Edge-face colouring of plane graphs with maximum degree nine

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Abstract

An edge-face-colouring of a plane graph with edge set E and face set F is a colouring of the elements of $E \cup F$ so that adjacent or incident elements receive different colours. Borodin proved that every plane graph of maximum degree $\Delta \geq 10$ can be edge-face-coloured with $\Delta + 1$ colours. We extend Borodin's result to the case where $\Delta = 9$.

1 Introduction

Let G be a plane graph with vertex set V, edge set E and face set F. Given a positive integer k, a *k-edge-face-colouring of G* is a mapping $\lambda : E \cup F \to \{1, 2, \ldots, k\}$ such that

(*i*) $\lambda(e) \neq \lambda(e')$ for every pair (e, e') of adjacent edges;

(*ii*) $\lambda(e) \neq \lambda(f)$ for edge e and every face f incident to e;

(*iii*) $\lambda(f) \neq \lambda(f')$ for every pair (f, f') of adjacent faces with $f \neq f'$.

The requirement in (*iii*) that f and f' be distinct is only relevant for graphs containing a cut-edge; such graphs would not have an edge-face colouring otherwise.

Edge-face colourings appear to have been first studied by Jucovič [8] and Fiamčík [7], who considered 3- and 4-regular graphs. Mel'nikov [10]

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conjectured that every plane graph of maximum degree Δ has a $(\Delta+3)$-edge-face-colouring. This was proved by Borodin [2, 4] for $\Delta \leq 3$ and $\Delta \geq 8$, and the general case was proved by Waller [14], and independently by Sanders and Zhao [11]. In fact, Borodin [4] proved the upper bound of $\Delta+1$ for plane graphs of maximum degree $\Delta \geq 10$. The bound is tight, as can be seen by considering trees. Borodin asked [4, Problem 9] to determine the exact upper bound for plane graphs with maximum degree $\Delta \leq 9$. We solve the problem for the case $\Delta = 9$ by proving the following theorem.

Theorem 1. *Every plane graph of maximum degree* 9 *has a* 10*-edge-face-colouring.*

Borodin's problem remains open for graphs of maximum degree $\Delta \in \{4, 5, \ldots, 8\}$. Note that Sanders and Zhao [12] have proved that plane graphs of maximum degree $\Delta \geq 7$ are $(\Delta+2)$-edge-face colourable.

Let us briefly mention the closely related concept of *total colouring*: given a graph $G = (V, E)$, we colour the elements of $V \cup E$ so that adjacent or incident elements receive different colours. The well-known *Total Colouring Conjecture* of Behzad [1] and Vizing [13] states that every graph of maximum degree Δ admits a $(\Delta+2)$-total-colouring. For planar graphs of maximum degree $\Delta \geq 14$, Borodin [3] proved the stronger bound $\Delta+1$. This bound was subsequently extended to graphs of maximum degree $\Delta \in \{9, 10, \ldots, 13\}$ by Borodin, Kostochka and Woodall [5, 6], by Wang [15], and by Kowalik, Sereni and Škrekovski [9].

We prove Theorem 1 by contradiction. From now on, we let $G = (V, E, F)$ be a counter-example to the statement of Theorem 1 with as few edges as possible. That is, G is a plane graph of maximum degree 9 and no 10-edge-face-colouring, but every plane graph of maximum degree 9 with less than $|E|$ edges has a 10-edge-face-colouring. In particular, for every edge $e \in E$ the plane subgraph $G - e$ of G has a 10-edge-face-colouring. First, we establish various structural properties of G in Section 2. Then, relying on these properties, we use the Discharging Method in Section 3 to obtain a contradiction.

In the sequel, a vertex of degree d is called a *d-vertex*. A vertex is a *$(\leq d)$-vertex* if its degree is at most d; it is a *$(\geq d)$-vertex* if its degree is at least d. The notions of *d-face*, *$(\leq d)$-face* and *$(\geq d)$-face* are defined analogously as for the vertices, where the *degree* of a face is the number of vertices incident to it. A face of length 3 is called a *triangle*. For integers a, b, c, an *$(\leq a, \leq b, \leq c)$-triangle* is a triangle xyz of G with $\deg(x) \leq a, \deg(y) \leq b$ and

$\deg(z) \le c$. The notions of $(a, \le b, \le c)$-triangles, $(a, b, \ge c)$-triangles and so on, are defined analogously.

2 Reducible configurations

In this section, we establish some structural properties of the graph G. In particular, we prove that some plane graphs are *reducible configurations*, i.e. they cannot be part of the chosen embedding of G.

For convenience, we sometimes define configurations by depicting them in figures. We use the following conventions: 2- and 3-vertices are depicted by small black bullets and black triangles, respectively; vertices of degree at most 5 are represented by black pentagons, and white bullets represent vertices of degree at least as large as the one shown in the figure (and made precise in the text, if necessary). Furthermore, the colour of a face is shown in a box on that face, to avoid confusion with the colours of the edges.

Let λ be a (partial) 10-edge-face-colouring of G. For each element $x \in E \cup F$, we define $\mathcal{C}(x)$ to be the set of colours (with respect to λ) of the edges and faces incident or adjacent to x. Also, we set $\mathcal{F}(x) := \{1, 2, \dots, 10\} \setminus \mathcal{C}(x)$. If $x \in V$ we define $\mathcal{E}(x)$ to be the set of colours of the edges incident to x. Moreover, λ is *nice* if only some (≤ 4)-faces are uncoloured. Observe that every nice colouring can be greedily extended to a 10-edge-face-colouring of G, since $|\mathcal{C}(f)| \le 8$ for each (≤ 4)-face f, i.e. f has at most 8 forbidden colours. Therefore, in the rest of the paper, we shall always suppose that such faces are coloured at the very end. More precisely, every time we consider a partial colouring of G, we uncolour all (≤ 4)-faces, and implicitly colour them at the very end of the colouring procedure of G. We make the following observation about nice colourings.

Observation. Let e be an edge incident to two faces f and f'. There exists a nice colouring λ of $G - e$, and hence a partial 10-edge-face-colouring of G in which only e and f are uncoloured. Moreover, if f is an (≤ 4)-face, then it suffices to properly colour the edge e with a colour from $\{1, 2, \dots, 10\}$ to extend λ to a nice colouring of G.

This observation is used throughout the paper.

Lemma 2. *The graph G has the following properties.*

(*i*) *Let v be a vertex of G, and $v_1, v_2, \ldots, v_d$ its neighbours in clockwise order in the embedding of G. If v is a cut-vertex of G, then no component C of $G - v$ is such that the neighbourhood of v in C is contained in $\{v_i, v_{i+1}\}$ for some $i \in \{1, 2, \ldots, d\}$, where the index i is taken modulo d. In particular, G has no cut-edge.*

(*ii*) *If uv is an edge incident to a* 5*-face then* $\deg(u) + \deg(v) \geq 10$.

(*iii*) *Let uv be an edge, and let $x \in \{1, 2\}$ be the number of* ($\leq$4)*-faces incident to uv. Then* $\deg(u) + \deg(v) \geq 10 + x$.

Proof. (*i*). Suppose on the contrary that C is a component of $G - v$ such that the neighbourhood N of v in C is contained in, say, $\{v_1, v_2\}$.

First, assume that $N = \{v_1, v_2\}$. Then G is the edge-disjoint union of two plane graphs $G_1 = (C \cup \{v\}, E_1)$ and $G_2 = (V \setminus C, E_2)$. The outer face f_1 of G_1 corresponds to a face f_2 of G_2. By the minimality of G, the graph G_i has a 10-edge-face-colouring λ_i for $i \in \{1, 2\}$. Since both vv_1 and vv_2 are incident in G_1 to f_1, we may assume that $\lambda_1(f_1) = 1$, $\lambda_1(vv_1) = 9$ and $\lambda_1(vv_2) = 10$. Regarding λ_2, we may assume that $\lambda_2(f_2) = 1$. Furthermore, up to permuting the colours, we can also assume that the colours of the edges of G_2 incident to v are contained in $\{1, 2, \ldots, 8\}$, since there are at most 7 such edges.

We now define an edge-face-colouring λ of G as follows. For every edge e of G, set $\lambda(e) := \lambda_1(e)$ if $e \in E_1$ and $\lambda(e) := \lambda_2(e)$ if $e \in E_2$. To colour the faces of G, let f be the face of G incident to both vv_1 and vv_d (note that there is only one such face, since otherwise v would have degree 2, which would be a contradiction). Now, observe that there is a natural one-to-one correspondence between the faces of G_1 and a subset F_1 of the face set F of G that maps f_1 to f. Similarly, there is a natural one-to-one correspondence between the faces of G and a subset F_2 of F that maps f_2 to f. Note that $F_1 \cap F_2 = \{f\}$. Now, we can colour every face $f \in F_i$ using λ_i. This is well defined since $\lambda_1(f_1) = \lambda_2(f_2) = 1$.

Let us check that λ is proper. Two adjacent edges of G are assigned different colours. Indeed, if the two edges belong to E_i for some $i \in \{1, 2\}$, then it comes from the fact that λ_i is a proper edge-face-colouring of G_i. Otherwise, both edges are incident with v, and one is in G_1 and the other in G_2. The former is coloured either 9 or 10, and the latter with a colour of $\{1, 2, \ldots, 8\}$ by the choice of λ_1 and λ_2. Two adjacent faces in G necessarily correspond to two adjacent faces in G_1 or G_2, and hence are assigned different colours. Last, let g be a face of G and e an edge incident to g in G. If $g \neq f$, then g and e are incident in G_1 or G_2, and hence coloured differently. Otherwise e is incident to f_i in G_i for some $i \in \{1,2\}$, and hence $\lambda(e) = \lambda_i(e) \neq \lambda_i(f_i) = 1 = \lambda(f)$.

The case where $N = \{v_1\}$, i.e. vv_1 is a cut-edge, is dealt with in the very same way so we omit it.

(ii). Let $e = uv$ be an edge with $\deg(u) + \deg(v) \leq 9$ and let f and f' be the two faces incident to e. Suppose that f is a 5-face. By the minimality of G, the graph $G - e$ has a nice colouring λ. Let f'' be the face of $G - e$ corresponding to the union of the two faces f and f' of G after having removed the edge e. We obtain a partial 10-edge-face-colouring of G in which only e, f and the ($\leq$4)-faces are uncoloured by just assigning the colour $\lambda(f'')$ to f', and keeping all the other assignments. Since f is a 5-face and e is uncoloured, $|\mathcal{C}(f)| \leq 9$. Thus, we can properly colour f. Now, $|\mathcal{C}(e)| \leq \deg(u)+\deg(v)-2+2 \leq 9$. Hence, the edge e can be properly coloured, which yields a nice colouring of G; a contradiction.

(iii). Suppose on the contrary that $\deg(u) + \deg(v) \leq 9 + x$. Let f and f' be the two faces incident to uv, with f being an ($\leq$4)-face. By the minimality of G, the graph $G - e$ has a nice colouring. We obtain a partial 10-edge-face-colouring of G as above (in particular, only e and the ($\leq$4)-faces are uncoloured). Consequently, $|\mathcal{C}(uv)| \leq \deg(u) + \deg(v) - 2 + 2 - x \leq 9$. Hence, we can properly colour the edge uv, thereby obtaining a nice colouring of G; a contradiction. □

Lemma 3. *The graph G satisfies the following assertions.*

(i) *Two vertices of degree 2 are not adjacent.*

(ii) *A triangle is not incident to a 2-vertex.*

(iii) *A 4-face incident to a 2-vertex is not incident to another ($\leq$3)-vertex.*

Proof. (i). Suppose on the contrary that v_1 and v_2 are two adjacent 2-vertices. For $i \in \{1,2\}$, let u_i be the neighbour of v_i other than v_{3-i}. Note that $u_1 \neq u_2$ by Lemma 2(i). In particular v_1 and u_2 are not adjacent. Let G' be the plane graph obtained from G by *suppressing* v_2, i.e. removing v_2 and adding an edge between v_1 and u_2. By the minimality of G, the graph G' has a 10-edge-face-colouring λ'. Observe that λ naturally defines a partial 10-edge-face-colouring λ of G in which only the edge v_1v_2 is uncoloured.

Indeed, every face f of G naturally corresponds to a face f' of G', so setting $\lambda(f) := \lambda'(f')$ yields a proper colouring of the faces of G. If e is an edge not incident to v_2 then e is also an edge of G', and we set $\lambda(e) := \lambda'(e)$. Next, we colour u_2v_2 with $\lambda'(u_1v_2)$. Now, $|\mathcal{C}(v_1v_2)| = \deg(v_1) + \deg(v_2) - 2 + 2 = 4 < 10$, so we can greedily colour v_1v_2, thereby obtaining a 10-edge-face-colouring of G.

(*ii*). Suppose that $f := uvw$ is a 3-face and v a 2-vertex. By Lemma 2(*iii*), the vertices u and v both have degree 9. By the minimality of G, the graph $G' := G - uv$ has a nice colouring. Thus, we obtain a partial 10-edge-face-colouring of G in which only uv and the ($\leq$4)-faces are uncoloured. We want to obtain a nice colouring of G, which would yield a contradiction. In particular, we may assume that $|\mathcal{C}(uv)| = 10$, and thus, up to a permutation of the colours, the colouring is the one shown in Figure 1(a). Let $x \in \{1, 2, \ldots, 7\}$. If $x \notin \mathcal{E}(w)$, then we can recolour the edge vw with x, and colour uv with 9 to obtain a nice colouring of G. Thus, $\mathcal{E}(w) = \{1, 2, \ldots, 9\}$.

Now, let g be the face incident to uw other than f, so the colour of g is α. We assert that $\alpha \neq 10$. To see this, let f' be the face of G' corresponding to the union of the two faces incident to uv. Observe that in G', the faces f' and g are not the same, for otherwise u would be a cut-vertex that contradicts Lemma 2(i). Therefore f' and g are adjacent in G', and thus the assertion holds. Consequently, we can recolour uw with 10 and colour uv with 8 to obtain a nice colouring of G; a contradiction.

(*iii*). Suppose on the contrary that $vuu'v'$ is a 4-face incident to a 2-vertex u and an ($\leq$3)-vertex. By Lemma 2(*iii*), we may assume that v' is an ($\leq$3)-vertex and v and u' are 9-vertices. By the minimality of G, the graph $G-uv$ has a nice colouring, from which we infer a partial 10-edge-face-colouring of G in which only the edge uv and the ($\leq$4)-faces are uncoloured. It suffices to properly colour the edge uv to obtain a nice colouring of G, and hence a contradiction. If we cannot do this greedily, then $|\mathcal{C}(uv)| = 10$ so we can assume without loss of generality that the colouring is the one shown in Figure 1(b) and (c).

Notice that $\{9, 10\} \subset \mathcal{C}(vv')$, otherwise we can recolour vv' with 9 or 10 and then colour uv with 1. Moreover, if there is a colour $x \in \{1, 2, \ldots, 8\} \setminus \mathcal{E}(u')$, we recolour uu' with x and then colour uv with 9, thereby obtaining a nice colouring of G. Thus, $\mathcal{E}(u') = \{1, 2, \ldots, 9\}$. In particular $\delta \notin \{9, 10\}$. So if v' is a 2-vertex then $\{9, 10\} \not\subseteq \mathcal{C}(vv')$; a contradiction.

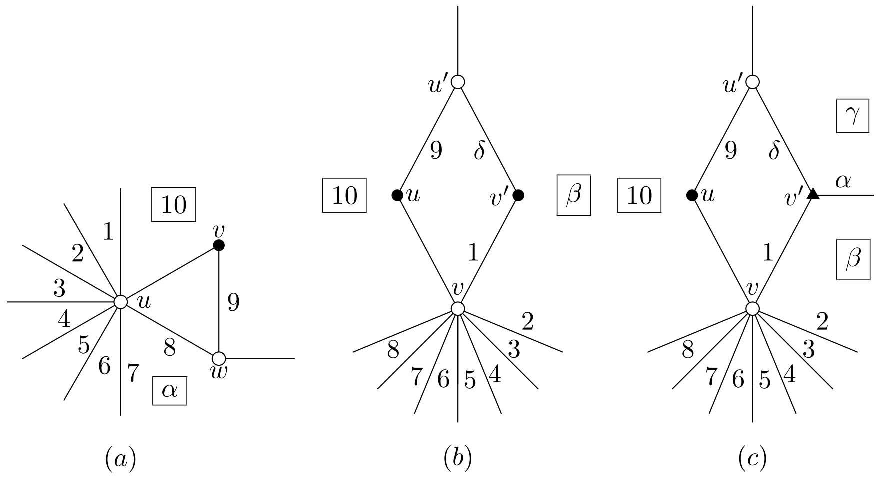


Figure 1: Reducible configurations for the proof of Lemma 3.

Thus, we may now assume that v' is a 3-vertex, and hence $\{9, 10\} = \{\alpha, \beta\}$. Recalling that G has no cut-edge by Lemma 2(i), we deduce that $\gamma \notin \{9, 10\}$. We colour uv with 9 and recolour uu' with δ (recall that $\delta \notin \{9, 10\}$). But now $\mathcal{C}(u'v') = \{1, 2, \ldots, 8, \alpha, \gamma\}$, so we can properly recolour $u'v'$ with $\beta \in \{9, 10\} \setminus \{\alpha\}$. This yields a nice colouring of G, and the desired contradiction. $\square$

Lemma 4. *Let uvw be a 3-face of G such that* $\deg(u) + \deg(v) \leq 11$. *Then* $\deg(w) \geq 8$. *In particular, G has no* $(4, 7, 7)$*-triangle and no* $(5, 6, \leq 7)$*-triangle.*

Proof. Suppose on the contrary that $\deg(w) \leq 7$, and assume without loss of generality that $\deg(u) \leq \deg(v)$. We obtain from a nice colouring of $G - uv$ a partial 10-edge-face-colouring λ of G in which only uv and the ($\leq$4)-faces are uncoloured. To extend λ to a nice colouring of G, it suffices to properly colour the edge uv. If we cannot do this greedily, it means that $|\mathcal{C}(uv)| = 10$. Thus, since $|\mathcal{C}(uv)| \leq \deg(u) + \deg(v) - 2 + 1 = 10$, we deduce that $\mathcal{E}(u) \cap \mathcal{E}(v) = \emptyset$.

We obtain a contradiction by using a counting argument. We assert that $\eta := |\mathcal{E}(u) \cap \mathcal{E}(w)| \leq \deg(u) + \deg(w) - 10$. To see this, observe that if $|\mathcal{C}(uw)| \leq 8$, then uw can be properly recoloured with a colour different from $c := \lambda(uw)$, and subsequently uv can be coloured with c; a contradiction. But $|\mathcal{C}(uw)| = (\deg(u) - 2) + (\deg(w) - 1) - (\eta - 1) + 1$. So $\eta \leq \deg(u) + \deg(w) - 10$, as asserted. Similarly, $\eta' := |\mathcal{E}(v) \cap \mathcal{E}(w)| \leq \deg(v) + \deg(w) - 10$.

Now set $\nu := |\mathcal{E}(u) \cup \mathcal{E}(v) \cup \mathcal{E}(w)|$. Since $\mathcal{E}(u) \cap \mathcal{E}(v) = \emptyset$, we obtain

$$\begin{aligned}\nu =& (\deg(u) - 1) + (\deg(v) - 1) + \deg(w) - \eta - \eta' \\ \geq& 20 - 2 - \deg(w) \\ \geq& 11\,;\end{aligned}$$

a contradiction. □

Lemma 5. *Let v be an 8-vertex of G, and $v_1, v_2, \ldots, v_8$ the neighbours of v in anti-clockwise order in the embedding of G. For $i \in \{1, 2, \ldots, 8\}$, let f_i be the face of G incident to vv_i and to vv_{i+1}, where the index is taken modulo 8. Assume that f_1 is a $(3, 8, \geq 8)$-triangle.*

- *(i) The face f_2 is not a $(3, 8, \geq 8)$-triangle; and*
- *(ii) if the face f_4 is a $(3, 8, \geq 8)$-triangle and f_2 and f_3 are 3-faces, then* $\deg(v_3) \geq 6$.

Proof. We proceed by contradiction in both cases, by assuming that v contradicts the considered statement.

(i). First, note that v_2 cannot be a 3-vertex by Lemma 2(iii). So both v_1 and v_3 are 3-vertices. By the minimality of G, the graph $G - vv_1$ has a nice colouring, which we extend to a partial 10-edge-face-colouring of G in which only vv_1 and the (≤ 4)-faces are uncoloured. We obtain a contradiction by properly colouring vv_1, thereby exhibiting a nice colouring of G. If vv_1 cannot be coloured greedily, then we may assume without loss of generality that the colouring is the one shown in Figure 2(a).

First, observe that $\{\alpha, \beta, \gamma\} = \{8, 9, 10\}$, for otherwise we can recolour vv_3 with $x \in \{8, 9, 10\} \setminus \{\alpha, \beta, \gamma\}$ and then colour vv_1 with 2. Consequently, if $\delta \neq 1$ then we can interchange the colours of vv_2 and v_2v_3, i.e. recolour vv_2 with α and v_2v_3 with 1. Now vv_1 can be properly coloured with 1. Thus, $\delta = 1$.

Since $\beta \neq \gamma$, there exists a colour $c \in \{\beta, \gamma\} \setminus \{9\}$. Note that $c \in \{8, 10\}$ and $c \neq \alpha$. Hence, $c \in \mathcal{E}(v_2) \setminus \{1, 9, \alpha\}$, for otherwise we recolour vv_2 with c and then colour vv_1 with 1 to obtain a nice colouring of G. Similarly, if there is a colour $x \in \{2, 3, \dots, 7\} \setminus (\mathcal{E}(v_2) \cup \{\varepsilon\})$, we recolour v_1v_2 with x and colour vv_1 with 9. Thus, $\mathcal{E}(v_2) \cup \{\varepsilon\} = \{1, 2, \dots, 7, 9, \alpha, c\}$, and $\varepsilon \in \{2, 3, \dots, 7\}$. Since $\deg(v_2) \leq 9$, we deduce that $\varepsilon \notin \mathcal{E}(v_2)$. We recolour v_2v_3 with ε and vv_2 with α. If the obtained colouring is proper, then we colour vv_1 with 2 to obtain a nice colouring of G. Otherwise, we infer that $\varepsilon = 2$. In this case, we recolour vv_3 with 1 and colour vv_1 with 2, which yields a nice colouring of G.

(ii). Lemma 2(iii) implies that none of v_2, v_3 and v_4 is a 3-vertex. So both v_1 and v_5 are 3-vertices. As in (i), we obtain a partial 10-edge-face-colouring of G in which only vv_3 and the ($\leq$4)-faces are uncoloured. Without loss of generality, we assume that the colouring is the one shown in Figure 2(b). Note that $\{8, 9, 10\} \subset \mathcal{E}(v_3)$, for otherwise we could greedily colour vv_3. Hence, at least one of $\{1, 2\}$ and $\{3, 4\}$ is disjoint from $\mathcal{E}(v_3)$. We may assume by symmetry that $\{1, 2\} \cap \mathcal{E}(v_3) = \emptyset$. We can now proceed as in (i). More precisely, we first note that $\{\alpha, \beta, \gamma\} = \{8, 9, 10\}$, since otherwise we could recolour vv_1 with $8, 9$ or 10 and then colour vv_3 with 1. Moreover, if $\delta \neq 2$, then we can interchange the colours of vv_2 and v_1v_2 (i.e. recolour vv_2 with α and v_1v_2 with 2), and colour vv_3 with 2. So $\delta = 2$.

Now we observe that no edge incident to v_2 is coloured 1. Indeed, since $\deg(v_2) \leq 9$, there is a colour x not assigned to an edge incident to v_2. If $x \in \{8, 9, 10\}$, we recolour vv_2 with x and then colour vv_3 with 2. If $x \in \{3, 4, \dots, 7\}$, we recolour v_1v_2 with x, vv_1 with α and then colour vv_3 with 1. Therefore $x = 1$ (since 2 is assigned to vv_2). As a result, we can safely interchange the colours of vv_1 and v_1v_2, and then colour vv_3 with 1. □

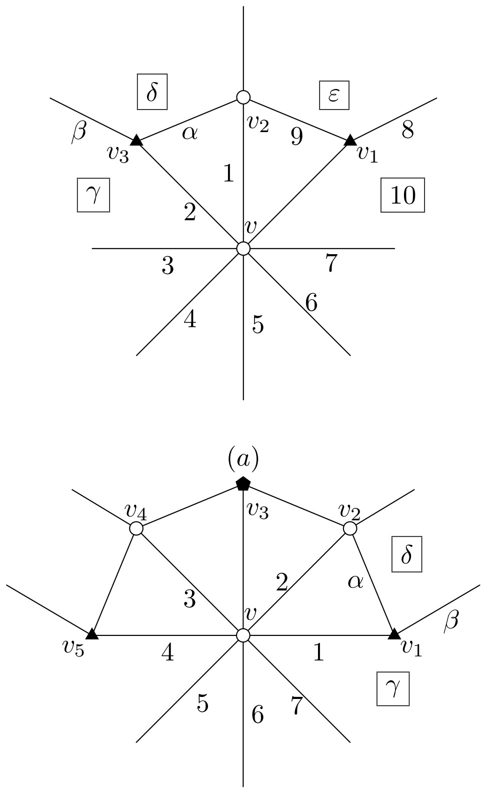
Lemma 6. *The graph G satisfies the following assertions.*

(i) *The configuration of Figure 3(a) is reducible.*

(ii) *The configuration of Figure 3(b) is reducible.*

(iii) *The configuration of Figure 3(c) is reducible.*

Proof. (i). Suppose on the contrary that G contains the configuration of Figure 3(a). By the minimality of G, the graph $G - vv_9$ has a nice colouring, from which we obtain a partial 10-edge-face-colouring with only the edge vv_9 and the ($\leq$4)-faces left uncoloured. It suffices to properly colour the edge vv_9 to obtain a nice colouring of G, which would lead to a contradiction.



(b)

Figure 2: Reducible configurations for the proof of Lemma 5.

If the edge vv_9 cannot be coloured greedily, then $|\mathcal{C}(vv_9)| = 10$, so we may assume the colouring is the one shown in Figure 3(a).

First, note that $\{9, 10\} = \{\alpha, \beta\}$, for otherwise we could recolour vv_7 with 9 or 10 and then colour vv_9 with 7. Similarly, $\beta \in \mathcal{E}(v_8)$, otherwise we recolour vv_8 with β and colour vv_9 with 8. Now $\gamma = 8$, for otherwise we interchange the colours of v_7v_8 and vv_8 (i.e. we recolour v_7v_8 with 8 and vv_8 with α) and next we colour vv_9 with 8. Since $\deg(v_8) \leq 9$ and $\{8, 9, 10\} = \{8, \alpha, \beta\} \subset \mathcal{E}(v_8)$, there exists a colour $x \in \{1, 2, \ldots, 7\} \setminus \mathcal{E}(v_8)$. As $\gamma = 8$, we can recolour v_7v_8 with x and vv_7 with α, and colour vv_9 with 7 to obtain a nice colouring of G.

(ii). Suppose on the contrary that G contains the configuration of Figure 3(b). By the minimality of G, the graph $G - vv_3$ has a nice colouring, from which we obtain a partial 10-edge-face-colouring with only vv_3 and the ($\leq$4)-faces uncoloured. It suffices to properly colour the edge vv_3 to obtain a nice colouring of G, and therefore a contradiction. If the edge vv_3 cannot be coloured greedily, then $|\mathcal{C}(vv_3)| = 10$ and thus we may assume that the colouring is the one shown in Figure 3(b).

First, note that $\{9, 10\} \subseteq \{\alpha, \beta, \gamma\}$, for otherwise we could recolour vv_1 with 9 or 10 and then colour vv_3 with 7. Furthermore, $10 \in \mathcal{E}(v_2)$, otherwise we recolour vv_2 with 10 and colour vv_3 with 8. If there exists a colour $x \in \{1, 2, \ldots, 7\} \setminus (\{\varepsilon\} \cup \mathcal{E}(v_2))$ then we recolour v_2v_3 with x and colour vv_3 with 9. Hence, $\{1, 2, \ldots, 10\} = \{\varepsilon\} \cup \mathcal{E}(v_2)$. Since $|\mathcal{E}(v_2)| = \deg(v_2) \leq 9$, we deduce that $\varepsilon \notin \mathcal{E}(v_2)$, and in particular $\varepsilon \neq 8$.

Suppose that $\alpha \neq 10$, and thus $\{\beta, \gamma\} = \{9, 10\}$. We recolour v_2v_3 with 8, vv_2 with 9, vv_1 with 8, and then colour vv_3 with 7 (note that $\alpha \neq 8$).

Hence, $\alpha = 10$. So $8 \in \{\beta, \gamma\}$, otherwise we recolour vv_1 with 8, vv_2 with 9, v_2v_3 with 8, and then colour vv_3 with 7. Thus, $\{\beta, \gamma\} = \{8, 9\}$, and consequently $\delta \notin \{8, 9, 10\}$. If $\beta = 9$ we recolour v_1v_2 with 8, vv_2 with 10 and then colour vv_3 with 8. If $\beta = 8$ we recolour v_1v_2 with 9, v_2v_3 with 8, vv_2 with 10 and then colour vv_3 with 9.

(iii). Suppose on the contrary that G contains the configuration of Figure 3(c). By the minimality of G, the graph $G - vv_1$ has a nice colouring, from which we infer a partial 10-edge-face-colouring of G in which only vv_1 and the ($\leq$4)-faces are left uncoloured. We now obtain a nice colouring of G by showing that the edge vv_1 can be properly coloured. If vv_1 cannot be coloured greedily, then $|\mathcal{C}(vv_1)| = 10$ and, up to permuting the colours, we may assume that the colouring is the one shown in Figure 3(c).

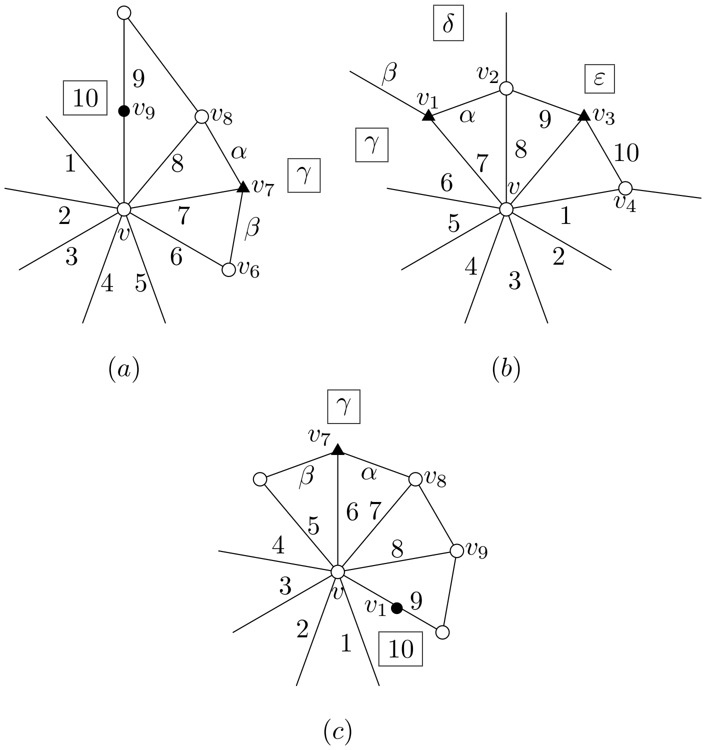


Figure 3: Reducible configurations of Lemma 6.

First, note that $\{\alpha, \beta\} = \{9, 10\}$, otherwise vv_7 can be recoloured with 9 or 10 and then vv_1 can be coloured with 6. Similarly, $\beta \in \mathcal{E}(v_8)$ for otherwise we recolour vv_8 with β and colour vv_1 with 7. Moreover, if $\gamma \neq 7$ then we interchange the colours of vv_8 and v_7v_8 and colour vv_1 with 7. Thus, $\gamma = 7$.

Since $\deg(v_8) \leq 9$, there exists a colour $x \in \{1, 2, \ldots, 10\} \setminus \mathcal{E}(v_8)$. Note that $x \notin \{7, 9, 10\} = \{7, \alpha, \beta\} \subset \mathcal{E}(v_8)$. Thus, since $\gamma = 7$, we can recolour v_7v_8 with x and vv_7 with α, and colour vv_1 with 6 to obtain a nice colouring of G. $\square$

We end this section with a lemma that will help us deal with ($\geq$6)-faces. An edge uv is *light* if $\deg(u) + \deg(v) \leq 9$.

Lemma 7. *Let f be a d-face of G for $d \geq 6$. Let q be the number of 2-vertices and ℓ the number of light edges incident to f. If $\ell \geq 1$ then $q + \ell \leq 2d - 10$.*

Proof. Suppose on the contrary that $\ell \geq 1$ and $q + \ell \geq 2d - 9$. Let $\mathscr{L}$ be the set of light edges incident to f. Since $|\mathscr{L}| = \ell \geq 1$, let $e_0 \in \mathscr{L}$. By the minimality of G, the graph $G - e_0$ has a nice colouring, from which we obtain a partial 10-edge-face-colouring of G in which only e_0, f and the ($\leq$4)-faces are uncoloured. We furthermore uncolour all the edges in $\mathscr{L}$. Now it suffices to properly colour the face f and the edges in $\mathscr{L}$ to obtain a nice colouring of G, and hence a contradiction.

First, note that f is adjacent to at most $d - q$ other faces. Furthermore, f is incident to at most $d - \ell$ coloured edges. Hence, $\mathcal{C}(f) \leq 2 \cdot d - q - \ell \leq 9$. So we can greedily colour f.

It remains to colour the edges of $\mathscr{L}$. To this end, we build an auxiliary graph H with vertex set $\mathscr{L}$, and for every pair $(e, e') \in \mathscr{L}^2$, we add an edge in H between e and e' if and only if e and e' are adjacent in G. Recall that $\mathcal{F}(e) = \{1, 2, \ldots, 10\} \setminus \mathcal{C}(e)$. Observe that properly colouring the edges of G in $\mathscr{L}$ amounts to properly colouring the vertices of H so that each $e \in \mathscr{L}$ is assigned a colour from $\mathcal{F}(e)$. Such a colouring of H is an *$\mathcal{F}$-colouring*.

For each edge $e = uv \in \mathscr{L}$, note that $|\mathcal{C}(e)| \leq \deg(u) + \deg(v) - 2 + 2 - \deg_H(e) \leq 9 - \deg_H(e)$ since e is light. Hence, $|\mathcal{F}(e)| \geq 1 + \deg_H(e)$. As a result, we can (properly) greedily colour each vertex e of H with a colour from $\mathcal{F}(e)$. Indeed, given any partial colouring of H and any $e \in \mathscr{L}$, the number of colours available to colour e is at least $|\mathcal{F}(e)| - \deg_H(e) \geq 1$. This concludes the proof. $\square$

3 Discharging part

Recall that $G = (V, E)$ is a plane graph that is a minimum counter-example to the statement of Theorem 1, in the sense that $|E|$ is minimum. (In particular, a planar embedding of G is fixed.) We obtain a contradiction by using the Discharging Method. Here is an overview of the proof. Each vertex and face of G is assigned an initial charge; the total sum of the charges is negative by Euler's Formula. Then vertices and faces send or receive charge according to certain redistribution rules. The total sum of the charges remains unchanged, but at the end we infer that the charge of each vertex and face is non-negative; a contradiction.

Initial charge. We assign a charge to each vertex and face. For every vertex $v \in V$, we define the initial charge $\mathrm{ch}(v)$ to be $2 \cdot \deg(v) - 6$, while for every face $f \in F$, we define the initial charge $\mathrm{ch}(f)$ to be $\deg(f) - 6$. The total sum is

$$\sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = -12\,.$$

Indeed, by Euler's formula $|E| - |V| - |F| = -2$. Thus, $6|E| - 6|V| - 6|F| = -12$. Since $\sum_{v \in V} \deg(v) = 2|E| = \sum_{f \in F} \deg(f)$, it follows that

$$\begin{aligned} -12 =& 4 \cdot |E| - 6 \cdot |V| + \sum_{f \in F} (\deg(f) - 6) \\ =& \sum_{v \in V} (2 \deg(v) - 6) + \sum_{f \in F} (\deg(f) - 6)\,. \end{aligned}$$

Rules. We need the following definitions to state the discharging rules. A 3-face incident to a 3-vertex is *very-bad*, and a 3-face incident to a 4- or 5-vertex is *bad*. Furthermore, let u be a 2-vertex and f a 4-face incident to u. If v is a neighbour of u then f is *very-bad for v*.
A face that is neither bad nor very-bad (for some vertex v) is *safe* (for v). Note that a very-bad 3-face cannot be bad by Lemma 2(iii). Recall that a 3-face with vertices x, y and z is a $(\deg(x), \deg(y), \deg(z))$-triangle.

Rule R0. *An* (≥ 4)*-face sends* 1 *to every incident* 2*-vertex.*

Rule R1. *An* (≥ 8)*-vertex sends* $3/2$ *to each of its incident very-bad faces;* $5/4$ *to each incident bad face; and* 1 *to each incident safe face.*

Rule R2. *A* 7*-vertex sends* $8/7$ *to each incident face.*

Rule R3. *A* 6*-vertex sends* 1 *to each incident face.*

Rule R4. *A* 5*-vertex sends* $4/5$ *to each incident face.*

Rule R5. *A* 4*-vertex sends* $2/3$ *to each incident* $(4, 7, \geq 8)$*-triangle; and* $1/2$ *to each incident* $(4, \geq 8, \geq 8)$*-triangle.*

In the sequel, we prove that the final charge $\mathrm{ch}^*(x)$ of every $x \in V \cup F$ is non-negative. Hence, we obtain

$$-12 = \sum_{x \in V \cup F} \mathrm{ch}(x) = \sum_{x \in V \cup F} \mathrm{ch}^*(x) \geq 0\,,$$

a contradiction. This contradiction establishes the theorem.

Final charge of faces. Let f be a d-face. Our goal is to show that $\mathrm{ch}^*(f) \geq 0$. Recall that the initial charge of f is $\mathrm{ch}(f) = \deg(f) - 6$.

We first focus on the case where $d \geq 6$. Let $v_1, v_2, \ldots, v_d$ be the vertices incident to f in clockwise order. Let p be the number of ($\geq$6)-vertices incident to f, and q the number of 2-vertices incident to f. Lemma 3(i) implies that $q \leq \left\lfloor \frac{d}{2} \right\rfloor$. Hence, by Rules R0–R5, the final charge of f is $\mathrm{ch}^*(f) \geq d - 6 + p - q$. In particular, $\mathrm{ch}^*(f) \geq 0$ provided $p - q \geq 6 - d$. We now prove that $p - q \geq 6 - d$ for all $d \geq 6$.

If $d \geq 11$ then $d - q \geq \left\lceil \frac{d}{2} \right\rceil \geq 6$ since $q \leq \left\lfloor \frac{d}{2} \right\rfloor$. Thus, $p - q \geq 6 - d$, as wanted. We obtain the desired conclusion for the cases where $d \in \{6, 7, 8, 9, 10\}$ by applying Lemma 7. Note that $\mathrm{ch}^*(f) \geq 0$ if $q = 0$ since $d \geq 6$. So we assume that $q \geq 1$. Let ℓ be the number of light edges of f. If $\ell = 0$, then $p \geq q$ and hence $\mathrm{ch}^*(f) \geq 0$ since $d \geq 6$. So we assume that $\ell \geq 1$. Therefore Lemma 7 implies that $q + \ell \leq 2d - 10$. Observe that $\ell \geq 2(q - p)$. Hence, $2(q - p) \leq 2d - 10 - q$. Consequently, if $q \geq 1$ then $2d - 10 - q \leq 2d - 11$ and hence $q - p \leq d - 6$ because $q - p$ is an integer.

Suppose now that $d = 5$. By Lemma 2(ii), the face f has no light edge. Hence, either f is incident only to ($\geq$5)-vertices, in which case $\mathrm{ch}^*(f) \geq 5 - 6 + 5 \cdot \frac{4}{5} = 3 > 0$ by Rules R1–R4, or f is incident to an ($\leq$4)-vertex v, in which case the two neighbours of v on f are ($\geq$6)-vertices by Lemma 2(ii) and therefore

$$\mathrm{ch}^*(f) \geq \min\left(5 - 6 - 1 \cdot 1 + 2 \cdot 1, 5 - 6 - 2 \cdot 1 + 3 \cdot 1\right) = 0$$

by Rules R0–R3.

We now suppose that $d = 4$, i.e. f is a 4-face. If f is not incident to a 2-vertex, then by Lemma 2(iii) the face f is incident to at least two ($\geq$6)-vertices. Therefore its final charge is $\mathrm{ch}^*(f) \geq 4 - 6 + 2 \cdot 1 = 0$.

If f is incident to a 2-vertex, then f is very-bad. By Lemma 3(iii), all the other vertices incident to f have degree at least 3. Furthermore, Lemma 2(iii) implies that the face f is incident to (at least) two 9-vertices u and v, namely the two neighbours of the 2-vertex. Thus, f is very-bad for

u and for v. Therefore f receives at least $2 \cdot \frac{3}{2} = 3$ by Rule R1, and sends 1 to its incident 2-vertex by Rule R0. Consequently, the final charge of f is $\mathrm{ch}^*(f) \geq 4 - 6 + 2 \cdot \frac{3}{2} - 1 = 0$.

Finally, assume that f is an (x, y, z)-triangle, with $x \leq y \leq z$. First, Lemma 3(ii) implies that f is not incident to a 2-vertex. Thus, Rule R0 does not apply to f, and therefore f sends nothing. We consider several cases regarding the value of x.

$x \geq 6$. Then by Rules R1, R2 and R3, the final charge of f is $\mathrm{ch}^*(f) \geq 3 - 6 + 3 \cdot 1 = 0$.

$x = 5$. In this case, f is bad. Moreover, Lemma 2(iii) implies that $y \geq 6$. If $y \geq 7$, then the final charge of f is $\mathrm{ch}^*(f) \geq 3 - 6 + \frac{4}{5} + 2 \cdot \frac{8}{7} = \frac{3}{35} > 0$, by Rules R1, R2 and R4. If $y = 6$, then Lemma 4 implies that $z \geq 8$. Therefore by Rules R1, R3 and R4, the final charge of f is $\mathrm{ch}^*(f) = 3 - 6 + \frac{4}{5} + 1 + \frac{5}{4} = \frac{1}{20} > 0$.

$x = 4$. Then f is bad. It follows from Lemmas 2(iii) and 4 that $y \geq 7$ and $z \geq 8$. If $y \geq 8$ then by Rules R1 and R5 the final charge of f is $\mathrm{ch}^*(f) \geq 3 - 6 + \frac{1}{2} + 2 \cdot \frac{5}{4} = 0$. If $y = 7$ then f receives $2/3$ from its 4-vertex by Rule R5. Furthermore, f receives $8/7$ from its 7-vertex by Rule R2, and $5/4$ from its ($\geq$8)-vertex by Rule R1. Thus, its final charge is $\mathrm{ch}^*(f) = 3 - 6 + \frac{2}{3} + \frac{8}{7} + \frac{5}{4} = \frac{5}{84} > 0$.

$x = 3$. The face f is very-bad, and it follows from Lemma 2(iii) that $y \geq 8$. Therefore by Rule R1 the face f receives $2 \cdot \frac{3}{2} = 3$. Thus, its final charge is $\mathrm{ch}^*(f) \geq 3 - 6 + 2 \cdot \frac{3}{2} = 0$.

Final charge of vertices. Let v be an arbitrary vertex of G. Our goal is to show that $\mathrm{ch}^*(v) \geq 0$. Recall that the initial charge of v is $\mathrm{ch}(v) = 2 \cdot \deg(v) - 6$. Moreover, $\deg(v) \geq 2$ by Lemma 2(i).

If $\deg(v) = 2$, then v is incident to two distinct ($\geq$4)-faces by Lemmas 2(i) and 3(ii). Each of those two faces gives 1 to v by Rule R0. Thus, the final charge of v is $\mathrm{ch}^*(v) = -2 + 2 = 0$.

If $\deg(v) = 3$, then v neither sends nor receives any charge. Hence, the final charge of v is $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 0$.

Suppose now that $\deg(v) = 4$. If v is not incident to a $(4, 7, \geq 8)$-triangle then by Rule R5 the final charge of v is $\mathrm{ch}^*(v) \geq 2 - 4 \cdot \frac{1}{2} = 0$. If v is incident to a $(4, 7, \geq 8)$-triangle, then Lemma 2(iii) implies that the edge between v

and the 7-vertex is incident to an ($\geq$5)-face. Hence, v is incident to at most three 3-faces, and therefore the final charge of v is $\mathrm{ch}^*(v) \geq 2 - 3 \cdot \frac{2}{3} = 0$.

Suppose that $\deg(v) \in \{5, 6, 7\}$. By Rules R2, R3 and R4 the vertex v sends $\frac{\mathrm{ch}(v)}{\deg(v)}$ to each of its incident faces. Therefore the final charge of v is $\mathrm{ch}^*(v) = 0$.

Suppose that $\deg(v) = 8$. By Lemma 2(iii), every very-bad face incident to v is a $(3, 8, \geq 8)$-triangle. Thus, Lemmas 2(iii) and 5(i) imply that v is incident to at most 4 very-bad faces. Let $f_1, f_2, \ldots, f_8$ be the faces incident to v, in clockwise order. For every $i \in \{1, 2, \ldots, 8\}$, observe that by Lemma 2(iii) at least one of f_{i-1}, f_i, f_{i+1} is safe or very-bad, where the index is taken modulo 8. In other words, there are no three consecutive bad faces. Furthermore, let us note that Lemma 2(iii) implies that every very-bad face is adjacent to a safe face. We consider several cases regarding the number x of very-bad faces for v. Recall that $x \leq 4$.

$x = 0$. Then the final charge of v is $\mathrm{ch}^*(v) \geq 10 - 8 \cdot \frac{5}{4} = 0$.

$x = 1$. Then, as noted above, v is incident to a safe face, so $\mathrm{ch}^*(v) \geq 10 - \frac{3}{2} - 6 \cdot \frac{5}{4} - 1 \cdot 1 = 0$.

$x = 2$. We assert that v is incident to at least two safe faces. This yields the desired conclusion since it implies that $\mathrm{ch}^*(v) \geq 10 - 2 \cdot \frac{3}{2} - 4 \cdot \frac{5}{4} - 2 \cdot 1 = 0$. Without loss of generality, suppose that f_8 is very-bad. Since every very-bad face incident to v is adjacent to a safe face, we can assume without loss of generality that f_1 is safe. Moreover, we also assume that f_2 is the second very-bad face, otherwise the assertion holds. But as we observed earlier, at least one face among f_3, f_4, f_5 is either very-bad or safe. Since none is very-bad, we deduce that v is incident to at least two safe faces, as asserted.

$x = 3$. We assert that v is incident to at least three safe faces. This yields the desired conclusion since then $\mathrm{ch}^*(v) \geq 10 - 3 \cdot \frac{3}{2} - 2 \cdot \frac{5}{4} - 3 \cdot 1 = 0$. Since every very-bad face incident to v is adjacent to a safe face, we infer the existence of an index i such that f_{i-1} and f_{i+1} are very-bad, and f_i is safe (for otherwise the conclusion holds). Without loss of generality, we may assume that f_1 and f_7 are very-bad and f_8 is safe. Let f_j be the third very-bad face incident to v. Then by symmetry $j \in \{2, 3, 4\}$. First, $j \neq 2$ by Lemmas 2(iii) and 5(i). If $j = 3$, then necessarily f_2 is safe for v by Lemma 2(iii). Furthermore, at least one face among f_4, f_5, f_6 is safe, since there cannot be three consecutive bad faces and none of these faces is very-bad. Finally, assume that $j = 4$. Then at least one face among f_2 and f_3 is safe by Lemmas 2(iii) and 5(ii). Similarly, at least one face among f_5 and f_6 is safe, so v is incident to three safe faces, as asserted.

$x = 4$. Lemmas 2(iii) and 5(i) imply that v is incident to four safe faces, so $\mathrm{ch}^*(v) \geq 10 - 4 \cdot \frac{3}{2} - 4 \cdot 1 = 0$.

Finally, assume that $\deg(v) = 9$. Let $v_1, v_2, \ldots, v_9$ be the neighbours of v in clockwise order, and for $i \in \{1, 2, \ldots, 9\}$ let f_i be the face incident to vv_i and vv_{i+1}, where the index is taken modulo 9.

Lemmas 2(iii), 3(iii), 6(i) and 6(ii), imply that v is not incident to three consecutive very-bad faces, i.e. there is no index i such that all of f_{i-1}, f_i and f_{i+1} are very-bad (where i is taken modulo 9). To see this, suppose that f_1, f_2 and f_3 are all very-bad for v. The face f_2 cannot be a 4-face, for otherwise one of v_2 and v_3 would be a 2-vertex, and so one of f_1 and f_3 would be safe for v by Lemma 2(iii). Hence, f_2 is a very-bad triangle and thus v_2 or v_3 is a 3-vertex. By symmetry, we may assume that v_3 is a 3-vertex, which implies that v_2 is an (≥ 8)-vertex by Lemma 2(iii). Consequently, we infer from Lemma 3(iii) that f_3 is not a very-bad 4-face for v. So, f_3 is a very-bad triangle. Now, f_1 can neither be a very-bad triangle by Lemma 6(ii), nor a very-bad 4-face for v by Lemma 6(i); a contradiction. As a result, the number x of very-bad faces for v is at most 6.

First, assume that v is not incident to two consecutive very-bad faces. Thus, $x \leq 4$. We consider two cases depending on the value of x.

$x \leq 3$. Then the final charge of v is $\mathrm{ch}^*(v) \geq 12 - 3 \cdot \frac{3}{2} - 6 \cdot \frac{5}{4} = 0$.

$x = 4$. Without loss of generality, the very-bad faces incident to v are f_1, f_3, f_5 and f_7. If v is incident to at least one very-bad 4-face, then it is incident to at least one safe face by Lemma 2(iii), and the final charge of v is $\mathrm{ch}^*(v) \geq 12 - 4 \cdot \frac{3}{2} - 4 \cdot \frac{5}{4} - 1 = 0$. So we may assume all the very-bad faces incident to v are triangles. Consequently, f_2, f_4 and f_6 are safe by Lemma 2(iii). Therefore the final charge of v is $\mathrm{ch}^*(v) \geq 12 - 4 \cdot \frac{3}{2} - 2 \cdot \frac{5}{4} - 3 = \frac{1}{2} > 0$.

It remains to deal with the case where v is incident to two consecutive very-bad faces.

First, let us suppose that v is incident to two consecutive very-bad faces one of which is a 4-face. Without loss of generality, assume that f_1 and f_2

are very-bad for v, and f_1 is a 4-face. Since f_2 is very-bad for v, we deduce from Lemma 2(iii) that v_1 is a 2-vertex. So f_9 is safe for v by Lemma 2(iii), and v_2 is an ($\geq$4)-vertex by Lemma 3(iii). Moreover, f_3 is also safe for v by Lemmas 2(iii) and 6(i). Thus, if $x \leq 5$ then $\mathrm{ch}^*(v) \geq 12 - 5 \cdot \frac{3}{2} - 2 \cdot \frac{5}{4} - 2 = 0$. If $x = 6$ then, since v is not incident to three consecutive very-bad faces, we deduce that f_4, f_5, f_7 and f_8 are very-bad for v. We prove that f_6 is then safe for v. This would yield that $\mathrm{ch}^*(v) = 12 - 6 \cdot \frac{3}{2} - 3 \cdot 1 = 0$, as wanted. If at least one of f_4 and f_5 is a 4-face, then we infer as above that f_6 is safe for v (and so is f_3). The same holds if one of f_7 and f_8 is a 4-face. So we may assume that f_4, f_5, f_7 and f_8 are all triangles. Consequently, Lemma 2(iii) ensures that f_6 is safe. This concludes our analysis in this case since $x \leq 6$, as noted earlier.

Finally, assume that v is not incident to two consecutive very-bad faces one of which is a 4-face, but v is incident to two consecutive very-bad faces. Without loss of generality, assume that f_1 and f_2 are two very-bad triangles. We consider several cases depending on the value of x. Recall that $x \leq 6$ since there are no three consecutive very-bad faces for v.

$x \leq 3$. Then the final charge of v is $\mathrm{ch}^*(v) \geq 12 - 3 \cdot \frac{3}{2} - 6 \cdot \frac{5}{4} = 0$.

$x = 4$. We assert that v is incident to a safe face. This yields the result since then the final charge of v is $\mathrm{ch}^*(v) \geq 12 - 4 \cdot \frac{3}{2} - 4 \cdot \frac{5}{4} - 1 = 0$. It remains to prove the assertion. Suppose on the contrary that v is not incident to a safe face. Then all faces incident to v are triangles (since every very-bad 4-face for v is adjacent to a face that is safe for v by Lemma 2(iii)). As a result, Lemma 6(ii) implies that $\deg(v_2) = 3$. By Lemma 2(iii), both v_1 and v_3 are ($\geq$8)-vertices. Since $f_3 = vv_3v_4$ and $f_9 = vv_9v_1$ are bad faces, both v_4 and v_9 are ($\leq$5)-vertices. Lemma 2(iii) then implies that v_5 and v_8 are ($\geq$7)-vertices. Since $x = 4$, one of v_6 and v_7 is a 3-vertex; by symmetry we may assume it is v_6. But now $\deg(v_7) \geq 8$ by Lemma 2(iii), so vv_7v_8 is a safe face; a contradiction.

$x = 5$. We assert that v is incident to at least two safe faces. This yields the conclusion since then $\mathrm{ch}^*(v) \geq 12 - 5 \cdot \frac{3}{2} - 2 \cdot \frac{5}{4} - 2 = 0$. Note that each of f_3 and f_9 is either bad or safe for v, since there are no three consecutive very-bad faces for v. Suppose first that none of f_3 and f_4 is safe for v. In particular, f_3 is bad. It then follows from Lemma 2(iii) that v_2 is a 3-vertex and v_4 an ($\leq$5)-vertex. Consequently, f_4 cannot be a very-bad triangle. Moreover, f_4 is not a very-bad 4-face for v by

Lemma 6(*iii*). So f_4 is a bad face. Now, three faces among f_5, f_6, f_7 and f_8 are very-bad for v, since f_9 cannot be very-bad for v. If both f_5 and f_6 are very-bad for v, then both are triangles and f_8 is also very-bad. Since f_4 is bad, we deduce that v_6 is a 3-vertex. Consequently f_8 must be a 4-face, for otherwise both f_7 and f_9 are safe. Thus, one of f_7, f_9 is safe. Further, the other face cannot be bad either by Lemma 6(*iii*), and hence it is also safe for v. Consequently, the assertion holds if none of f_3 and f_4 is safe for v. By symmetry, the same argument applies to f_8 and f_9, and thus v is incident to at least two safe faces, as asserted.

$x = 6$. Then the very-bad faces are f_1, f_2, f_4, f_5, f_7 and f_8. Hence, they are all very-bad triangles. Consequently, f_3, f_6 and f_9 are safe for v by Lemma 2(*iii*). Hence, the final charge of v is $\mathrm{ch}^*(v) = 12 - 6 \cdot \frac{3}{2} - 3 = 0$.

This establishes that the final charge of every vertex is non-negative, so the proof of Theorem 1 is now complete. □

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