# An Eberhard-like theorem for pentagons and heptagons

Matt DeVos
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
mdevos@sfu.ca

Agelos Georgakopoulos[*†]
Technische Universität Graz
Steyrergasse 30, 8010
Graz, Austria
georgakopoulos@tugraz.at

Bojan Mohar[‡§]
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
mohar@sfu.ca

Robert Šámal[¶]
KAM & ITI
Charles University
Prague, Czech Republic
samal@kam.mff.cuni.cz

[*]Supported in part by a GIF grant and in part by an FWF grant.

[†]This work was conceived during a visit of the second author at the Simon Fraser University; we thank the SFU for supporting this visit.

[‡]Supported in part by the Research Grant P1–0297 of ARRS, by an NSERC Discovery Grant and by the Canada Research Chairs program.

[§]On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

[¶]Supported in part by PIMS postdoctoral fellowship at the Simon Fraser University, by Institute for Theoretical Computer Science (ITI) and by Department of Applied Mathematics.

**Abstract**

Eberhard proved that for every sequence $(p_k), 3 \leq k \leq r, k \neq 5, 7$ of non-negative integers satisfying Euler's formula $\sum_{k\geq 3}(6-k)p_k = 12$, there are infinitely many values $p_6$ such that there exists a simple convex polyhedron having precisely $p_k$ faces of length $k$ for every $k \geq 3$, where $p_k = 0$ if $k > r$. In this paper we prove a similar statement when non-negative integers $p_k$ are given for $3 \leq k \leq r$, except for $k = 5$ and $k = 7$. We prove that there are infinitely many values $p_5, p_7$ such that there exists a simple convex polyhedron having precisely $p_k$ faces of length $k$ for every $k \geq 3$. We derive an extension to arbitrary closed surfaces, yielding maps of arbitrarily high face-width. Our proof suggests a general method for obtaining results of this kind.

# 1 Introduction

Let $G$ be a cubic plane graph, and let $p_k$ $(k \geq 1)$ denote the number of its $k$-gonal faces. It is a simple corollary of Euler's formula that

$$\sum_{k\geq 1}(6-k)p_k = 12. \tag{1}$$

It is natural to ask for which sequences $(p_k)_{k\geq 1}$ satisfying (1) there exists a cubic plane graph whose face lengths comply with the sequence $(p_k)$. This question is even more interesting when additional restrictions on the graph are given. The most important case is to consider graphs of 3-dimensional convex polyhedra, so called *polyhedral graphs*. By Steinitz's Theorem, this is the same as requiring the graphs to be 3-connected.

The general problem about the existence of polyhedral graphs with given face lengths is still wide open. However, there are many special cases that have been solved. For example [9, Theorem 13.4.1], it is known that there exists a simple polyhedron with six quadrangular faces and $p_6$ faces of size six if and only if $p_6 \neq 1$; and there exists a simple polyhedron with twelve pentagonal faces and $p_6$ faces of size six (a "fullerene" graph) if and only if $p_6 \neq 1$. A similar case of four triangular faces and $p_6$ faces of length 6 has infinitely many exceptions: such a polyhedron exists if and only if $p_6$ is even. We refer to [9] for a complete overview. The most fundamental result in this area is the following classical theorem of Eberhard [3], stating that there is always a solution provided we are allowed to replace $p_6$ (whose value does not affect the satisfaction of (1)) by a large enough integer. Call a polyhedron *simple* if its graph is cubic.

**Theorem 1.1 (Eberhard [3]).** *For every sequence* $(p_k), 3 \le k \le r, k \ne 6$ *of non-negative integers satisfying (1), there are infinitely many values* $p_6$ *such that there exists a simple convex polyhedron having precisely* $p_k$ *faces of length* $k$ *for every* $k \ge 3$*, where* $p_k = 0$ *if* $k > r$.

Eberhard's proof is not only long and messy but also some of its parts may not satisfy today's standards of rigor. Grünbaum [9] gave a simpler complete proof utilizing graphs and Steinitz's Theorem. This result was strengthened by Fisher [5] who proved that there is always a value of $p_6$ that satisfies $p_6 \le p_3 + p_4 + p_5 + \sum_{k \ge 7} p_k$.

Grünbaum also considered a 4-valent analogue of Eberhard's theorem. Fisher [6] proved a similar result for 5-valent polyhedra, establishing existence for all admissible sequences of face lengths if $p_4 \ge 6$.

Various other generalizations of Eberhard's theorem have been discovered. Papers by Jendrol' [10, 11] give a good overview and bring some of today's most general results in this area. Some other relevant works include [1, 2, 4, 8, 13]. Several papers treat extensions of Eberhard's theorem to the torus [7, 12, 15, 16] and more general surfaces [10]. It is worth pointing out that on the torus there is precisely one admissible sequence (namely $p_5 = p_7 = 1$ and $p_i = 0$ for $i \notin \{5, 7\}$), for which an Eberhard-type result with added hexagons does not hold [12].

In this paper we consider a similar problem that is also motivated by (1). Let us suppose that we are given face lengths as before but we are only allowed to change $p_5$ and $p_7$ (or $p_{6-t}$ and $p_{6+t}$ for some $t$, $1 \le t \le 3$). In this case, we think of $p_k$ (for $k \ge 3$, $k \ne 5, 7$) as being fixed and $p_5, p_7$ as being free to choose. Equation (1) determines the difference $s = p_7 - p_5$, and we are asking if there exist $p_5$ and $p_7 = p_5 + s$ with a polyhedral realization. We give an affirmative answer to this question, and derive an extension solving the corresponding problem on an arbitrary closed surface. Our construction gives simple polyhedral maps on a surface, and one can impose the additional conditions that these maps have large face-width and their graphs be 3-connected. More precisely, we prove

**Theorem 1.2.** *Let* $(p_k), 3 \le k \le r, k \ne 5, 7$ *be a sequence of non-negative integers, let* $S$ *be a closed surface, and let* $w$ *be a positive integer. Then there exist infinitely many pairs of integers* $p_5$ *and* $p_7$ *such that there is a 3-connected map realizing* $S$*, with face-width at least* $w$*, having precisely* $p_k$ *faces of length* $k$ *for every* $k \in \{3, \dots, r\}$.

It is worth observing that the extension of Eberhard's Theorem to a surface $S$ other than the sphere needs an adjustment in (1); the right hand side has to be replaced by $6\chi(S)$. However, in our setting the formula adjusts itself by using an appropriate number of pentagons and heptagons.

Finally, as we point out in Section 4, our proof suggests a general method for obtaining results of this kind.

## 2 Definitions

A finite sequence $p = (p_3, p_4, \ldots, p_r)$ is *plausible* for a closed surface $S$ if

$$\sum_{3 \leq k \leq r} (6-k)p_k = 6\chi(S) \tag{2}$$

where $\chi(S)$ is the Euler characteristic of $S$. By Euler's formula, (2) is a necessary condition for the existence of a cubic graph embeddable in $S$ with precisely $p_k$ $k$-gons for $3 \leq k \leq r$ and no other faces. If there exists a cubic graph which is 2-cell embeddable in $S$ with precisely $p_k$ faces of length $k$ for $3 \leq k \leq r$ and no other faces, then we say that $p$ is *realizable* in $S$. If $\sum_{3 \leq k \leq r}(6-k)p_k = 0$, then we call $p$ a *neutral* sequence. For any two such sequences, one can consider their sum which is defined in the obvious way. Let us observe that the sum of a neutral sequence and a plausible sequence is a plausible sequence. We would like to understand in this context which plausible sequences are realizable, and try to do so by asking when a sum of a plausible sequence with an appropriate neutral sequence is realizable. For the neutral sequence $(0, 0, 0, 1)$ this is Eberhard's theorem.

The most important building block in both Eberhard's as well as our proofs is a construction called a triarc. A *triarc* is a plane graph $T$ such that the boundary $C$ of the outer face of $T$ is a cycle, and moreover the following conditions are satisfied (examples are the graphs in Figure 3 with the half-edges in the outer face removed):

- every vertex of $T - C$ has degree 3 in $T$;
- $C$ contains distinct vertices $x, y, z$ of degree 2 (called the *corners* of the triarc) such that the degrees (in $T$) of the vertices on each of the three paths in $C - \{x, y, z\}$ alternate between 2 and 3, starting and ending with a vertex of degree 2.

A *side* of a triarc $T$ as above is a subpath of $C$ that starts and ends at distinct corners of $T$ and does not contain the third corner. The *length* of a side $P$ of $T$ is the number of inner vertices of degree 2 on $P$; note that although the corners of a triarc have degree 2, they are not counted when calculating the lengths of its sides. A triarc with sides of lengths $a$, $b$, $c$ is called an $(a, b, c)$*-triarc*. Of course, we can flip or rotate such a triarc and consider it, for example, as a $(b, a, c)$-triarc.

Triarcs are very versatile tools. Firstly, if the length of some side of a triarc $T$ equals the length of some side of another triarc $R$, then $T$ and $R$ can be glued together along those sides to yield a new plane graph with all inner vertices having degree 3; see for example Figure 9. Secondly, every triarc $T$ has zero total curvature; to see this, take two copies of $T$, turn one of them upside down, glue them along a common side to obtain a 'parallelogram' (see Figure 9 again), and identify opposite sides of this parallelogram to obtain a graph embeddable in the torus. But perhaps the most important property of triarcs is the possibility to 'glue' them together to obtain larger triarcs; we describe this operation below.

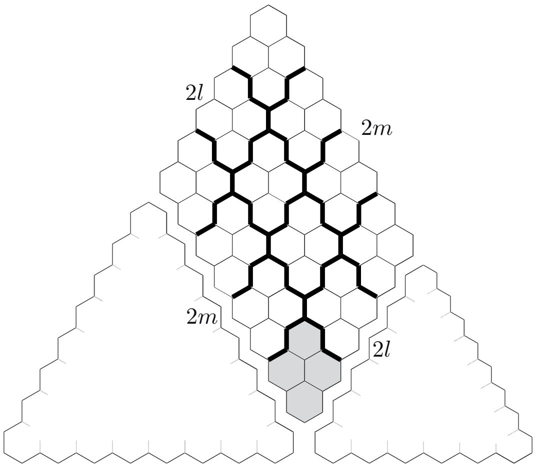


Figure 1: Glueing two triarcs with two sides of even length together using the tile of Figure 2.

Suppose we have an $(a_1, b_1, c_1)$-triarc and an $(a_2, b_2, c_2)$-triarc such that $b_1 = 2m$ and $c_2 = 2l$ are even. Then, we may combine these triarcs (and

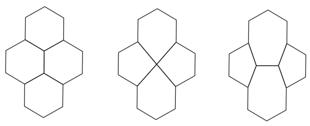

Figure 2: In a configuration of 4 hexagons we may contract the central edge and then 'uncontract' it in the other direction. A 'tile' consisting of two pentagons and two heptagons results; we use such tiles in Figure 1.

several pentagons and heptagons) to construct an $(a_1+a_2, b_1+b_2, c_1+c_2)$-triarc. To do this, we identify a corner (and an incident edge) of the first triarc with a corner (and an edge) of the second triarc —see Figure 1— so that the two identified corners yield a vertex of degree 3 on a side of length $a_1+a_2$ in a new triarc. Then, we can add a *"parallelogram"* consisting of hexagons to obtain an $(a_1+a_2, b_1+b_2, c_1+c_2)$-triarc. However, we do not want to add hexagons. Instead, we decompose the parallelogram into tiles each consisting of four hexagons as depicted in Figure 1, and replace each of these tiles by two pentagons and two heptagons as indicated in Figure 2.

We are going to use this operation of glueing two triarcs into a larger one several times in the following section.

# 3 Proof of Theorem 1.2

We are ready to state and prove our main result. Let us observe that, unlike Eberhard's Theorem, we do not need to assume that the given face-lengths form a plausible sequence (although we make this assumption in the formulation of the theorem) because given a sequence $(p_k), 3 \leq k \leq r, k \neq 5, 7$, the sequence can always be appended by appropriate values $p_5$ and $p_7$ to become plausible.

**Theorem 3.1.** *Let $p = (p_3, p_4, \ldots, p_r)$ be a plausible sequence for the sphere. Then there exist infinitely many integers $n \in \mathbb{N}$ such that the sequence $p + n \cdot (0, 0, 1, 0, 1)$ is realizable in the sphere.*

*Proof.* We will give an explicit construction of a cubic graph embeddable in the sphere whose face sequence is of the form $p + n \cdot (0, 0, 1, 0, 1)$. The rough plan for this is as follows. For each face imposed by the sequence $p$, we create a *basic triarc* containing this face as well as some pentagons and heptagons. Then, we glue all these triarcs together and extend to a triarc with sides of suitable lengths. Finally, we construct a new triarc having the same side lengths, and glue these two triarcs together (as explained later) to obtain the desired graph embedded in the sphere.

To construct a basic triarc for a $k$-gon (we will make $p_k$ copies of it), we surround the $k$-gon by three heptagons and $k-3$ pentagons as shown in the right half of Figure 3 (where the $k$-gon we are surrounding happens to be a pentagon). Note that we can always make the basic triarc isosceles with the equal sides having even length. We call the $k$-gon we started with the *nucleus* of this triarc.

Having constructed all basic triarcs, our next step is to glue them all together to obtain a single triarc $T$ containing them all. We do so recursively, attaching one basic triarc at a time as shown in Figure 1, where we use many copies of the 'tile' in Figure 2 in order to build the parallelogram needed. Each time we use this glueing operation we are assuming that both triarcs in Figure 1 are isosceles, with the equal sides having even length, and align them so that the two equal even sides are the upper left and upper right side. Note that the resulting triarc is also isosceles with two equal sides of even length. Thus, we can continue recursively to glue all basic triarcs into one isosceles triarc $T$.

Our next aim is to enlarge $T$ into an equilateral triarc $T'$ with sides of length $n$, where $n$ is a multiple of 8 and satisfies $n \equiv 2 \pmod 3$, using only pentagons and heptagons. To this end, we will use the glueing operation of Figure 1 and many copies of a $(4,4,3)$-triarc and a $(2,2,4)$-triarc. Figure 3 shows how to construct those triarcs with pentagons and heptagons only.

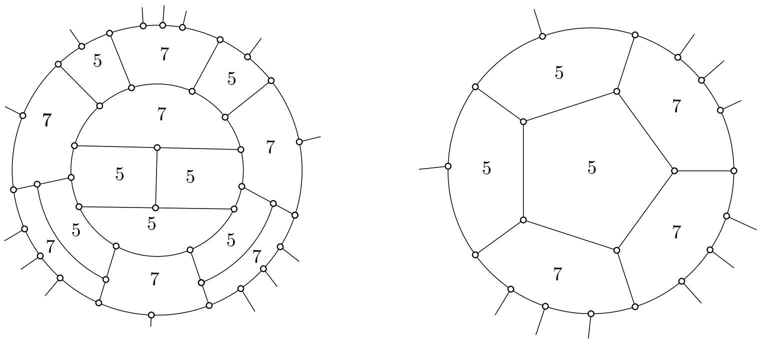


Figure 3: A $(4,4,3)$-triarc and a $(2,2,4)$-triarc.

Note that glueing $T'$ with a $(4, 4, 3)$-triarc (as in Figure 1) keeps it isosceles and decreases the difference of lengths between the "base" and the other two sides by 1, while glueing with a $(2, 2, 4)$-triarc increases that difference by 2. Thus, recursively glueing with such triarcs we can enlarge $T$ into an equilateral triarc $S$ with sides of even length.

Moreover, using the glueing operation of Figure 1 three times, once with a $(2, 2, 4)$-triarc and twice with a $(4, 4, 3)$-triarc, we can increase the side-lengths by $(2, 2, 4) + (4, 4, 3) + (4, 4, 3) = (10, 10, 10)$. Thus we can increase the length of each side of $S$ by 10. Since $10 \equiv 1 \pmod 3$, we can use this operation to enlarge $S$ into an equilateral triarc $S'$ with even sides of length $2 \pmod 3$. Moreover, since performing this operation three times increases the length of each side by 30, and $30 \equiv 6 \pmod 8$, we can enlarge $S'$ into an equilateral triarc $T'$ with the length of each side being a multiple of 8 and congruent to 2 modulo 3.

Next, we are going to construct a triarc $R$ that has the same side lengths as $T'$ but consists of pentagons and heptagons only. By glueing together a $(2, 2, 4)$-triarc, a $(2, 4, 2)$-triarc and a $(4, 2, 2)$-triarc (that is, the same triarc in three different rotations) we get an $(8, 8, 8)$-triarc, which we will call $D$. Since the sides of $T'$ have length a multiple of 8, by glueing copies of $D$ together recursively as in Figure 1 we can indeed construct a triarc $R$ that has the same dimensions as $T'$.

We can now combine $R$ and $T'$ together to produce a cubic graph tiling the sphere as shown in Figure 4. By construction, this graph has for every $k \in \mathbb{N}\backslash\{0, 1, 2, 5, 7\}$ precisely $p_k$ faces of size $k$, and moreover it has at least $p_5$ pentagons and at least $p_7$ heptagons. Thus its face sequence is of the form $p + (0, 0, n, 0, m)$ for some $n, m \in \mathbb{N}_+$. Since both $p$ and $p + (0, 0, n, 0, m)$ satisfy Euler's formula (the former by assumption, the latter because the plane graph we just constructed implements it), we have $n = m$.

This completes the construction and shows the existence of one particular value of $n$ as desired. However, observe that the construction of $T'$ and $R$ allows us to make the side lengths of these triarcs arbitrarily large. This shows that we can get examples for infinitely many values of $n$ and thus completes the proof. □

We now turn from planar graphs to maps on arbitrary (compact) surfaces. A *map* on a surface $S$ is a graph together with a 2-cell embedding in $S$. A map is *polyhedral* if all faces are closed disks in the surface and the intersection of any two faces is either empty, a common vertex or a common edge. If the graph of the map is cubic, then we say that the map is *simple*.

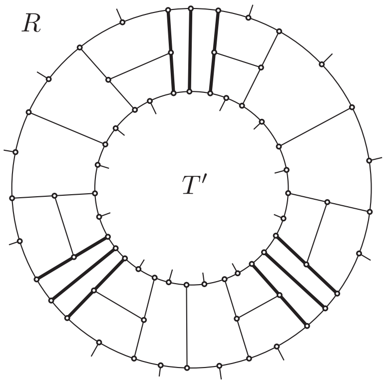


Figure 4: Glueing $R$ and $T'$ together along a "ring" consisting of pentagons and heptagons. This operation is possible because we made sure that every side of $T'$, and thus also of $R$, has length congruent to 2 (mod 3).

A cycle contained in the graph of a map is *contractible* if it bounds a disk on the surface. The *edge-width* of a map $M$ is the length of a shortest non-contractible cycle in $M$. The *face-width* of $M$ is the minimum number of faces, the union of whose boundaries contains a non-contractible cycle. We refer the reader to [14] for more about the basic properties and the importance of these parameters of maps. At this point we only note that a map is polyhedral if and only if its graph is 3-connected and its face-width is at least three, see [14, Proposition 5.5.12]. We also note that if $r$ is the largest length of a face of $M$, then the edge-width of $M$ cannot exceed $\frac{r}{2}$ times the face-width of $M$.

We now restate and prove our main result, Theorem 1.2.

**Corollary 3.2.** *Let $(p_k), 3 \leq k \leq r, k \neq 5, 7$ be a sequence of non-negative integers, let $S$ be a closed surface, and let $w$ be a positive integer. Then there exist infinitely many pairs of integers $p_5$ and $p_7$ such that the sequence $(p_3, p_4, p_5, \ldots, p_r)$ is realizable in $S$ and there is a 3-connected realizing map of face-width at least $w$.*

*Proof.* Let us first describe a construction that does not necessarily achieve the desired face-width; we will later explain how to modify this construction in order to get large face-width.

The rough sketch of this construction is as follows. Firstly, we increase the number of hexagons in the sequence $(p_k)$ to $p_6' := p_6 + 2h + c$, where

$h$ is the number of handles of $S$ and $c$ the number of its crosscaps (by the surface classification theorem we may assume that one of $h, c$ is zero, but we do not have to). It follows from Theorem 3.1 that we can increase the numbers $p_5$ and $p_7$ of this sequence to some appropriate values so that the resulting sequence $p'$ is realized by a map on the sphere. We will then use the $2h + c$ auxilliary hexagons of this map we added above to introduce some handles and/or crosscaps. After doing so, all auxilliary hexagons will have disappeared, and we obtain a map on $S$ whose sequence of faces differs from $(p_k)$ by some pentagons and heptagons only.

More precisely, similarly to the proof of Theorem 3.1, we construct a basic triarc for each face in $p'$, but with one modification: for each hexagon we construct a triarc like the one in Figure 5 (on the left) rather than one with two even sides of equal lengths (in fact, we need this modification for the auxiliary hexagons only, but we might as well use it for the original hexagons in $p$ as well).

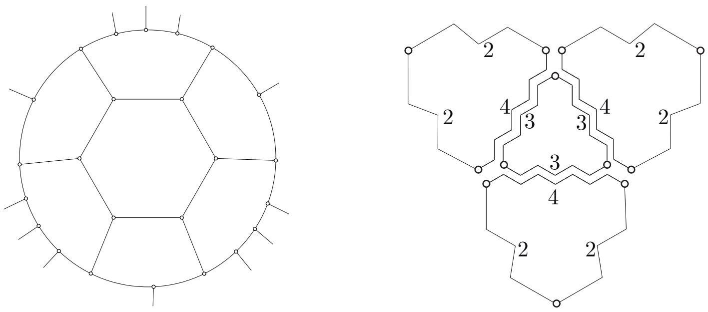


Figure 5: On the left: the new basic triarc for a hexagon. On the right: extending the triarc from the left into an equilateral triarc with even sides.

Next, we proceed as in Theorem 3.1 to glue all basic triarcs together into one triarc $T$. However, since we now have basic triarcs with all sides odd (the ones of Figure 5), the glueing operation of Figure 1 will not work for these triarcs. For this reason, we first extend each such triarc into an equilateral triarc with even sides using three copies of the $(2, 2, 4)$-triarc of Figure 3 as shown in Figure 5 (right).

We continue imitating the proof of Theorem 3.1 to obtain a cubic graph $G$ embedded in a homeomorphic copy $S'$ of the sphere that contains all basic triarcs. We will now perform some cutting and glueing operations on both $S'$ and $G$ to obtain a new surface, homeomorphic to $S$, with a cubic

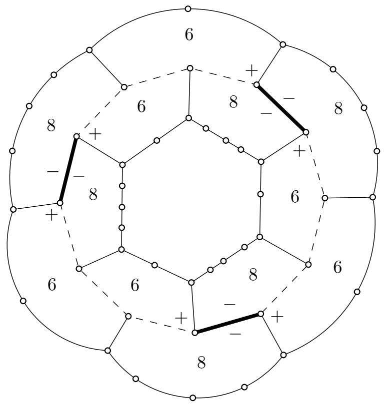


Figure 6: The situation arising after introducing a handle. The 12-cycle $C$ consists of the dashed and the thick edges.

graph $G'$ embedded in it.

Suppose that $h > 0$. Then, pick $h$ pairs $(F_1, F_1'), \ldots, (F_h, F_h')$ of hexagonal faces of $G$, such that all the faces $F_i$ and $F_i'$ are distinct (there are enough hexagonal faces by our choice of the sequence $p'$). Now for each pair $(F_i, F_i')$ perform the following operations. Cut out the two discs of $S'$ corresponding to $F_i, F_i'$, and glue their boundaries together with a half-twist; that is, each vertex of the boundary of $F_i$ is identified with the midpoint of an edge of $F_i'$ and vice-versa. This operation creates a handle in $S'$, and the embedded graph remains cubic; however, it also gives rise to some unwanted faces: the length of each face that was incident to $F_i$ or $F_i'$ has now been increased by 1. We thus have the situation depicted in Figure 6, where $C$ is the cycle of length 12 resulting from the boundaries of $F_i$ and $F_i'$. Recall that since every hexagon is put in a basic triarc like the one in Figure 5, the lengths of the faces on each side of $C$ alternate between 6 and 8 as shown in Figure 6. But now, contracting and uncontracting each of the three thick edges (in the way explained in Figure 2) turns each of the faces incident with $C$ into a heptagon.

On the other hand, if $c > 0$, then pick $c$ distinct hexagonal faces $F_1, \ldots, F_c$, and for every $i$ cut out the disc corresponding to $F_i$ and glue in its place the outside of the hexagon of Figure 7 with a half twist. Each such operation gives rise to a new crosscap, but also to unwanted faces just like in Figure 6. But again, contracting and uncontracting each of the three

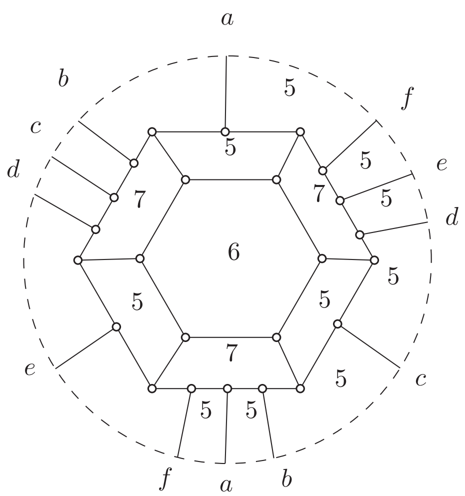


Figure 7: The gadget used to create a crosscap inside a hexagon.

thick edges we can turn all these unwanted hexagons and octagons into heptagons.

Thus, after all these operations have been completed, we obtain a surface with $h$ handles and $c$ crosscaps with a cubic graph embedded in it whose face sequence is $p+(0,0,n,0,m)$ for some $n,m \in \mathbb{N}_+$. Note that all auxiliary hexagons in $p'-p$ have disappeared after the above operations. As in the previous proof, $n=m$ must hold by Euler's formula.

Clearly, our maps are 3-connected. It remains to discuss how to modify this construction to obtain maps with arbitrarily large face-width. By the remark preceding Corollary 3.2, it suffices to construct maps with arbitrarily large edge-width $z$ since the face lengths are bounded from above by $r$. This is achieved as follows.

First of all, we make every basic triarc used in the construction large enough that the distance from its nucleus to the boundary of the triarc is at least $z$ and the length of each side of each triarc is at least $3z$. This can be achieved by the method we used in the proof of Theorem 3.1 to enlarge $T$ into an equilateral triarc $T'$.

Next, we replace the auxiliary hexagons used in order to add handles and crosscaps with $6N$-gons, where $N$ is odd and greater than $z/2$. Of course, this will force us to add some more pentagons to our sequence $p_k$ to make it plausible. Note that we can generalize the triarc on the left of Figure 5 so that the inner 6-gon is replaced by a $6N$-gon surrounded by three heptagons and $6N-3$ pentagons, arranged in a symmetric way so that any two heptagons separate $2N-1$ pentagons from the rest. We will make use of the fact that $2N-1$ is odd. We need to adapt the right half of Figure 5 as well, since the inner triarc has now grown larger. For this, note that each side of the inner triarc has now length $2N+1$, and so in order to use the method of the right half of Figure 5 the three peripheral triarcs must have a base of length $2N+2$ (in addition to having their other two sides of equal length). Since we chose $N$ to be odd, it turns out that $2N+2$ is a multiple of four, and so we can construct the required peripheral triarcs by glueing several $(2,2,4)$-triarcs together using Figure 1 into a $(N+1, N+1, 2N+2)$-triarc.

Moreover, the crosscap gadget shown in Figure 7 can be generalized so that the inner 6-gon is replaced by a $6N$-gon that is surrounded by $3N$ heptagons and $3N$ pentagons, arranged alternatingly around the $6N$-gon (here it is also important that we chose $N$ to be odd).

When the time comes to insert crosscaps or glue pairs of such $6N$-gons together (after a half-twist), we obtain a similar configuration as in Figure 6, but with $3N$ thick edges. Some of these thick edges are surrounded by faces of lengths $8, 6, 8, 6$ (as in Figure 6), while others are surrounded by four hexagons or by one octagon and three hexagons. Note, however, that for parity reasons we can make sure that every octagon is incident with a thick edge, and still every fourth edge on the dashed cycle is thick. Finally, the contract-uncontract operation of Figure 2 turns these faces into pentagons and heptagons only.

Let us now argue that the resulting map $G$ has edge-width at least $z$. Recall that the surface $S$ is obtained from a plane graph $G'$, embedded in the sphere, that is composed of large basic triarcs $T_1, \ldots, T_m$, some large parallelograms used to glue the basic triarcs together into a large triarc $T$, and a remainder $X$ comprising the material we used to enlarge $T$ into $T'$, the ring of Figure 4, and the triarc $R$. Let $L_i$ be the nucleus of $T_i$. Then $S$ is obtained from $G'$ by glueing the crosscap gadget into some of the $6N$-gons $L_i$, and/or by identifying some pairs $L_i, L_j$ of the $6N$-gons to create handles.

We claim that for every basic triarc $T_i$ such that the nucleus $L_i$ of $T_i$ is a $6N$-gon, and

$$\text{for every side } P \text{ of } T_i\text{, there is a set of } z \text{ pairwise disjoint } L_i\text{–}P \textit{ paths}. \quad (3)$$

Indeed, recall that in order to construct $T_i$, we first surrounded $L_i$ by several pentagons and heptagons, $6N$ in total, to obtain a triarc $T_i^1$, then we performed the operation of the right half of Figure 5 to obtain a triarc $T_i^2$, and finally we enlarged this into a larger triarc $T_i^3 = T_i$ using the operation of Figure 1 several times. Now given any side $P'$ of $T_i^2$ it is possible to find, within $T_i^2$, a set of $z$ pairwise disjoint $L_i$–$P'$ *paths*, see Figure 8. Then, every time we use the operation of Figure 1 while enlarging $T_i^2$ into

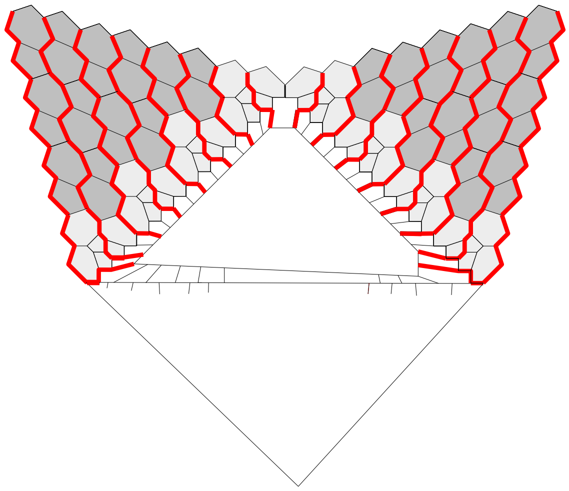

Figure 8: Constructing $z$ disjoint $L_i$–$P'$ *paths*, in the case that the auxilliary hexagons are replaced with 42-gons ($6N$ for $N = 7$). In light gray are the $(2, 2, 4)$-triarcs, in dark gray the modified hexagonal tiles from the gluing operation of Figures 1 and 2. The 16 thick paths are the ones we need to prove that our graphs have large face-width.

$T_i^3$, it is possible to recursively propagate those paths to reach the side of $T_i^3$ corresponding to $P'$; if $P'$ is included within a side of $T_i^3$ then nothing needs to be done, and if not then we can propagate our paths through the parallelogram of Figure 1 while keeping them disjoint (this is true even after performing the contract-uncontract operations of Figure 2). This proves our claim (3).

Next, we claim that any two nuclei $L_i, L_j$ can be joined by $z$ pairwise disjoint paths in $G'$. Indeed, this follows easily from (3) and the fact that whenever we glue two triarcs $T, T'$ together as in Figure 1 by a parallelogram $R$ with side-lengths $m, n$, then we can find a set of $m$ pairwise disjoint paths within $R$ joining its two opposite sides of length $m$, as well as a set of $\min(m, n)$ pairwise disjoint paths within $R$ joining the sides of $T$ and $T'$ incident with $R$.

Since, easily, any non-contractible cycle $C$ in $G$ must yield a cycle $C'$ in $G'$, with $|C'| \leq |C|$, that separates some nucleus $L_i$ from some other nucleus $L_j$ in $G'$, the above observation implies that $|C| \geq z$ as desired.

□

# 4 Other neutral sequences

We would be tempted to conjecture that an Eberhard-like theorem holds for any neutral sequence, not just for $(0, 0, 1, 0, 1)$. However at this point we believe that some additional global constraints may also apply. Nevertheless, some parts of our construction can be extended and it is the purpose of this section to explain this.

In Section 3 we showed that every plausible sequence can be extended into a realizable one by adding pentagons and heptagons only. We expect that our construction will help yield more general results: in what follows we are going to give a rough sketch of a proof that an arbitrary neutral sequence $s$ can be used to extend any plausible sequence into a realizable one under the assumption that a couple of basic building blocks can be constructed using precisely the faces that appear in some multiple of $s$.

So let $p = (p_3, p_4, \ldots, p_r)$ be a plausible sequence for the sphere or the torus, and let $s = (p'_3, p'_4, \ldots, p'_t)$ be a neutral sequence. In order to prove that there is some $n$ so that $p + ns$ is realizable, it suffices to find some $k \in \mathbb{N}$ for which it is possible to construct the following building blocks using precisely the faces that appear in some multiple of $s$:

(i) a $(k, k, k)$-triarc;

(ii) a $(k, k, k-1)$-triarc;

(iii) for every non-zero entry $p_l$ in $p$, a triarc containing a face of size $l$, such that the length of two of the sides of this triarc is a multiple of $k$;

(iv) a “ring” like the one in Figure 4 (using the faces from $s$ in the right proportion rather than pentagons and heptagons) for combining two equilateral triarcs.

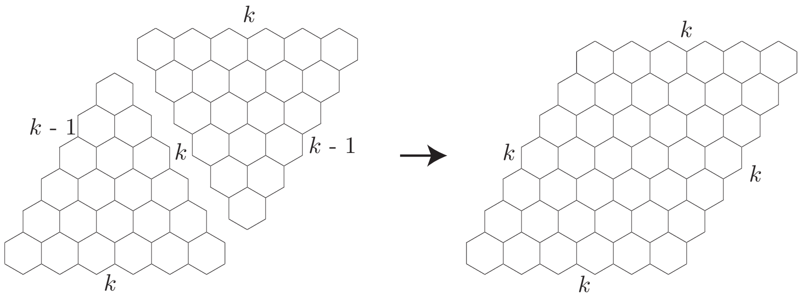


Figure 9: Constructing a parallelogram out of two $(k, k, k-1)$-triarcs.

Indeed, to begin with, construct a parallelogram with all sides of length $k$ out of two $(k, k, k-1)$-triarcs (supplied by (ii)) as shown in Figure 9. (In figures explaining our construction, we shall use triarcs made of hexagonal faces, but this is for illustration purposes only; in fact they have to be made of multiples of $s$.) This also allows us to construct any parallelogram with dimensions $mk, lk$ for every $m, l \in \mathbb{N}$.

Next, similarly to the construction in Theorem 3.1, construct a ‘basic’ triarc as in (iii) for each face-length $l$ for which $p_l \neq 0$; in fact, we construct $p_l$ copies of this basic triarc for every $l$. Then, using the parallelograms we constructed earlier, we recursively glue all those triarcs together into a single triarc $T$, in a manner very similar to the operation of Figure 1.

By recursively glueing the resulting triarc with a $(k, k, k-1)$-triarc provided by (ii) using the glueing operation of Figure 1, we can transform $T$ into an equilateral $(mk, mk, mk)$-triarc $T'$ for some (large) $m \in \mathbb{N}$.

Using the glueing operation of Figure 1 it is possible to construct a triarc $R$ with the same side-lengths as $T'$, using only $(k, k, k)$-triarcs (provided by (i)) and the above parallelograms; see Figure 10.

In the case of the sphere, we can combine $R$ and $T'$ by using the “ring” provided by (iv) to complete the construction.

If the underlying surface $S$ is the torus, we glue $R$ and $T'$ together along one of their sides to obtain a parallelogram, and glue two opposite sides of this parallelogram together to obtain a cylinder $C$ both of whose bounding

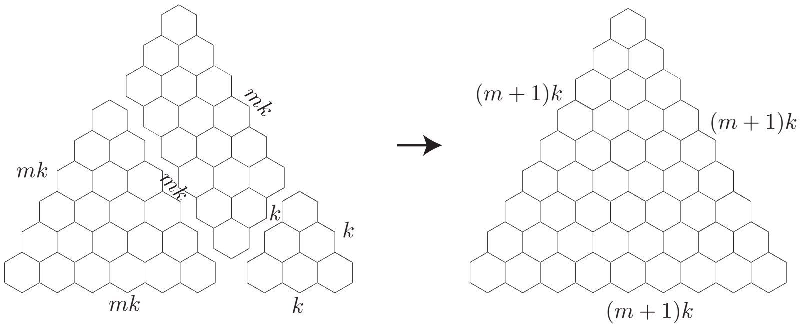


Figure 10: Increasing the length each side of a triarc by $k$.

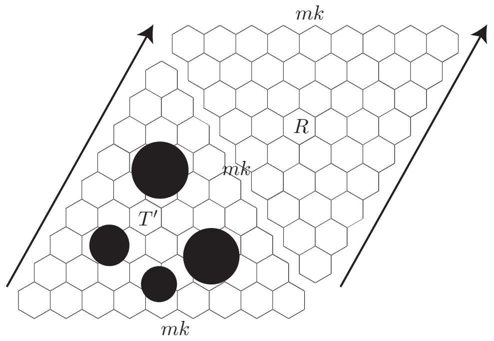


Figure 11: Glueing $R$ and $T'$ together. The black dots depict the faces imposed by the sequence $p$.

cycles are in-out alternating cycles of length $mk$, see Figure 11. We then glue the two bounding cycles of $C$ together to obtain a realization of a torus.

If $p$ is plausible for some other surface $S$, then we would need additional gadgets like those used in the proof of Corollary 3.2.

# 5 Outlook

Trying to achieve a better understanding of the implications of Euler's formula, we studied the question of whether, given a plausible sequence $p$, and a neutral sequence $q$, it is possible to combine $p$ and $q$ into a realizable sequence $p + nq$, but we did so in very restricted cases. The general problem remains wide open; in particular, we would be interested to see an answer to the following problem.

**Problem 5.1.** *Given a closed surface $S$, is it true that for every plausible sequence $p$ for $S$, and every neutral sequence $q$, there is an $n \in \mathbb{N}$ such that $p+nq$ is realizable in $S$ with the exception of only finitely many pairs $(p, q)$?*

(As mentioned in the introduction, if $S$ is the torus then the list of exceptional pairs $(p, q)$ cannot be empty.)

# References


[1] D. Barnette, On $p$-vectors of 3-polytopes. J. Combinatorial Theory 7 (1969), 99–103.

[2] D.W. Barnette, P. Gritzmann, R. Höhne, On valences of polyhedra. J. Combin. Theory Ser. A 58 (1991), no. 2, 279–300.

[3] V. Eberhard, Zur Morphologie der Polyeder, Teubner, Leipzig, 1891.

[4] T.C. Enns, 4-valent graphs. J. Graph Theory 6 (1982), no. 3, 255–281.

[5] J.C. Fisher, An existence theorem for simple convex polyhedra. Discrete Math. 7 (1974), 75–97.

[6] J.C. Fisher, Five-valent convex polyhedra with prescribed faces. J. Combin. Theory, Ser. A 18 (1975), 1–11.

[7] P. Gritzmann, The toroidal analogue to Eberhard's theorem. Mathematika 30 (1983), no. 2, 274–290.

[8] B. Grünbaum, Some analogues of Eberhard's theorem on convex polytopes. Israel J. Math. 6 (1968), 398–411.

[9] B. Grünbaum, Convex polytopes. Second edition. Springer-Verlag, New York, 2003.

[10] S. Jendrol', On face-vectors and vertex-vectors of polyhedral maps on orientable 2-manifolds. Math. Slovaca 43 (1993), 393–416.

[11] S. Jendrol', On face vectors and vertex vectors of convex polyhedra. Discrete Math. 118 (1993), 119–144.

[12] S. Jendrol', E. Jucovic, On the toroidal analogue of Eberhard's theorem. Proc. London Math. Soc. (3) 25 (1972), 385–398.

[13] S. Jendrol', E. Jucovic, On a conjecture by B. Grünbaum. Discrete Math. 2 (1972), 35–49.

[14] B. Mohar and C. Thomassen, Graphs on Surfaces. Johns Hopkins University Press, Baltimore, 2001.

[15] J. Zaks, The analogue of Eberhard's theorem for 4-valent graphs on the torus. Israel J. Math. 9 (1971), 299–305.

[16] J. Zaks, 6-valent analogues of Eberhard's theorem. Israel J. Math. 18 (1974), 19–29.