Note on bipartite graph tilings

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Abstract

Let $s < t$ be two fixed positive integers. We study what are the minimum degree conditions for a bipartite graph G, with both color classes of size $n = k(s+t)$, which ensure that G has a $K_{s,t}$-factor. Exact result for large n is given.

Our result extends the work of Zhao, who determined the minimum degree threshold which guarantees that a bipartite graph has a $K_{s,s}$-factor.

1 Introduction

For two (finite, loopless, simple) graphs H and G, we say that G contains an *H-factor* if there exist $v(G)/v(H)$ vertex-disjoint copies of H in G. As a synonym, we say that there exists an *H-tiling of G*. Obviously, if G contains an H-factor, then $v(G)$ is a multiple of $v(H)$. For a fixed graph H, necessary and sufficient conditions on the minimum-degree of G which guarantee that G contains an H-factor were studied extensively. Results in this spirit include the Tutte 1-factor Theorem (see [7]), the Hajnal-Szemerédi Theorem [4], and series of improving results by Alon and Yuster [1, 2], Komlós [5], Zhao and Shokoufandeh [8], and by Kühn and Osthus [6]. In [6] the answer to the problem is settled for any H and a large graph G. It

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was shown that the relevant parameters are the chromatic number and the critical chromatic number of H.

The additional information that G is r-partite might help to decrease the minimum-degree threshold for G containing an H-factor. There has been considerable attention devoted to study of this area. The conjectured r-partite version of the Hajnal-Szemerédi Theorem [3] is such an example. (Let us mention that recently, a proof of the approximate version of the r-partite Hajnal-Szemerédi Theorem was announced by Csaba.) In this paper we determine the threshold for the minimum-degree of a balanced bipartite graph G which guarantees that G contains a $K_{s,t}$-factor, for arbitrary integers $s < t$. If the cardinalities of the two color-classes of G are both equal to n, a necessary condition for G having a $K_{s,t}$-factor is that n is a multiple of $s+t$. The sufficient minimum-degree condition is given in Theorem 3, and a matching lower bound is provided in Theorem 4. Our work can be seen as an extension of the work of Zhao [9], who investigated the case $s = t$.

Theorem 1 (Zhao, [9]). *For each $s \geq 2$ there exists a number k_0 such that if $G = (A, B; E)$ is a bipartite graph, $|A| = |B| = n = ks$, where $k > k_0$, and*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } k \text{ is even,} \\ \frac{n+3s}{2} - 2 & \text{if } k \text{ is odd,} \end{cases}$$

then G has a $K_{s,s}$-factor.

Moreover, Zhao showed that the bound in Theorem 1 is tight.

Theorem 2 (Zhao, [9]). *For each $s \geq 2$ there exists a number k_0 such that for any $k \in \mathbb{N}, k > k_0$, there exists a bipartite graph $G = (A, B; E)$, with $|A| = |B| = ks = n$, such that*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } k \text{ is even,} \\ \frac{n+3s}{2} - 3 & \text{if } k \text{ is odd,} \end{cases}$$

and G does not have a $K_{s,s}$-factor.

Our main results are the two theorems listed below.

Theorem 3. *Let $1 \leq s < t$ be fixed integers. There exists a number $k_0 \in \mathbb{N}$ such that if $G = (A, B; E)$ is a bipartite graph, $|A| = |B| = n = k(s+t)$, with $k > k_0$, and*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } k \text{ is even,} \\ \frac{n+t+s}{2} - 1 & \text{if } k \text{ is odd,} \end{cases}$$

then G has a $K_{s,t}$-factor.

On the other hand, we show that these bounds are best possible.

Theorem 4. *Let $1 \leq s < t$ be fixed integers. There exists a number $k_0 \in \mathbb{N}$ such that for every $k > k_0$ there exists a bipartite graph $G = (A, B; E)$, $|A| = |B| = k(s+t) = n$, such that*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } k \text{ is even,} \\ \frac{n+t+s}{2} - 2 & \text{if } k \text{ is odd,} \end{cases}$$

and G does not have a $K_{s,t}$-factor.

The bounds in Theorem 3 and 4 exhibit a somewhat surprising phenomenon: for the case when k is even the bound is free of the value t, while for the case k is odd, the minimum-degree condition depends on t.

Given a graph G and two disjoint sets $A, B \subset V(G)$ we define

$$\delta(A, B) = \min\{\deg(v, B) : v \in A\}, \quad \Delta(A, B) = \max\{\deg(v, B) : v \in A\},$$

and $\quad d(A, B) = \frac{e(A,B)}{|A||B|}$.

2 Lower bound

In this section we prove Theorem 4. We treat three cases (based on the parity of k, and on the relation between s and t) separately. The proof of Theorem 4 is constructive, i.e., we will construct a graph G with the demanded minimum-degree and then argue that G does not have a $K_{s,t}$-factor.

The building blocks of our constructions are the graphs $P(m, p)$, where $m, p \in \mathbb{N}$. The graphs $P(m, p)$ were introduced in [9]. We just state their properties, which will be used throughout the section. The keen reader may find a proof of their existence in [9].

Lemma 5. *For any $p \in \mathbb{N}$ there exists a number m_0 such that for any $m \in \mathbb{N}, m > m_0$ there exists a bipartite graph $P(m, p) = (P_1, P_2; E_P)$ satisfying*

- $|P_1| = |P_2| = m$,
- *$P(m, p)$ is p-regular, and*
- *$P(m, p)$ does not contain a copy of $K_{2,2}$.*

In all constructions we assume that n is large enough.

2.1 Case k is even

For two integers m and q we write $Q(m, q)$ to denote (any of possibly many) bipartite graph $Q(m, q) = (Q_1, Q_2; E_Q)$ with the following properties:

- $|Q_1| = m, |Q_2| = m - 2$,
- $Q(m, q)$ does not contain any $K_{2,2}$,
- $\deg(x) \in \{q - 1, q\}$ for any vertex $x \in Q_1$, and
- $\deg(y) = q$ for any vertex $y \in Q_2$.

Such graphs $Q(m, q)$ do exist for fixed q and large m. One way to construct them is to take the graph $P(m, q) = (P_1, P_2; E_P)$ from Lemma 5, find two vertices $w_1, w_2 \in P_2$ such that they do not share a common neighbor in P_1, and then take $Q(m, q)$ to be the subgraph of $P(m, q)$ induced by the vertex sets P_1, $P_2 \setminus \{w_1, w_2\}$. In particular, the graph $Q(m, 0)$ is the empty graph.

Now we describe the construction of the graph G. Partition $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_1| = \frac{n}{2} + 1$, $|A_2| = |B_2| = \frac{n}{2} - 1$. The graph G is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$, and
- $G[A_1, B_2] \cong G[B_1, A_2] \cong Q(n/2 + 1, s - 1)$.

We have $\delta(G) = \frac{n}{2} + s - 2$. The fact that there exists no $K_{s,t}$-factor is implied immediately by the fact that there is no subgraph isomorphic to $K_{s,t}$ whose vertices would touch both A_1 and B_2, or A_2 and B_1.

2.2 Case k is odd, $t > s + 1$

Let $k = 2l + 1$, $n = k(s + t)$. Note that $\frac{n-t+s+2}{2}$ is an integer. Partition $A = A_1 + A_2 + A_*$, $B = B_1 + B_2 + B_*$, $|A_1| = |A_2| = |B_1| = |B_2| = \frac{n-t+s+2}{2}$, $|A_*| = |B_*| = t - s - 2$. The graph G is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$,
- $G[A_*, B_i]$ and $G[B_*, A_i]$ are complete bipartite graphs for $i = 1, 2$,
- $G[A_1, B_2] \cong G[A_2, B_1] \cong P(\frac{n-t+s+2}{2}, s - 1)$,
- the graph $G[A_*, B_*]$ is empty.

We have $\delta(G) = \frac{n+t+s}{2} - 2$. To see that G does not have a $K_{s,t}$-factor, we argue as follows. Suppose for contradiction that G has a $K_{s,t}$-factor. Fix a $K_{s,t}$-factor of G. First, observe that there cannot be a copy isomorphic to $K_{s,t}$ touching both $A_1 \cup B_1$ and $A_2 \cup B_2$. Let r_1 and r_2 be the number of copies of $K_{s,t}$ in the tiling whose color class of size t touches A_1 and B_1, respectively. Let A_c and B_c be vertices covered by these $r_1 + r_2$ copies. It holds

$$\begin{aligned} A_1 \subset A_c \subset A_1 \cup A_* \text{ , and} \\ B_1 \subset B_c \subset B_1 \cup B_* \text{ .} \end{aligned} \tag{1}$$

If $r_1 \neq r_2$ then $||A_c| - |B_c|| \geq t - s$, which contradicts (1). Thus, $r_1 = r_2$. We conclude that

$$\frac{l(s+t)+s+1}{s+t} \leq r_1 \leq \frac{l(s+t)+t-1}{s+t},$$

a contradiction to the integrality of r_1.

2.3 Case k is odd, $t = s + 1$

By $R(m, q)$ we denote (any of possibly many) bipartite graph $R(m, q) = (R_1, R_2; E_R)$ with the following properties:

- $|R_1| = m, |R_2| = m - 1$,
- $R(m, q)$ does not contain any $K_{2,2}$,
- for any vertex x in R_1, it holds $\deg(x) \in \{q - 1, q\}$, and
- for any vertex y in R_2, it holds $\deg(y) = q$.

For fixed q and large m existence of such graph $R(m, q)$ follows by a construction analogous to the construction of the graph $Q(m, q)$.

Let $k = 2l + 1$. Partition $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_1| = l(s + t) + s$, $|A_1| = |B_1| = l(s + t) + s + 1$. The graph G is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$,
- $G[B_2, A_1] \cong G[A_2, B_1] \cong R((n + 1)/2, s - 1)$.

One immediately sees that $\delta(G) = \frac{n+t+s}{2} - 2$ and no $K_{s,t}$-tiling of G exists.

3 Upper bound

We prove Theorem 3 in this section. The proof of Theorem 3 utilizes heavily previous work of Zhao [9]. We will need the following technical lemma, which allows us to find many vertex disjoint copies of certain stars. For $h \in \mathbb{N}$, an h-star is a graph $K_{1,h}$, its *center* is the unique vertex in the part of size one.

Lemma 6 (Zhao, [9]). *Let $1 \le h \le \delta \le M$ and $0 < c < 1/(6h+7)$. Suppose that $H = (U_1, U_2; E_H)$ is a bipartite graph such that $||U_i| - M| \le cM$ for $i = 1, 2$. If $\delta = \delta(U_1, U_2) \le cM$ and $\Delta = \Delta(V_2, V_1) \le cM$, then we can find a family of vertex-disjoint h-stars, $2(\delta - h + 1)$ of which have centers in U_1 and $2(\delta - h + 1)$ of which have centers in U_2.*

As in [9] we distinguish between an extremal and a non-extremal case. If we find a $K_{s+t,s+t}$-factor in G we are done, as each copy of $K_{s+t,s+t}$ can be split into two copies of $K_{s,t}$ and hence we have a $K_{s,t}$-factor. Thus the theorem stated next is just a corollary of Theorem 9 in [9].

Theorem 7 (Zhao, [9]). *For every $\alpha > 0$ and positive integers $s < t$, there exist $\beta > 0$ and a positive integer k_0 such that the following holds for all $n = k(s+t)$ with $k > k_0$. Given a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$, if $\delta(G) > (\frac{1}{2} - \beta)n$, then either G contains a $K_{s,t}$-factor, or there exist*

$$A_1 \subset A, \quad B_1 \subset B \quad \text{such that} \quad |A_1| = |B_1| = \lfloor n/2 \rfloor, \quad d(A_1, B_1) < \alpha.$$

Therefore, we reduce the problem to the extremal case. Let $\alpha = \alpha(t) > 0$ be small. As in the proof of Theorem 11 in [9], define

$$\begin{aligned}
A'_1 &= \left\{ x \in A : \deg(x, B_1) < \alpha^{\frac{1}{3}}, \tfrac{n}{2} \right\} & B'_1 &= \left\{ x \in B : \deg(x, A_1) < \alpha^{\frac{1}{3}}, \tfrac{n}{2} \right\} \\
A'_2 &= \left\{ x \in A : \deg(x, B_1) > (1 - \alpha^{\frac{1}{3}}) \tfrac{n}{2} \right\} & B'_2 &= \left\{ x \in B : \deg(x, A_1) > (1 - \alpha^{\frac{1}{3}}) \tfrac{n}{2} \right\} \\
A_0 &= A - A'_1 - A'_2 & B_0 &= B - B'_1 - B'_2 \\
G_1 &= G[A'_1, B'_1] & G_2 &= G[A'_2, B'_2].
\end{aligned}$$

Similarly as in the proof of Theorem 11 in [9], we assume that removing any edge from G would violate the minimum-degree condition and then change A'_i and B'_i a little so that $\Delta(G_1), \Delta(G_2) < \alpha^{\frac{1}{9}} n$. Vertices in $A_0 \cup B_0$ are called *special.*

3.1 k is even

To exhibit the existence of a tiling in this case, it is sufficient to translate carefully the proof of Case I of Theorem 11 from [9]. We give a sketch the proof below and refer the reader to the corresponding places in [9] for more details.

Set $\mathcal{V} = (A'_1, B'_1, A'_2, B'_2)$. First assume, that no member of $\mathcal{V}$ contains more than $n/2$ vertices. We add vertices from A_0 and B_0 into sets of $\mathcal{V}$ in such a way, that every set has size exactly $n/2$. Then, we may apply arguments used in [9], based on Hall's Marriage Theorem, to find a $K_{s+t,s+t}$ tiling.

Next, assume that there is only one set in $\mathcal{V}$ which has more than $n/2$ elements. Without loss of generality, assume that it is A'_2, i.e., $|A'_2| = c > n/2$. Lemma 6 applied to the graph $G[A'_2, B'_2]$ yields the existence of $c-n/2$ disjoint s-stars with centers in A'_2. We move the centers of the stars into A'_1 and extend each of the stars into a copy of $K_{s,t}$ (each of these copies lies entirely in $A'_1 \cup B'_2$, with the color class of size s being contained in B'_2). We distribute vertices of B_0 into B'_1 and B'_2 so, that $|B'_1| = |B'_2| = n/2$. Then, it is easy to finish the entire tiling. This is done in three steps. In the first step, we find in an arbitrary manner $c - n/2$ copies of $K_{s,t}$ (disjoint with the previous ones) in $G[A'_1, B'_2]$ placed in such a way, that the color-class of size s lies in A'_1. This is obviously always possible. This step ensures us, that the cardinalities of untiled (i.e., those vertices which are not covered by the partial $K_{s,t}$-factor) vertices in the both color-classes of $G[A'_1, B'_2]$ are equal and divisible by $s + t$. In the second step, all yet untiled vertices of $G[A'_1, B'_2]$ which were originally special vertices are tiled. In the third step, the tiling is in an analogous manner defined for $G[A'_2, B'_1]$.

Now, assume that two diagonal sets of $\mathcal{V}$, say A'_2 and B'_1 have sizes more than $n/2$. Then we apply separately Lemma 6 to $G[A'_2, B'_2]$ and $G[A'_1, B'_1]$ to obtain families $\mathcal{S}_A$ and $\mathcal{S}_B$ of disjoint s-stars with centers in A'_2 and B'_1, such that $|A'_2| - |\mathcal{S}_A| = |B'_1| - |\mathcal{S}_B| = n/2$. We move the centers of the stars to A'_1 and B'_2 and proceed as in the previous case.

The remaining case is when two non-diagonal sets from $\mathcal{V}$ have size more than $n/2$. Assume these are A'_2 and B'_1. We apply Lemma 6 to the graph $G[A'_2, B'_2]$ to obtain families $\mathcal{S}_A, \mathcal{S}_B$ of disjoint s-stars with centers in A'_2 and B'_2, such that $|A'_2| - |\mathcal{S}_A| = |B'_2| - |\mathcal{S}_B| = n/2$. We proceed as in the previous cases.

3.2 k is odd

Let $k = 2l + 1$. We say that a set of special vertices (A_0 and/or B_0) is *small* if its size is less than $t - s$. Otherwise, it is called *big*.

We distinguish four cases.

- *Both A_0 and B_0 are small.* Then there exist $i, j \in \{1, 2\}$, such that $|A'_i|, |B'_j| \geq l(s+t)+s+1$. If $i = j$, then we apply Lemma 6 to the graph G_i and find families $\mathcal{S}_A$, $\mathcal{S}_B$ of pairwise disjoint s-stars with centers in A'_i and B'_i respectively, so that $|A'_i| - |\mathcal{S}_A| = |B'_i| - |\mathcal{S}_B| = l(s+t)+s$. Move the centers of the stars in A'_{3-i} and B'_{3-i}. After the changes we shall tile two graphs: $G[A'_1, B'_2]$ and $G[A'_2, B'_1]$. Note, that both those graphs are not balanced. The tiling procedure is analogous to the previous cases (when k is even); the only difference is that one copy of $K_{s,t}$ has to be found in the graphs first to make each of them balanced.

  If $i \neq j$, we can assume that $|A'_j|, |B'_i| \leq l(s+t)+s$. Because, if not then we could change one index and continue as in the case when $i = j$. We will show that one can add vertices to A'_j and to B'_i so that $|A'_j| = l(s+t)+s$ and $|B'_i| = l(s+t)+t$. Then, the existence of the tiling will follow by standard arguments. We apply Lemma 6 to the graph G_j to obtain a family of $|B'_j| - (l(s+t)+s)$ vertex disjoint s-stars with centers in B'_j and end-vertices in A'_j. If we moved all the centers to B'_i and all the vertices of B_0, the cardinality of B'_i would be
  $$|B'_i| + (|B'_j| - (l(s+t)+s)) + |B_0| = l(s+t)+t\ .$$
  The same applies for A'_j. Therefore, by removing some of the vertices, we may attain $|A'_j| = l(s+t)+s$ and $|B'_i| = l(s+t)+t$. Then, the existence of a tiling follows.

- *A_0 is small and B_0 is big.* Then at least one B'_i (say B'_2) has at most $l(s+t)+s$ vertices. Lemma 6 asserts that we can find a family $\mathcal{S}_B$ of disjoint s-stars with centers in B'_1 and end-vertices in A'_1, such that $|B'_1| - |\mathcal{S}_B| \leq l(s+t)+s$. This implies, that we can find vertices (in B_0 or centers of the stars of $\mathcal{S}_B$) which can be moved to B'_2 so that $|B'_2| = l(s+t)+t$.

  As A_0 is small, one of A'_1 and A'_2 must have at least $l(s+t)+s+1$ vertices. The tiling can be found by standard arguments if we achieve to have $|A'_1| = l(s+t)+s$. If $|A'_1| > l(s+t)+s$, Lemma 6 yields existence of a family $\mathcal{S}_A$ of disjoint s-stars with centers in A'_1 and end-vertices in B'_1 such that $|A'_1| - |\mathcal{S}_A| = l(s+t)+s$. Moving the centers to A'_2, we achieve $|A'_1| = l(s+t)+s$. Assume that $|A'_1| \leq l(s+t)+s$. The size of A'_2 is $k(s+t) - |A'_1| - |A_0| > l(s+t)+s$. Lemma 6 yields existence of a family $\mathcal{S}_A$ of disjoint s-stars in G_2 centered in A'_2 with the property that $|A'_1| + |\mathcal{S}_A| = l(s+t)+s$. Moving the centers to A'_1 yields demanded $A'_1 = l(s+t)+s$.

- *A_0 is big and B_0 is small.* The analysis of this case is analogous to the previous one.

- *Both A_0 and B_0 are big.* We shall show in the next paragraph, that we can achieve A'_1 to be of size $l(s+t)+s$ and of size $l(s+t)+t$. An analogous procedure can be used to show the same property for the set B'_1. Then, the existence of the tiling follows immediately; one prescribes the cardinalities of A'_1 and B'_1 to be equal to the same number $l(s+t)+s$.

  If $|A'_i \cup A_0| < l(s+t)+t$ for some $i \in \{1,2\}$, then we have $|A'_{3-i}| > l(s+t)+s$. Appealing to Lemma 6 we can remove centers of s-stars from A'_{3-i} in such a way that $|A'_{3-i}| = l(s+t)+s$. Also, by moving $t-s$ vertices from the big set A_0 to A'_{3-i} arrive at $|A'_{3-i}| = l(s+t)+t$. Then, the partial $K_{s,t}$-tiling can be extended to a $K_{s,t}$-factor.

  Finally, if both $|A'_1| \leq l(s+t)+s$ and $|A'_2| \leq l(s+t)+s$ then we redistribute some vertices (again, appealing to Lemma 6, and using the set A_0) to arrive at the situation when $|A'_1| = l(s+t)+s$, $|A'_2| = l(s+t)+t$. Then the tiling can be extended as before.

Acknowledgement

We thank a careful referee for suggesting several improvements in the presentation.

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