A note on edge-colourings avoiding rainbow K_4 and monochromatic K_m

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Abstract

We study the mixed Ramsey number $maxR(n, K_m, K_r)$, defined as the maximum number of colours in an edge-colouring of the complete graph K_n, such that K_n has no monochromatic complete subgraph on m vertices and no rainbow complete subgraph on r vertices. Improving an upper bound of Axenovich and Iverson, we show that $maxR(n, K_m, K_4) \leq n^{3/2}\sqrt{2m}$ for all $m \geq 3$. Further, we discuss a possible way to improve their lower bound on $maxR(n, K_4, K_4)$ based on incidence graphs of finite projective planes.

1 Introduction

A subgraph of an edge-coloured graph is *monochromatic* if all of its edges receive the same colour, and it is *rainbow* if all the edge colours are distinct. Ramsey theory was born with the observation that every sufficiently large

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*The Institute for Theoretical Computer Science (ITI) is supported by project 1M0545 of the Czech Ministry of Education.

complete graph whose edges are coloured by k colours, where k is a fixed integer, contains a large monochromatic complete subgraph [15, 8]. In the following decades, it evolved into a rich part of graph theory with strong links to combinatorial number theory [13] and combinatorial geometry (see, e.g., [14]). There are many Ramsey-type problems that involve monochromatic substructures of various combinatorial structures, but some are of a different type.

Among them is the question asked by Erdős, Simonovits and Sós [7]: Given a graph H and an integer n, what is the maximum number of colours in an edge-colouring of a complete graph K_n such that no copy of H in K_n is rainbow? This number is the *anti-Ramsey number* (for H and n) (see also [11, 12]).

As a combination of the two problems, Axenovich and Iverson [1] defined the *mixed Ramsey numbers* $maxR(n, G, H)$ and $minR(n, G, H)$ as the maximum (respectively, minimum) number of colours in an edge-colouring of K_n such that no monochromatic subgraph of K_n is isomorphic to G and no rainbow subgraph is isomorphic to H. They noted that the numbers are well-defined whenever the edges of G do not induce a star and H is not a forest (see also [9]). Their results asymptotically determine the behaviour of $maxR(n, G, H)$ in most cases and exhibit a close relation between this number and the *vertex arboricity* $a(H)$ of H, defined as the least number of parts in a decomposition of $V(H)$ into sets inducing acyclic subgraphs of H.

In the present paper, we will be concerned with bounds on the number $maxR(n, K_m, K_4)$ for $m \geq 3$. Let us briefly recall some of the results of [1] related to $maxR(n, G, H)$. Assume that the edges of G do not induce a star. If $a(H) \geq 3$, then $maxR(n, G, H)$ is quadratic in n, namely

$$maxR(n, G, H) = \frac{n^2}{2}\Big(1 - \frac{1}{a(H) - 1}\Big)(1 + o(1)).$$

On the other hand, if $a(H) = 2$, then $maxR(n, G, H)$ is subquadratic:

$$maxR(n, G, H) = O(n^{2-\frac{1}{\epsilon}}),$$

for some ϵ which depends on G and H. There exists a more explicit upper bound if $V(H)$ can be decomposed into two sets inducing forests, one of which is of order at most 2. In this case,

$$maxR(n, G, H) \leq n^{5/3}(1 + o(1)). \tag{1}$$

In the special case that H is a cycle, $maxR(n, G, H)$ can be determined for non-bipartite graphs G:

$$maxR(n, G, C_k) = n\Big(\frac{k-2}{2} + \frac{1}{k-1}\Big) + O(1).$$

As for the lower bounds, Axenovich and Iverson [1] prove that if G is non-bipartite and the minimum degree of H is at least 3, then

$$maxR(n, G, H) \geq n \log n. \tag{2}$$

If we restrict to the case where G and H are complete graphs, the above results asymptotically determine $maxR(n, G, H)$ in all cases except $H = K_4$, where we only have the bounds (1) and (2). In particular, the problem of determining $maxR(n, K_4, K_4)$ is referred to in [1] as 'one of the most intriguing' in this area.

The purpose of this note is to improve the upper bound on the number $maxR(n, K_m, K_4)$ for all $m \geq 3$:

Theorem 1.

$$maxR(n, K_m, K_4) \leq n^{3/2}\sqrt{2m}.$$

We prove this bound in Section 2. In Section 3, we discuss a possible way to improve the lower bound (for $m = 4$) based on incidence graphs of finite projective planes, and present some open problems.

2 The upper bound

Let $m \geq 3$ be a fixed integer throughout this section. Let us call an edge-colouring of a complete graph K_n *admissible* if K_n has no monochromatic complete subgraph on m vertices and no rainbow complete subgraph on 4 vertices. For an admissible edge-colouring c of K_n and disjoint sets $A, B \subset V(K_n)$, we define $S_c(A, B)$ as the set of colours that are used by c, but only on edges joining A to B. Furthermore, we set $\sigma_c(A, B) = |S_c(A, B)|$.

To prove the following lemma, one could use a suitable version of the Zarankiewicz theorem (e.g., that in [10, Exercise 2.6]). For the reader's convenience, we give a self-contained proof.

Lemma 2. *Let c be an admissible edge-colouring of K_n and A, B disjoint subsets of $V(K_n)$ each of size at most k. Then*

$$\sigma_c(A, B) \leq k^{3/2}\sqrt{m}.$$

Proof. For each colour $s \in S_c(A,B)$, choose an edge of colour s and let Y be the set of all chosen edges. Define H to be the spanning subgraph of K_n with edge set Y. Observe that $|Y| = \sigma_c(A,B)$. We claim that every two vertices $x, y \in B$ have fewer than m common neighbors in the graph H.

For any two vertices $x, y \in B$, let A_{xy} be the set of common neighbours of x and y in the graph H. Consider $z_1, z_2 \in A_{xy}$. Since the induced subgraph of K_n on $\{x, y, z_1, z_2\}$ is not rainbow, we must have $c(xy) = c(z_1 z_2)$. But then all the edges on A_{xy} have colour $c(xy)$. Since G contains no monochromatic complete subgraph of order m, we have $|A_{xy}| \leq m-1$ for every $x, y \in B$.

Let N be the number of all triples xyz with $x, y \in B$ and $z \in A_{xy}$. By the above,

$$N \leq (m-1)\binom{|B|}{2}.$$

On the other hand, if we set $d_1, \ldots, d_\ell$ to be the degrees of the vertices in A in the graph H ($\ell = |A|$), we find that N equals the sum of $\binom{d_i}{2}$ and therefore

$$\sum_{i=1}^{\ell} \binom{d_i}{2} \leq (m-1)\binom{k}{2}. \tag{3}$$

Since the function $f(x) = x(x-1)/2$ is convex, we may use Jensen's inequality to derive

$$\sum_{i=1}^{\ell} \binom{d_i}{2} \geq \ell \cdot \frac{(\sum_{i=1}^{\ell} d_i)/\ell \cdot ((\sum_{i=1}^{\ell} d_i)/\ell - 1)}{2}.$$

Observing that the sum of the d_i is $|Y|$ and combining with (3), we obtain

$$|Y|\,(|Y| - \ell) \leq k(k-1)(m-1)\ell.$$

Furthermore, ℓ may be replaced with k on both sides of the inequality as $\ell \leq k$. This leads to the following quadratic inequality in $|Y|$:

$$|Y|^2 - k\,|Y| - k^2(k-1)(m-1) \leq 0. \tag{4}$$

Solving (4), we find

$$|Y| \leq k \cdot \frac{1 + \sqrt{1 + 4(k-1)(m-1)}}{2}. \tag{5}$$

The fraction in the right hand side of (5) is easily seen to be at most $\sqrt{km}$ by a direct calculation, so $|Y| \leq k^{3/2}\sqrt{m}$ and the statement of the lemma is true. □

It is now easy to derive our upper bound on $maxR(n, K_m, K_4)$:

Proof of Theorem 1. We proceed by induction on n. It is easy to check that for $n \leq 21$, $\binom{n}{2}$ is less than $n^{3/2}\sqrt{2m}$ for $m \geq 3$, so we may assume that $n \geq 22$. Set $\alpha = (1 + 1/22)^{3/2}$ and note that $(n+1)^{3/2} \leq \alpha n^{3/2}$.

Let c be an admissible colouring of K_n. For $X \subset V(K_n)$, define $\ell(X)$ as the number of colours used for edges on X. We need to prove that $\ell(V(K_n)) \leq n^{3/2}\sqrt{2m}$. To this end, partition $V(K_n)$ arbitrarily into sets A and B such that $|A| \leq n/2$ and $|B| \leq (n+1)/2$. By Lemma 2 and the induction, we then have

$$\begin{aligned}
\ell(V(K_n)) &\leq \ell(A) + \ell(B) + \sigma_c(A, B) \\
&\leq \left(\frac{n}{2}\right)^{3/2}\sqrt{2m} + \left(\frac{n+1}{2}\right)^{3/2}\sqrt{2m} + \left(\frac{n+1}{2}\right)^{3/2}\sqrt{m} \\
&\leq n^{3/2}\sqrt{m} \cdot \frac{\alpha(\sqrt{2}+1)+\sqrt{2}}{2\sqrt{2}} \\
&< n^{3/2}\sqrt{2m}.
\end{aligned}$$

□

3 Lower bounds

Theorem 1 improves the asymptotic upper bound for $maxR(n, K_4, K_4)$ to $O(n^{3/2})$, but this is still far from the lower bound $n \log n$ of (2). We now discuss a possible way to improve the lower bound, which is based on incidence graphs of finite projective planes. (See, e.g., [4] for background on finite geometries.)

Throughout this section, let q be a prime power and $n(q) = 2(q^2+q+1)$. Recall that there is a projective plane $PG(2, q)$ of order q. The incidence graph of $PG(2, q)$ is a $(q+1)$-regular bipartite graph L_q whose vertices are the points and the lines of $PG(2, q)$, and whose edges join each point p to the lines containing p. Since $PG(2, q)$ has $q^2 + q + 1$ points and the same number of lines, we can (and will) consider L_q as a spanning subgraph of the complete graph on $n(q)$ vertices.

One way to obtain an admissible colouring of $K_{n(q)}$ using $\Omega(n^{3/2})$ colours is to first colour L_q, assigning each of its edges a colour of its own (one that does not appear on any other edge of L_q), and then try to extend this colouring to an admissible colouring of $K_{n(q)}$. Since $L(q)$ has $\Omega(n(q)^{3/2})$ edges, the number of colours is as requested.

Among the colourings obtained this way, we looked for ones satisfying a mild additional restriction (which may make them somewhat easier to find). Call a colouring c of $K_{n(q)}$ *special* if no edge of $L_q \subset K_{n(q)}$ has a colour which is used on another edge of $K_{n(q)}$. Note that to describe the colouring up to a permutation of colours, it suffices to specify the colours of the edges not in L_q.

Figure 1 shows that special colourings do exist in the case $q = 2$, where we obtain the well-known Heawood graph on 14 vertices as the graph L_2. One method to find such colourings is as follows. Regarding the vertices of K_{14} as points and lines of $PG(2, 2)$, choose a 7-cycle $C \subset K_{14}$ on the points and a 7-cycle $C' \subset K_{14}$ on the lines such that every edge of C and every edge of C' are at distance 1 in L_2. (It is not difficult to show that such a choice is possible.) Assign colour 0 to all edges of K_{14} that are included in C or C', or join a point of $PG(2, 2)$ to a line. Colour the other edges of K_{14} with colour 1. Easy case analysis confirms that the associated special colouring is indeed admissible.

In general, a rotational symmetry such as that of Figure 1 may be useful when looking for special colourings. The first question to be addressed, however, is whether the graphs L_q ($q \geq 3$) themselves admit a rotationally symmetric drawing. More precisely, let us call a Hamilton cycle $v_0 v_1 \dots v_{n(q)-1}$ in L_q *rotational* if for each $i, j \in \{0, \dots, n(q) - 1\}$, $v_i v_j \in E(L_q)$ if and only if $v_{i+2} v_{j+2} \in E(L_q)$ (indices taken modulo $n(q)$). Somewhat surprisingly, it turned out that all the graphs L_q, where q is a prime power, have rotational Hamilton cycles. We suppose that this is a known result, but since our search in the literature did not reveal anything, we briefly sketch the proof. The only result in this direction we are aware of is the result of Brown [3] that the graphs L_q, for prime q, are Hamiltonian. (We are indebted to Geoff Exoo for this information.)

Proposition 3. *For any prime power q, the graph L_q admits a rotational Hamilton cycle.*

Proof. The proof is inspired by a proof of Erdős [6] concerning so-called Sidon sets (see also [5]), which in turn builds on a proof of Singer [16] of the existence of perfect difference sets. We take a primitive element α in

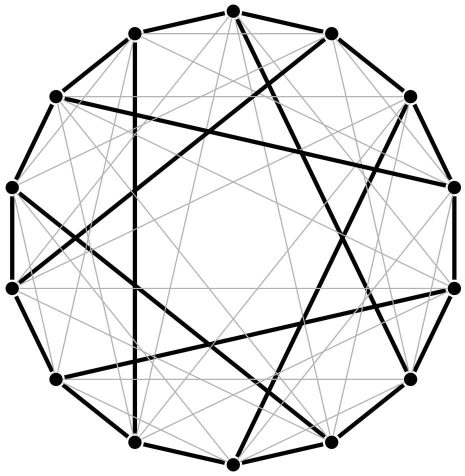

Figure 1: A special admissible colouring of K_{14} by 23 colours. Thick edges are those of L_2 (hence each of them has a colour of its own, say from $\{2,\ldots,22\}$), missing edges represent colour 0, grey edges represent colour 1.

the field $GF(q^3)$ and view the field as a vector space V of dimension 3 over $F = GF(q)$. Recall that points and lines of $PG(2,q)$ correspond one-to-one to subspaces of V of dimension 1 and 2, respectively. In particular, for $i = 0,\ldots,q^2+q$, let p_i be the point of $PG(2,q)$ corresponding to the line in V through 0 and α^i. All of these points are distinct, since α^j is a scalar multiple of α^i if and only if $j-i$ is a multiple of q^2+q+1.

Let ℓ_i ($i = 0,\ldots,q^2+q-1$) be the unique line of $PG(2,q)$ through α^i and α^{i+1}, and similarly let ℓ_{q^2+q} be the line through α^{q^2+q} and 1.

To verify that $H = (p_0,\ell_0,p_1,\ldots,p_{q^2+q},\ell_{q^2+q})$ is a Hamilton cycle in L_q, all we need to show is that all the lines ℓ_i are distinct. Suppose not. Then for some $0 \le i < j \le q^2+q$, the set $\left\{0,\alpha^i,\alpha^{i+1},\alpha^j,\alpha^{j+1}\right\}$ is contained in a plane P of V (note that this holds even in the boundary case $j = q^2+q$).

Without loss of generality, we may assume that $i = 0$ (multiplying the equation of P by α^{-i} if necessary), which implies that $\left\{0,1,\alpha,\alpha^j,\alpha^{j+1}\right\}$ is contained in P. Since 1 and α span P, we may write $\alpha^j = c\alpha + d$, where $c,d \in F$. Hence $\alpha^{j+1} = c\alpha^2 + d\alpha$, and since α^{j+1} is in P, so is α^2. However, a similar argument then shows that α^3 and all the successive powers of α

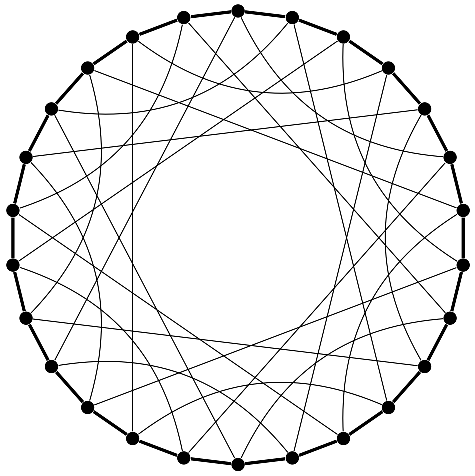

Figure 2: A rotational Hamilton cycle in L_3 (bold).

are also in P, a contradiction with the fact that V is 3-dimensional.

What remains to be shown is that the Hamilton cycle H is rotational, i.e. that if a point p_i lies on a line ℓ_j, then p_{i+1} lies on ℓ_{j+1}. A key observation is that p_i lies on ℓ_j if and only if α^i, α^j and α^{j+1} are linearly dependent in V. Assuming this condition holds, it is clear that α^{i+1}, α^{j+1} and α^{j+2} are also linearly dependent, i.e. p_{i+1} lies on ℓ_{j+1} as claimed. $\square$

An example of a rotational Hamilton cycle constructed using Proposition 3 is shown in Figure 2.

Let $v_0 v_1 \dots v_{n(q)-1}$ be a rotational Hamilton cycle in L_q and let c be an edge-colouring of $K_{n(q)}$ with colours in a set Y. For $i \neq j$ in $\{0, \dots, n(q)-1\}$, we define a symbol $\bar{c}_{i,j} \in Y \cup \{*\}$ by

$$\bar{c}_{i,j} = \begin{cases} * & \text{if } v_i v_j \text{ is an edge of } L_q, \\ c(v_i v_j) & \text{otherwise.} \end{cases}$$

For $i = 0, \dots, n(q)-1$, we define the words

$$c(v_i) = (\bar{c}_{i,i+1} \bar{c}_{i,i+2} \dots \bar{c}_{i,i-1}),$$

where the indices are taken modulo $n(q)$. We extend the above terminology and call the colouring c *rotational* if $c(v_i) = c(v_0)$ for all even i, and $c(v_i) = c(v_1)$ for all odd i.

This is the case in Figure 1, where we have

$$c(v_0) = (*001*1010100*),$$
$$c(v_1) = (*1010000*101*).$$

For $q = 3$, we found a number of rotational colourings by a computer search. One of these, for instance, is determined by the words

$$c(v_0) = (*00001*001*1110100110010*),$$
$$c(v_1) = (*0100110010111*100*10000*).$$

Note that the words in the latter case have the additional curious property that $c(v_1)$ is the reverse of $c(v_0)$.

In general, we had to leave the following problem open:

Problem 1. *For $q \geq 3$, are there any admissible rotational colourings of $K_{n(q)}$? Are there any admissible special colourings?*

We think that even a negative answer to Problem 1 may shed some light on the question whether the upper bound given in Theorem 1 is asymptotically tight.

Acknowledgment

We are indebted to Martin Klazar for pointing us to the results on Sidon sets and perfect difference sets which led us to the proof of Proposition 3. Geoff Exoo kindly informed us about the paper of Brown [3].

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