Interval valued bimatrix games

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Abstract

Payoffs in (bimatrix) games are usually not known precisely, but it is often possible to determine lower and upper bounds on payoffs. Such interval valued bimatrix games are considered in this paper. There are many questions arising in this context. First, we discuss the problem of existence of an equilibrium being common for all instances of interval values. We show that this property is equivalent to solvability of a certain linear mixed integer system of equations and inequalities. Second, we characterize the set of all possible equilibria by mean of a linear mixed integer system.

Notation

e	a vector of all ones (with convenient dimension)
e_k	the k-th basis vector (with convenient dimension), i.e., the k-th column of the identity matrix
$x \leq y$, $A \leq B$, $A < B$, ...	vector and matrix relations are understood componentwise
$A_{i,\bullet}$	the i-th row of a matrix A

1 Introduction

Competitive situations arise in many part of real life and game theory gives a mathematical background for dealing with such conflicting events. There are diverse kinds of mathematical games; we focus on matrix games.

A bimatrix game is a two player game, each of them has finite number of strategies. The payoff function is determined by two matrices $A, B \in \mathbb{R}^{m \times n}$. When player I chooses the i-th strategy and player II his j-th strategy then a_{ij} and b_{ij} are payoffs of the player I and II, respectively. Thus, a bimatrix game is determined by a pair of matrices (A, B). Rational behavior of players is assumed, that is, each of them attempts to maximize his/her reward.

An equilibrium point in pure strategies needn't exist, so we have to consider mixed strategies. *Mixed strategy* is a probability vector over a set of pure strategies. That is, a mixed strategy for player I is a vector $x \in \mathbb{R}^m$, $x \geq 0$, $e^T x = 1$. Likewise, a mixed strategy for player II is a vector $y \in \mathbb{R}^n$, $y \geq 0$, $e^T y = 1$. Now, the expected reward for player I and II is $x^T A y$ and $x^T B y$, respectively. A pair of mixed strategies $(\hat{x}, \hat{y})$ is called *(Nash) equilibrium* if

$$\begin{aligned} \hat{x}^T A \hat{y} &\geq x^T A \hat{y}, \\ \hat{x}^T B \hat{y} &\geq \hat{x}^T B y \end{aligned}$$

for any mixed strategy x and y. In other words, no player has a motivation to change his/her strategy. In mixed strategies, at least one equilibrium always exists [8].

Equilibria can be computed by solving a linear complementarity problem [11], however, for our purposes it is more convenient to use the following result by Audet et al. [3]. It says that equilibria corresponds one-to-one to solutions of a particular mixed integer programming problem. Note that variables α and β are respectively payoffs of the player I and II.

Theorem 1. *Let*

$$\begin{aligned} L_1 &:= \max_{i,j} a_{ij} - \min_{i,j} a_{ij}, \\ L_2 &:= \max_{i,j} b_{ij} - \min_{i,j} b_{ij}. \end{aligned}$$

The set of equilibria is the set of mixed strategies (x, y) for which there are

$\alpha, \beta \in \mathbb{R}$ *and vectors* $u \in \{0,1\}^m$ *and* $v \in \{0,1\}^n$ *satisfying*

$$e^T x = 1,\ x \geq 0, \tag{1}$$

$$e^T y = 1,\ y \geq 0, \tag{2}$$

$$\alpha e - L_1 u \leq Ay \leq \alpha e, \tag{3}$$

$$\beta e - L_2 v \leq B^T x \leq \beta e, \tag{4}$$

$$x + u \leq e, \tag{5}$$

$$y + v \leq e. \tag{6}$$

In many economical situations the payoffs are not known precisely. Uncertainty in mathematical problems can be dealt by various ways. An interval-based approach is considered in this papers. Herein, we assume that we have lower and upper bounds on the imprecise data. Such interval-valued games have already been studied in the recent years. Yager & Kreinovich [12] showed how fair division under interval uncertainty can be performed. Zero-sum interval matrix games were considered by Liu & Kao [7], Collins & Hu [4, 5], Levin [6], and by Shashikhin [10]. While the former authors Liu & Kao discussed the range of possible payoffs, the others solved the interval problem by imposing a binary relation on intervals. Cooperation under interval uncertainty was dealt with by Alparslan-Gök et al. [1, 2].

To the best of our knowledge, bimatrix games with interval-valued entries has never been investigated. To formalize the problem, we define *an interval matrix*[9] to be a family of matrices

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}\ \forall i,j\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ are given matrices. An interval matrix $\boldsymbol{B}$ is defined by analogy. For more results on linear systems with interval values see e.g. Rohn [9]. In fact, we were inspired by his approach to handle (1)–(6). Nevertheless, the direct utilization is not possible as (1)–(6) incurs some discrete variables and dependencies (double appearance of A and B).

By *an instance* we mean a bimatrix game (A, B) with certain $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$.

Surprisingly, not always the maximal payoffs (entries of A and B) result in real maximal expected rewards. For example, let

$$\boldsymbol{A} = \begin{pmatrix} 5 & 0 \\ [4,6] & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 5 & [4,6] \\ 0 & 1 \end{pmatrix}.$$

The bimatrix game $(\underline{A}, \underline{B})$ has three equilibria (e^1, e^1), (e^2, e^2) and $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$. The corresponding rewards are respectively 5, 1 and $\frac{5}{2}$ for both players. The first equilibrium dominates the others, so rational behavior of the players should tend to the first equilibrium with the maximal reward 5. However, the game $(\overline{A}, \overline{B})$ has only one equilibrium (e^2, e^2) and both players earn merely 1.

2 Equilibrium preservation

A natural question is whether there exists a pair of mixed strategies which forms an equilibrium being common for all instances of interval data. Such an equilibrium is called *strong equilibrium.* How to check its existence? And what are properties of the corresponding strategies? First, we observe that the strategies must be pure provided we perturb all matrix entries.

Proposition 1. *Let $(\hat{x}, \hat{y})$ is a strong equilibrium. If $\underline{A} < \overline{A}$ then $\hat{x}$ is a pure strategy. If $\underline{B} < \overline{B}$ then $\hat{y}$ is a pure strategy.*

Proof. For contradiction and without loss of generality assume that the first two components of $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$, are positive. Put $A := \frac{1}{2}(\overline{A} - \underline{A})$. By definition of equilibrium, $\hat{x}_1 A\hat{y} = \hat{x}_2 A\hat{y}$. Perturbing the first row of A, however, the equation will not hold true. That is, $(\hat{x}, \hat{y})$ is not equilibrium when we change A to $A + \varepsilon e^1 e$ with $\varepsilon > 0$ small enough. That contradicts the assumption. Likewise for the strategy $\hat{y}$ of player II. $\square$

Under the assumption of Proposition 1 any strong equilibrium must consist of pure strategies only. Thus it suffices to inspect all combination of pure strategies and check if they form a strong equilibrium. The following statement gives instructions for this approach.

Theorem 2 (Strong equilibrium in pure strategies). *There exists a strong equilibrium in pure strategies if and only if there is some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ such that*

$$\underline{a}_{ij} \geq \overline{a}_{kj} \quad \forall k = 1, \ldots, m,\ k \neq i, \tag{7}$$

$$\underline{b}_{ij} \geq \overline{b}_{ik} \quad \forall k = 1, \ldots, n,\ k \neq j. \tag{8}$$

In this case, (e^i, e^j) is a strong equilibrium.

Proof. Let $A \in \boldsymbol{A}$ and let $B \in \boldsymbol{B}$ and (7)–(8) be true. Then $a_{ij} \geq \underline{a}_{ij} \geq \overline{a}_{kj} \geq a_{kj}$ for all $k \neq i$. That is, $e^{i\,T} A e^j \geq e^{k\,T} A e^j$ for all $k \neq i$ and hence $e^{i\,T} A e^j \geq x^T A e^j$ for any mixed strategy x. Likewise one can show that $e^{i\,T} B e^j \geq e^{i\,T} B y$ for any mixed strategy y. Therefore (e^i, e^j) is an equilibrium.

Conversely, let (e^i, e^j) be a strong equilibrium. Then $e^{i\,T} A e^j \geq x^T A e^j$ must be true for any $A \in \boldsymbol{A}$ and any mixed strategy x. Put $a_{ij} := \underline{a}_{ij}$ and $a_{kj} := \overline{a}_{kj}$ for $k = 1, \ldots, m,\ k \neq i$; the other entries can be chosen arbitrarily. Next, put $x := e^k$ for $k \neq i$. Then the inequality reads $\underline{a}_{ij} \geq \overline{a}_{kj}$ and holds true for any $k \neq i$. This proves (7), and the second relation (8) can be proven accordingly. □

Now, we know how to check if there exists a strong equilibrium in pure strategies. The method proposed by Theorem 2 is easily computable and can be implemented in time complexity $\mathcal{O}(mn)$ (simply for each row of $\overline{B}$ and column of $\overline{A}$ find the two largest entries and then compare candidates only with them). Below, we propose an existence test for a strong equilibrium in mixed strategies that are not pure. We extend the definition of L_1 and L_2 as follows

$$
\begin{aligned}
L_1 &:= \max_{i,j} \overline{a}_{ij} - \min_{i,j} \underline{a}_{ij},\\
L_2 &:= \max_{i,j} \overline{b}_{ij} - \min_{i,j} \underline{b}_{ij}.
\end{aligned}
$$

Theorem 3 (Strong equilibrium in non-pure strategies). *A pair of mixed non-pure strategies $(\hat{x}, \hat{y})$ is a strong equilibrium if and only if there is some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ such that they solve the system*

$$e^T x = 1,\ x \geq 0, \tag{9}$$

$$e^T y = 1,\ y \geq 0, \tag{10}$$

$$\alpha e - L_1 u \leq \underline{A} y,\ \overline{A} y \leq \alpha e, \tag{11}$$

$$\beta e - L_2 v \leq \underline{B}^T x,\ \overline{B}^T x \leq \beta e, \tag{12}$$

$$x + u \leq e, \tag{13}$$

$$y + v \leq e. \tag{14}$$

Proof. One implication is easily observed. Let $\hat{x} \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^n$, $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ be a solution of the system (9)–(14), and let

$A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ be arbitrarily chosen. Then due to non-negativity of $\hat{y}$ we have

$$\hat{\alpha} e - L_1 \hat{u} \leq \underline{A} \hat{y} \leq A \hat{y} \leq \overline{A} \hat{y} \leq \hat{\alpha} e,$$

and likewise

$$\hat{\beta} e - L_2 \hat{v} \leq \underline{B}^T \hat{x} \leq B^T \hat{x} \leq \overline{B}^T \hat{x} \leq \hat{\beta} e.$$

Hence $\hat{x}$, $\hat{y}$, $\hat{\alpha}$, $\hat{\beta} \in \mathbb{R}$, $\hat{u}$ and $\hat{v}$ form a solution to the system (1)–(6), and therefore $(\hat{x}, \hat{y})$ is an equilibrium to any bimatrix game (A, B) with $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$.

To show the converse implication, we define $I(\hat{x}) := \{i \mid \hat{x} > 0\}$ and $I(\hat{y}) := \{i \mid \hat{y} > 0\}$. We claim that $\underline{a}_{ij} = \overline{a}_{ij}$ and $\underline{b}_{ij} = \overline{b}_{ij}$ for every $i \in I(\hat{x})$ and $j \in I(\hat{y})$. As $\hat{x}$ is not pure strategy, $I(\hat{x})$ contains more than one element. Thus, for any $i_1, i_2 \in I(\hat{x})$ and any $A \in \boldsymbol{A}$ we have ${e^{i_1}}^T A \hat{y} = {e^{i_2}}^T A \hat{y}$. This equation will not be true when we perturb any $a_{i_1 j}$ or $a_{i_2 j}$ with $j \in I(\hat{y})$. That is why $\underline{a}_{ij} = \overline{a}_{ij}$ holds for every $i \in I(\hat{x})$ and $j \in I(\hat{y})$. Likewise the second equation.

Let $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$. By Theorem 1 there are some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ satisfying (1)–(6). Obviously, $\hat{x}_i > 0$ implies $\hat{u}_i = 0$. For $\hat{x}_i = 0$ we can assume that $\hat{u}_i = 1$ (if $\hat{u}_i = 0$, then also $\hat{u}_i = 1$ solves the system). Let $i \in I(\hat{x})$. Then (3) implies $\hat{\alpha} \leq (A\hat{y})_i \leq \hat{\alpha}$, i.e., $\hat{\alpha} = (A\hat{y})_i$. As a_{ij} is constant for every $i \in I(\hat{x})$, $j \in I(\hat{y})$ and $A \in \boldsymbol{A}$ we obtain that the reward $\hat{\alpha}$ of player I is the same for all instances. Thus $\hat{x}$, $\hat{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{u}$ and $\hat{v}$ solve (1)–(6) for every $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$. Particularly, for $A := \underline{A}$ we get $\hat{\alpha} e - L_1 \hat{u} \leq \underline{A} \hat{y}$, and for $A := \overline{A}$ we get $\overline{A} \hat{y} \leq \hat{\alpha} e$. Hence (11) is satisfied, (12) holds accordingly, and the other equations and inequalities are satisfied trivially. $\square$

We should give some explanation to Theorem 3. Any non-pure strong equilibrium is described by (9)–(14). Conversely, any solution to the system gives some strong equilibrium, which is not necessary in non-pure strategies. That is, system (9)–(14) describes all non-pure strong equilibria and possibly some others strong equilibria. In general, not all strong equilibria are included there. Analogous remark could be given for Theorem 4 below.

Proof of Theorem 3 also shows what are rewards for the players. Let $\hat{x}$, $\hat{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{u}$ and $\hat{v}$ be a solution of the system (9)–(14). Then for every $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ the pair $(\hat{x}, \hat{y})$ is an equilibrium to (A, B) with constant rewards $\hat{\alpha}$

and $\hat{\beta}$ for player I and II, respectively. This is true just for equilibria in non-pure strategies, since in this case a reduction to the system (1)–(6) was used. Analogous statement holds for Theorem 4 for the second player's reward, and in Theorem 5 whenever non-pure strategy is considered. However, $\hat{\alpha}$ and $\hat{\beta}$ do not determine the players' payoffs and needn't be constant as long as we deal with pure strategies (for player I in Theorem 4 or for both players in Theorem 5); see Example 1.

Theorem 4 (Strong equilibrium in pure and non-pure strategy). *A pair $(\hat{x}, \hat{y})$ is a strong equilibrium consisting of pure strategy $\hat{x}$ and a non-pure strategy $\hat{y}$ if and only if there is some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ solving the system*

$$e^T x = 1,\ \ x \geq 0, \tag{15}$$

$$e^T y = 1,\ \ y \geq 0, \tag{16}$$

$$\alpha e - L_1 u \leq \underline{A} y,\ \ \overline{A} y \leq \alpha e + L_1(e-u), \tag{17}$$

$$e^T u = m - 1, \tag{18}$$

$$\beta e - L_2 v \leq \underline{B}^T x,\ \ \overline{B}^T x \leq \beta e, \tag{19}$$

$$x + u \leq e, \tag{20}$$

$$y + v \leq e. \tag{21}$$

Proof. Let $\hat{x} \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^n$, $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ be a solution of the system (15)–(21), and let $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ be arbitrarily chosen. Due to (18) and (20), $\hat{x}$ is a pure strategy, i.e., $\hat{x} = e^k$ for some $k \in \{1, \dots, m\}$. Put $\tilde{\alpha} := (A\hat{y})_k$. Then

$$\tilde{\alpha} e - L_1 \hat{u} \leq A\hat{y}.$$

Moreover,

$$(A\hat{y})_k = (\tilde{\alpha} e)_k,$$

and for $i \neq k$ we have according to (17)

$$(A\hat{y})_i \leq (\overline{A}\hat{y})_i \leq (\hat{\alpha} e)_i = \hat{\alpha} \leq (\underline{A}\hat{y})_k \leq (A\hat{y})_k = \tilde{\alpha} = (\tilde{\alpha} e)_i.$$

Thus,

$$\tilde{\alpha} e - L_1 \hat{u} \leq A\hat{y} \leq \tilde{\alpha} e,$$

proving (3). The inequalities (4) hold true as

$$\hat{\beta}e - L_2\hat{v} \leq \underline{B}^T\hat{x} \leq B^T\hat{x} \leq \overline{B}^T\hat{x} \leq \hat{\beta}e.$$

So $\hat{x}$, $\hat{y}$, $\tilde{\alpha}$, $\hat{\beta} \in \mathbb{R}$, $\hat{u}$ and $\hat{v}$ is a solution to the system (1)–(6), and therefore $(\hat{x}, \hat{y})$ is a strong equilibrium.

Conversely, let $(\hat{x}, \hat{y})$ be a strong equilibrium, let $\hat{x} = e^k$, $k \in \{1, \ldots, m\}$, be a pure strategy, and let $\hat{y}$ be a non-pure strategy. Let $B \in \boldsymbol{B}$ and define $A' \in \boldsymbol{A}$ in this way: $a'_{kj} = \underline{a}_{kj}$ for $j = 1, \ldots, n$, and $a'_{ij} = \overline{a}_{ij}$ for $i = 1, \ldots, m$, $i \neq k$, $j = 1, \ldots, n$. By Theorem 1 there are some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ satisfying (1)–(6). We can assume that $\hat{u} = e - \hat{x}$. Then (18) holds true. Moreover, the k-th inequality in (3) reads

$$\hat{\alpha} \leq (A'\hat{y})_k = (\underline{A}\hat{y})_k.$$

For $i = 1, \ldots, m$, $i \neq k$,

$$\hat{\alpha} - L_1 \leq (\underline{A}\hat{y})_i$$

holds as L_1 is large enough. Thus we have

$$\hat{\alpha}e - L_1\hat{u} \leq \underline{A}\hat{y}.$$

Similarly we show the second inequality system in (17): The k-th inequality

$$(\overline{A}\hat{y})_k \leq \hat{\alpha} + L_1$$

holds as L_1 is large enough, and

$$(\overline{A}\hat{y})_i = (A'\hat{y})_i \leq \hat{\alpha}$$

is true for every $i = 1, \ldots, m$, $i \neq k$ due to Theorem 1.

The remaining inequalities in (15)–(21) are satisfied as well; one can proceed as in proof of Theorem 3. □

Remark 1. The system (15)–(21) contains $m + n + 2$ continuous variables and $m + n$ binary variables. From the computational point of view it seems better to use the following condition; it consists of checking solvability of m mixed integer systems (for $k = 1, \ldots, m$), each of which has $n+1$ continuous variables and n binary variables.

Equivalent formulation of Theorem 4 states that pure strategy e^k and a non-pure strategy $\hat{y}$ form a strong equilibrium if and only if there is some $\hat{\beta} \in \mathbb{R}$, and $\hat{v} \in \{0,1\}^n$ solving the system

$$\begin{aligned}
e^T y &= 1,\ y \geq 0,\\
{e^i}^T \overline{A} y &\leq {e^k}^T \underline{A} y, \quad \forall i = 1,\ldots,m,\ i \neq k,\\
\beta e - L_2 v &\leq \underline{B}^T x,\ \overline{B}^T x \leq \beta e,\\
y + v &\leq e.
\end{aligned}$$

We have described all cases when a strong equilibrium consists of pure strategies, non-pure strategies, or their combination. Each case needed its own test, which is not convenient for processing. The next theorem covers all situations together and propose only one unifying test.

Theorem 5 (Strong equilibrium). *A pair $(\hat{x}, \hat{y})$ is a strong equilibrium if and only if there exists some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{\gamma}, \hat{\delta} \in \{0,1\}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ solving the system*

$$e^T x = 1,\ x \geq 0, \tag{22}$$
$$e^T y = 1,\ y \geq 0, \tag{23}$$
$$\alpha e - L_1 u \leq \underline{A} y,\ \overline{A} y \leq \alpha e + L_1(e - u), \tag{24}$$
$$\overline{A} y \leq \alpha e + L_1 \gamma e, \tag{25}$$
$$(m-1)\gamma \leq e^T u, \tag{26}$$
$$\beta e - L_2 v \leq \underline{B}^T x,\ \overline{B}^T x \leq \beta e + L_2(e - v), \tag{27}$$
$$\overline{B}^T x \leq \beta e + L_2 \delta e, \tag{28}$$
$$(n-1)\delta \leq e^T v, \tag{29}$$
$$x + u \leq e, \tag{30}$$
$$y + v \leq e. \tag{31}$$

Proof. Let $\hat{x} \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^n$, $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{\gamma}, \hat{\delta} \in \{0,1\}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ be a solution of the system (22)–(31). If $\hat{\gamma} = 0$ and $\hat{\delta} = 0$ then the variables solve also (9)–(14) and so $(\hat{x}, \hat{y})$ is a strong equilibrium in non-pure strategies by Theorem 3. If $\hat{\gamma} = 1$ and $\hat{\delta} = 0$ then the variables solve also (15)–(21) and so $(\hat{x}, \hat{y})$ is a strong equilibrium by Theorem 4. The situation when $\hat{\gamma} = 0$ and $\hat{\delta} = 1$ is dealt with accordingly. Eventually, if

$\hat{\gamma} = 1$ and $\hat{\delta} = 1$ then $(\hat{x}, \hat{y})$ is a strong equilibrium in pure strategies by Theorem 2: (26) and (29) imply that $\hat{x}$ and $\hat{y}$ are pure strategies, i.e., $\hat{x} = e^i$ and $\hat{y} = e^j$ for some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. By (24) we have $\underline{a}_{ij} \geq \hat{\alpha} \geq \overline{a}_{kj}$ for every $k = 1, \ldots, m$, $k \neq i$. By (27), $\underline{b}_{ij} \geq \hat{\beta} \geq \overline{b}_{ik}$ for every $k = 1, \ldots, n$, $k \neq j$.

Conversely, let $(\hat{x}, \hat{y})$ be a strong equilibrium. If it consists of non-pure strategies, then there is a solution to (9)–(14). Putting $\hat{\gamma} := 0$ and $\hat{\delta} := 0$ we get a solution to (22)–(31). If $\hat{x} = e^k$ for some $k \in \{1, \ldots, m\}$ and $\hat{y}$ is a non-pure strategy, then there is a solution to (15)–(21). Putting $\hat{\gamma} := 1$ and $\hat{\delta} := 0$ we get a solution to (22)–(31). Eventually, if $\hat{x} = e^i$ and $\hat{y} = e^j$ are pure strategies then we put $\hat{u} := e - \hat{x}$, $\hat{v} := e - \hat{y}$, $\hat{\alpha} := \underline{a}_{ij}$, $\hat{\beta} := \underline{b}_{ij}$, $\hat{\gamma} := 1$ and $\hat{\delta} := 1$. In this setting the system (15)–(21) is satisfied. $\square$

Example 1. Consider an interval bimatrix game $(\boldsymbol{A}, \boldsymbol{B})$ with

$$\boldsymbol{A} = \begin{pmatrix} 42 & [21, 24] & 21 \\ [49, 52] & [35, 38] & [14, 17] \\ 7 & [77, 80] & 35 \end{pmatrix}, \quad \boldsymbol{B} = \boldsymbol{A}^T.$$

In non-pure strategies, there exists just one strong equilibrium (x, y) with $x = y = (0.2857, 0, 0.7143)^T$ and players' rewards $\alpha = \beta = 27$. That is, x, y, α, β, $u = v = (0, 1, 0)^T$ and $\gamma = \delta = 0$ form a solution to (22)–(31).

In pure strategies, there is a unique strong equilibrium (x, y) with $x = y = (0, 0, 1)^T$. Players' rewards are 35. The corresponding solution to (22)–(31) consists of x, y, $u = v = (1, 1, 0)^T$, $\gamma = \delta = 1$ and any $\alpha, \beta \in [21, 35]$.

The interval game has no equilibrium in mixed pure and non-pure strategies, even though (22)–(31) has a solution with $\alpha = 0$ and $\beta = 1$. However, this solution corresponds merely to pure-strategies equilibrium.

3 Equilibria set

In this section we focus on the set of all equilibria for all instances of the interval bimatrix game. Theorem 6 shows that every equilibrium is a solution of a particular mixed integer linear system and vice versa.

Theorem 6. *The set of all equilibria for all bimatrix games (A, B) with $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ is described by the mixed integer linear system*

$$
\begin{aligned}
e^T x = 1,\ x \geq 0, \qquad & (32)\\
e^T y = 1,\ y \geq 0, \qquad & (33)\\
\alpha e - L_1 u \leq \overline{A} y,\ \underline{A} y \leq \alpha e, \qquad & (34)\\
\beta e - L_2 v \leq \overline{B}^T x,\ \underline{B}^T x \leq \beta e, \qquad & (35)\\
x + u \leq e, \qquad & (36)\\
y + v \leq e, \qquad & (37)\\
u \in \{0,1\}^m,\ v \in \{0,1\}^n. \qquad & (38)
\end{aligned}
$$

Proof. Let $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ and $(\hat{x}, \hat{y})$ be an equilibrium of the bimatrix game (A, B). By Theorem 1 there is some $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ solving (1)–(6). Then

$$\hat{\alpha} e - L_1 \hat{u} \leq A\hat{y} \leq \overline{A}\hat{y},$$

and

$$\underline{A}\hat{y} \leq A\hat{y} \leq \hat{\alpha} e.$$

Thus (34) holds true and the other inequalities are satisfied accordingly.

Conversely, let $\hat{x} \in \mathbb{R}^m$, $\hat{y} \in \mathbb{R}^n$, $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ be a solution to (32)–(38). We define $A \in \boldsymbol{A}$ in the following way. For i with $\hat{u}_i = 0$ we have

$$(\underline{A}\hat{y})_i \leq \hat{\alpha} \leq (\overline{A}\hat{y})_i.$$

If $(\underline{A}\hat{y})_i = (\overline{A}\hat{y})_i$ then the i-th row $A_{i,\bullet}$ of A can be arbitrarily chosen (in prescribed bounds), otherwise we put

$$A_{i,\bullet} := \underline{A}_{i,\bullet} + \frac{\hat{\alpha} - (\underline{A}\hat{y})_i}{(\overline{A}\hat{y})_i - (\underline{A}\hat{y})_i} (\overline{A}_{i,\bullet} - \underline{A}_{i,\bullet}).$$

Then $(A\hat{y})_i = A_{i,\bullet}\hat{y} = \hat{\alpha}$. For i with $\hat{u}_i = 1$ we put $A_{i,\bullet} := \underline{A}_{i,\bullet}$. Then $\hat{\alpha} - L_1 \hat{u}_i \leq (A\hat{y})_i \leq \hat{\alpha}$. Similarly we define $B \in \boldsymbol{B}$. Thus $\hat{x}$, $\hat{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{u}$ and $\hat{v}$ satisfy the system (1)–(6) and according to Theorem 1 the pair $(\hat{x}, \hat{y})$ forms an equilibrium to the bimatrix game (A, B). $\square$

Due to Theorem 6, testing if (x, y) is an equilibrium of some instance is an easy task. We put $u_i := 0$ if $x_i > 0$ and $u_i := 1$ otherwise; likewise for v. Finally, check solvability of the remaining bi-variate linear system (w.r.t. variables $\alpha, \beta \in \mathbb{R}$).

Theorem 6 also implies that the equilibria set consists of a union of finitely many convex polyhedra. For any $u \in \{0,1\}^m$ and $v \in \{0,1\}^n$ the system (32)–(37) describes a convex polyhedron, and so the equilibria set is composed of at most 2^{m+n} convex polyhedra. Moreover, for any $u \in \{0,1\}^m$ and $v \in \{0,1\}^n$ the system decomposes into two sub-systems: the first one consists of (32) and (35)–(36) with variables $\beta \in \mathbb{R}$ and $x \in \mathbb{R}^m$, and the second one comprises (33)–(34) and (37) with variables $\alpha \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

Proof of Theorem 6 reveals another useful aspect of the system (32)–(37). This system describes not only all equilibria, but also the corresponding payoffs for both players. If one desires to know lower and upper bounds for payoffs of player I for instance, then it suffices to respectively minimize and maximize α over the system (32)–(37). It means solving two mixed integer linear programs. However, when restricted on certain $u \in \{0,1\}^m$ and $v \in \{0,1\}^n$ both problems reduce to linear programs.

Example 2. Recall an interval bimatrix game $(\boldsymbol{A}, \boldsymbol{B})$ from Example 1. Using Theorem 6 we obtain that the equilibria set consists of the following polyhedra:

1. Let $u = v = (0,0,0)^T$. The convex polyhedron $\mathcal{X}$ described by (32), (35)–(36) has vertices

$$\begin{aligned}
(x^1, \beta^1) &= (0.2857, 0, 0.7143, 27.0000),\\
(x^2, \beta^2) &= (0.3714, 0.1000, 0.5286, 29.1000),\\
(x^3, \beta^3) &= (0.3671, 0.0886, 0.5443, 28.7089),\\
(x^4, \beta^4) &= (0.3676, 0.1029, 0.5294, 29.0294),\\
(x^5, \beta^5) &= (0.3636, 0.0909, 0.5455, 28.6364).
\end{aligned}$$

   The convex polyhedron $\mathcal{Y}$ associated (33)–(34) and (37) equals $\mathcal{X}$. Hence the Cartesian product $\mathcal{X} \times \mathcal{X}$ yields a set of equilibria with corresponding rewards for players.

2. For $u = (0,0,0)^T$ and $v = (0,1,0)^T$ we calculate the set of equilibria $\mathcal{X} \times \{(0.2857, 0, 0.7143, 27.0000)\}$. It is a subset of $\mathcal{X} \times \mathcal{X}$, so we can omit this case.

3. Situation $u = (0, 1, 0)^T$ and $v = (0, 0, 0)^T$ is symmetric to the previous and we drop it, too.
4. For $u = (0, 1, 0)^T$ and $v = (0, 1, 0)^T$ we obtain the set of equilibria

$$\{(0.2857, 0, 0.7143, 27.0000)\} \times \{(0.2857, 0, 0.7143, 27.0000)\}.$$

Also this case is covered by the first one.

5. Finally, let $u = (1, 1, 0)^T$ and $v = (1, 1, 0)^T$. Herein we get only one equilibrium (e_3, e_3) consisting of pure strategies. The reward is 35 for both players.

We enumerated all feasible cases; the other evaluations of u and v result in empty sets of equilibria. Summing up, the total equilibria set (with the corresponding rewards) is composed of union of the convex polyhedron $\mathcal{X} \times \mathcal{X}$ and the isolated point $(0, 0, 1, 35, 0, 0, 1, 35)$.

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