence subgraph of an equivalence covering of $L(G)$, it is easy to confirm that the result is an orientation covering of G. The result follows. □

The invariants $\sigma(G)$ and $eq(L(G))$ are not always equal. If G is a triangle with a pendant vertex, then $\sigma(G) = 3$ and $eq(L(G)) = 2$. We suspect that this may be the worst case, i.e. that $\sigma(G) \leq eq(L(G)) + 1$ for all connected G, unless G is a triangle, in which case $eq(L(G)) = 1$ and $\sigma(G) = 3$. For now we simply show that they are within a multiplicative constant of one another.

Theorem 4 *For any graph G,*

$$eq(L(G)) \leq \sigma(G) \leq 3\, eq(L(G)).$$

Proof. Consider k equivalence subgraphs $R_1, \dots, R_k$ of $L(G)$ covering all the edges of $L(G)$. Using Observation 2, we only need to prove that $\sigma(G) \leq 3k$. Take $i \in [k]$. Each component of R_i is either contained in C_v for some $v \in V(G)$, or corresponds to the edges of a triangle in G. Let T_i be the set of triangles of G corresponding to cliques in R_i. Observe that the triangles of T_i must be edge-disjoint, since otherwise the corresponding triangles of R_i would not be vertex-disjoint. Consequently there exist three orientations $\overrightarrow{T_i}^1, \overrightarrow{T_i}^2, \overrightarrow{T_i}^3$ of the edges of T_i such that for any triple (u, v, w) of vertices of G corresponding to a triangle of T_i, the edges are oriented $\overrightarrow{uv}$ and $\overrightarrow{uw}$ in one of the orientations.

We extend each $\overrightarrow{T_i}^j$ to an orientation $\overrightarrow{G_i}^j$ of H as in the proof of Proposition 3. That is, for every $w \in V(R_i)$ corresponding to an edge uv of G, we orient uv towards v if $N_{R_i}(w) \subseteq C_u$. If w has no neighbor in R_i, orient uv arbitrarily. This construction gives us an orientation covering $\{\overrightarrow{G_i}^j \mid i \in [k],\ j \in [3]\}$, so $\sigma(G) \leq 3k$. $\square$

This proves that $eq(L(G))$ and $\sigma(G)$ are within a multiplicative constant of one another. In the next section we prove that $\sigma(G)$ is within a multiplicative constant of $\log_2 \log_2 \chi(G)$.

3 Homomorphisms, eyebrows and elbows

The bounds that we prove in this paper are generally stated in terms of the chromatic number. There is a simple justification for this, which is that $\sigma(G)$ is monotone with respect to homomorphism[2]:

Proposition 5 *Let G and H be graphs such that there is a homomorphism from G to H. Then $\sigma(G) \leq \sigma(H)$.*

Proof. Consider a homomorphism $f : V(G) \to V(H)$ along with a minimum orientation covering of H. For each orientation of H we define an orientation of G such that $\overrightarrow{uv} \in \overrightarrow{E}(\overrightarrow{G})$ precisely if $\overrightarrow{f(u)f(v)} \in \overrightarrow{E}(\overrightarrow{H})$. It

[2] A homomorphism from a graph G to a graph H is a function $f : V(G) \to V(H)$ such that any two adjacent vertices of G get mapped to adjacent vertices of H.

is straightforward to confirm that the resulting orientations of G form an orientation covering, and we omit the details. $\square$

Corollary 6 *For any graph G with chromatic number k, $\sigma(G) \leq \sigma(K_k)$.*

It is not clear whether or not there exists a graph G for which $\sigma(G) < \sigma(K_{\chi(G)})$. However, this tightness does hold for a related invariant which we now introduce.

Consider the following weakening of an orientation covering: Instead of insisting that any two incident edges are out-oriented from their shared endpoint in some orientation, we merely insist that in some orientation they are either both out-oriented or both in-oriented. This weakening inspires a new invariant.

Definition 1 *The elbow number $elb(G)$ of a graph G is the minimum k for which there exist k orientations $\{\overrightarrow{G_i} \mid i \in [k]\}$ of G with the following property: For any path u, v, w of G, there is an i such that u, v, w is not a directed path in $\overrightarrow{G_i}$. Such a collection of orientations is an* elbow covering.

Our interest in the elbow number comes primarily from two desirable properties of the invariant. First and foremost, it is not too far from the orientation covering number:

Proposition 7 *For any graph G, $elb(G) \leq \sigma(G) \leq 2\,elb(G)$.*

Proof. Clearly $elb(G) \leq \sigma(G)$ because every orientation covering is also an elbow covering. If we take a minimum elbow covering along with the reversal of each of its orientations, we get an orientation covering of size at most $2\,elb(G)$. $\square$

Second, a straightforward modification of the proof of Proposition 5 tells us that the elbow number is also monotone under homomorphism:

Proposition 8 *Let G and H be graphs such that there is a homomorphism from G to H. Then $elb(G) \leq elb(H)$. Consequently $elb(G) \leq elb(K_{\chi(G)})$.*

We now characterize $elb(G)$ precisely, beginning with the lower bound.

Theorem 9 *For any graph G with $\chi(G) \geq 3$, $elb(G) \geq \lceil \log_2 \log_2 \chi(G) \rceil + 1$.*

Proof. Suppose $\chi(G) \geq 3$ and $elb(G) = k$. Using a minimum elbow covering of G, we will construct a proper coloring of G using $2^{2^{k-1}}$ colors.

Take an elbow covering of G using k orientations $\overrightarrow{G_1}, \ldots, \overrightarrow{G_k}$, and for every edge incidence (u, uv), let $o(u, uv)$ be the set of orientations for which uv is oriented out of u. That is, $o(u, uv) = \{i \mid \overrightarrow{uv} \in \overrightarrow{E}(\overrightarrow{G_i})\}$.

The following properties of $o(u, uv)$ follow from the definition of an elbow covering. First, for adjacent u and v, $o(u, uv) = [k] \setminus o(v, uv)$. Second, for u with neighbors v and w, $o(u, uv) \neq [k] \setminus o(u, uw)$. For if $o(u, uv)$ and $o(u, uw)$ partition $[k]$, then v, u, w is a directed path in every orientation, a contradiction.

For $X \subseteq [k]$, let G_X be the subgraph of G on those edges uv such that $o(u, uv) = X$ or $o(v, uv) = X$. Note that the vertices of $G_X = G_{[k]\setminus X}$ can be properly 2-colored with colors X and $[k] \setminus X$. Therefore G_X is bipartite. Choose a 2-coloring of every G_X; since $G_X = G_{[k]\setminus X}$ we can insist that G_X and $G_{[k]\setminus X}$ get the same 2-coloring. Call this 2-coloring c_X, and observe that any two adjacent vertices u and v get a different color in some c_X, namely $c_{o(u,uv)}$. Thus the product of c_X over every possible $X \subseteq [k]$ gives us a proper coloring of G. Since $c_X = c_{[k]\setminus X}$, the coloring uses $2^{2^{k-1}}$ colors. Therefore $\chi(G) \leq 2^{2^{k-1}}$. □

Now we prove that the lower bound is tight.

Theorem 10 *For any graph G with $\chi(G) \geq 3$, $elb(G) = \lceil \log_2 \log_2 \chi(G) \rceil + 1$.*

Proof. It suffices to show that $elb(G) \leq \lceil \log_2 \log_2 \chi(G) \rceil + 1$, and in particular it suffices to show this when $\chi(G) = 2^{2^\ell}$ for some nonnegative integer ℓ. Proposition 8 tells us that we can assume G is the complete graph on $n = 2^{2^\ell}$ vertices. We proceed by induction. If $\ell = 1$ then $n = 4$ and it is easy to confirm that $elb(K_4) = 2$.

So assume $elb(K_n) = k = \lceil \log_2 \log_2 n \rceil + 1$, let $G = K_n$, and let $\overrightarrow{G_1}, \ldots, \overrightarrow{G_k}$ be a minimum elbow covering of G. We will use this to construct an elbow covering of $G' = K_{n^2}$ as follows. Label the vertices of G as $\{v_i \mid 1 \leq i \leq n\}$, and label the vertices of G' as $\{v_i^j \mid 1 \leq i, j \leq n\}$. For each $\overrightarrow{G_i}$ we construct $\overrightarrow{G_i'}$ such that $\overrightarrow{v_a^b v_c^d} \in \overrightarrow{E}(\overrightarrow{G_i'})$ precisely if $\overrightarrow{v_a v_c} \in \overrightarrow{E}(\overrightarrow{G_i})$, or if $v_a = v_c$ and $\overrightarrow{v_b v_d} \in \overrightarrow{E}(\overrightarrow{G_i})$. Finally, we add an orientation $\overrightarrow{G_{k+1}'}$ such that $\overrightarrow{v_a^b v_c^d} \in \overrightarrow{E}(\overrightarrow{G_{k+1}'})$ precisely if $\overrightarrow{v_a v_c} \in \overrightarrow{E}(\overrightarrow{G_1})$, or if $v_a = v_c$ and $\overrightarrow{v_d v_b} \in \overrightarrow{E}(\overrightarrow{G_1})$.

In other words, we compose $\overrightarrow{G_i}$ with itself for each i, then we compose $\overrightarrow{G_1}$ with its reversal.

Now consider the possibility that v_i^j, v_a^b, v_c^d form a directed path in every orientation of G'. By the construction of our orientations of G', it is easy to see that $|\{i, a, c\}| = 2$. So assume without loss of generality that $i = a \neq c$. Since the edge $v_i^j v_a^b$ will be oriented differently in $\overrightarrow{G_1'}$ and $\overrightarrow{G_{k+1}'}$ and the edge $v_a^b v_c^d$ will be oriented the same, it follows that v_i^j, v_a^b, v_c^d cannot be a directed path in both orientations. Therefore we have an elbow covering of K_{n^2} of size $k+1$, and the theorem follows by induction. $\Box$

This gives us a bound on $\sigma(G)$:

Corollary 11 *For any graph G with $\chi(G) \geq 2$,*

$$\lceil \log_2 \log_2 \chi(G) \rceil + 1 \ \leq \ \sigma(G) \ \leq \ 2\lceil \log_2 \log_2 \chi(G) \rceil + 2.$$

Proof. When $\chi(G) \geq 3$, this follows immediately from Proposition 7 and the previous theorem. It is easy to see that any bipartite graph has orientation covering number at most two: if $V(G)$ is covered by two disjoint stable sets A and B, we simply choose one orientation in which all vertices in A are sources, and one orientation in which all vertices in B are sources. The result follows. $\Box$

We can actually improve the lower bound by exploiting properties of orientation coverings to refine the proof of Theorem 9:

Theorem 12 *Any graph G with $\sigma(G) = k \geq 3$ has $\chi(G) \leq k + 2^{2^{k-1}-k-1}$. Thus*

$$k \geq \log_2 \left(\log_2(\chi(G) - k) + k + 1 \right)$$

Proof. Let G be a minimum counterexample. We can assume G has no vertex of degree 1, since removing such a vertex will change neither $\sigma(G)$ nor $\chi(G)$. Consider an orientation covering $\overrightarrow{G_1}, \ldots \overrightarrow{G_k}$ of G, and let $\ell = k + 2^{2^{k-1}-k-1}$. We will construct an ℓ-coloring of G. As in the proof of Theorem 9, we set $o(u, uv) = \{i \mid \overrightarrow{uv} \in \overrightarrow{E}(\overrightarrow{G_i})\}$ for any incidence (u, uv). First, for $i \in [k]$ let S_i be the set of vertices v having a neighbor u such that $o(v, uv) = \{i\}$. Each S_i is a stable set. Let $S = \cup_i S_i$ and let $U = V(G) \setminus S$. We now proceed to color U using $\ell - k = 2^{2^{k-1}-k-1}$ colors.

We claim that for any adjacent vertices $u, v \in U$, $2 \leq |o(v, uv)| \leq k - 2$. Clearly $o(v, uv)$ cannot be empty or equal to $[k]$ by properties of an orientation covering, since G has minimum degree at least two. And $o(v, uv)$ cannot have size 1 or $k-1$, otherwise either u or v would be in S. Thus there are $2^k - 2k - 2$ possibilities for $o(v, uv)$, and for each possibility we get a bipartite graph, as in the proof of Theorem 9. And again as in the proof of Theorem 9, we actually get a bipartite graph for every complementary pair of subsets of $[k]$. Thus we color U by taking the product of 2-colorings of $2^{k-1} - k - 1$ bipartite subgraphs. This gives us an $(\ell - k)$-coloring of U and an ℓ-coloring of G. □

Although this bound on $\sigma(G)$ may seem ungainly, we will see in Section 4 that it is tight for small values of $\chi(G)$.

Just as we have bounded $\sigma(G)$ and $elb(G)$ in terms of $\chi(G)$, we can bound $\chi(G)$ in terms of $\sigma(G)$ and $elb(G)$.

Corollary 13 *For any graph G with $elb(G) \geq 2$ and $\sigma(G) \geq 3$,*

$$2^{2^{elb(G)-2}} \; < \; \chi(G) \; \leq \; 2^{2^{elb(G)-1}}$$

and

$$2^{2^{(\sigma(G)/2)-2}} \; < \; \chi(G) \; \leq \; \sigma(G) + 2^{(2^{(\sigma(G)-1)} - \sigma(G) - 1)}.$$

3.1 Elbows versus eyebrows

The elbow number of a graph is very closely related to the *eyebrow number* of a graph, studied by Kříž and Nešetřil [5] and defined thusly:

Definition 2 *The eyebrow number $eye_\pi(G)$ of a graph G is the minimum k for which there exist k permutations $\{\pi_i \mid i \in [k]\}$ on $V(G)$ with the following property: For any edge uv of G and third vertex w, there is an i such that $\pi_i(w)$ is not between $\pi_i(u)$ and $\pi_i(v)$.*

The connection between permutations and acyclic orientations is the following. For a permutation π of $[n]$ and a graph G with vertex set $\{v_1, \ldots, v_n\}$, define the following acyclic orientation $\overrightarrow{G_\pi}$ of G: an edge $v_i v_j$ of G is oriented from v_i to v_j in $\overrightarrow{G_\pi}$ precisely if $\pi(i) < \pi(j)$. Conversely, an acyclic orientation of G is a partial order on its vertices, and any linear extension of this order corresponds to a permutation of $[n]$. In [5], Kříž and Nešetřil proved that $eye_\pi(K_n) = \lceil \log_2 \log_2 n \rceil + 1$ using the tightness of a result of Erdős and Szekeres [4] that is closely related to our proof of the upper bound on $elb(G)$. This can be used to provide an alternative proof of the upper bound in Corollary 11. Like us, they were motivated by a different problem. They were interested in proving the existence of posets of bounded dimension whose Hasse diagrams could have arbitrarily high chromatic number. It follows immediately from Theorem 10 that for a complete graph, the eyebrow number and elbow number are equal. However, the eyebrow number is not monotonic under homomorphism – there are graphs for which $eye_\pi(G) = eye_\pi(K_{\chi(G)}) + 1 = elb(K_{\chi(G)}) + 1 = elb(G) + 1$ (for example, a sufficiently large complete and regular tripartite graph [5]). More fundamentally, the eyebrow number of a graph does not really reflect the structure of incident edge pairs. This is the first reason behind our interest in the elbow number as opposed to the eyebrow number.

The second reason is that we do not want to restrict ourselves to acyclic orientations. McClain [8] asked whether, for any n, $L(K_n)$ has a minimum equivalence covering in which every equivalence subgraph is the analogue of an acyclic orientation of K_n (recall we defined an analogue in Section 2). Theorem 9 answers the corresponding question in the affirmative for elbow coverings. That is, using orientations with cycles does not help in constructing a minimum elbow covering. However the question remains open for the orientation covering number.

4 Tightness and complexity

Let us consider our upper bound on $\sigma(G)$, which we got from the bound on $elb(G)$. Corollary 11 implies that $\sigma(K_{16}) \leq 6$, but as one might expect, this bound is not tight. The five (acyclic) orientations of K_{16} associated to the following five permutations show that $\sigma(K_{16}) \leq 5$:

i	1	2	3	4	5	6	7	8
$\pi_1(i)$	1	2	3	4	5	6	7	8
$\pi_2(i)$	13	11	10	6	4	9	5	3
$\pi_3(i)$	14	11	10	3	8	12	5	7
$\pi_4(i)$	15	7	8	9	6	4	3	12
$\pi_5(i)$	16	5	4	10	11	3	12	6
i	9	10	11	12	13	14	15	16
$\pi_1(i)$	9	10	11	12	13	14	15	16
$\pi_2(i)$	7	2	12	8	14	15	16	1
$\pi_3(i)$	2	9	4	6	15	16	1	13
$\pi_4(i)$	10	11	5	2	16	1	13	14
$\pi_5(i)$	9	7	2	8	1	13	14	15

Corollary 13 tells us that if $\sigma(G) = 3$, then $3 \leq \chi(G) \leq 4$ – the lower bound comes from the easy fact that $\sigma(G) \leq 2$ precisely if G is bipartite. The converse is also true by Corollary 6 and the orientation covering of size three of K_4 depicted in Figure 1. If $\sigma(G) = 4$ then Corollary 13 tells us that $\chi(G) \leq 12$. This is tight as well – an example due to McClain [8] implies that $\sigma(K_{12}) = 4$. So there is some evidence that the improved bound of Theorem 12 may be tight or nearly tight in general. As a consequence of these observations, we obtain the following two equivalences:

Theorem 14 *A graph G has $\sigma(G) = 3$ precisely if $3 \leq \chi(G) \leq 4$, and $\sigma(G) = 4$ precisely if $5 \leq \chi(G) \leq 12$.*

Blokhuis and Kloks [2] proved that $eq(G)$ is NP-complete to compute, even if it is at most four and G has maximum degree at most six and clique number at most three. As proved by Maffray and Preissmann [6], it is NP-complete to decide whether or not G is k-colorable for $k \geq 3$, even when G is triangle-free. As a consequence, $\sigma(G)$ is difficult to compute, as is $eq(L(G))$:

Theorem 15 *It is NP-complete to decide whether or not a triangle-free graph G has $\sigma(G) \leq 3$ (resp. $\sigma(G) \leq 4$). Equivalently, it is NP-complete to decide whether or not $eq(L(G)) \leq 3$ (resp. $eq(L(G)) \leq 4$).*

In fact, we conjecture that this also holds for all larger values of σ:

Conjecture 1 *For any $k \geq 3$, it is NP-complete to decide whether or not $\sigma(G) \leq k$.*

5 Conclusion

Theorem 4 implies that for any graph G, $\frac{1}{3}\sigma(G) \leq eq(L(G)) \leq \sigma(G)$. Applying Proposition 7, we obtain $\frac{1}{3}\,elb(G) \leq eq(L(G)) \leq 2\,elb(G)$. If $\chi(G) \geq 3$, Theorem 10 tells us that

$$\tfrac{1}{3}\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right) \leq eq(L(G)) \leq 2\left(\lceil \log_2 \log_2 \chi(G) \rceil + 1\right).$$

As a consequence, $eq(L(G))$ is unbounded, answering a question of [7]. Further, as the chromatic number is unbounded for triangle-free graphs, $eq(L(G))$ is not bounded above by three; this disproves a conjecture in [8]. The tighter Theorem 12 implies that

$$\log_2\left(\log_2\left(\chi(G) - 3eq(L(G))\right) + 3eq(L(G)) + 1\right) \leq eq(L(G)).$$

Otherwise if G is bipartite, then $eq(L(G)) = \sigma(G) = 2$, so the the previous inequalities still hold. Finally if $\chi(G) = 1$, the graph $L(G)$ has no vertices and the inequalities are meaningless.

There are several compelling problems that remain to be solved. First is an improved bound on $\sigma(K_n)$. We believe that it is closer to the lower bound than the upper bound, and we even think that the lower bound might be tight. The second question is that of bounding $\sigma(G)$ in terms of $eq(L(G))$. We suspect that they differ by at most an additive constant for any graph. Finally, we would like to know if there is some graph G for which $\sigma(G) < \sigma(K_{\chi(G)})$.

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