On Generalized Middle Level Problem*

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Abstract

Let G_n^k be the subgraph of the hypercube Q_n induced by levels between k and $n-k$, where $n \geq 2k+1$ is odd. The well-known middle level conjecture asserts that G_{2k+1}^k is Hamiltonian for all $k \geq 1$. We study this problem in G_n^k for fixed k. It is known that G_n^0 and G_n^1 are Hamiltonian for all odd $n \geq 3$. In this paper we prove that also G_n^2 is Hamiltonian for all odd $n \geq 5$, and we conjecture that G_n^k is Hamiltonian for every $k \geq 0$ and every odd $n \geq 2k+1$.

1 Introduction

Let G_n^k be the subgraph of the n-dimensional hypercube Q_n induced by the vertices in levels between k and $n-k$, where $n \geq 2k+1$ is odd. The level i consists of vertices with exactly i 1's. Note that n is required to be odd in order to have bipartite classes of equal size in G_n^k.

The well-know middle level conjecture, attributed to Havel [6], asserts that the graph G_{2k+1}^k consisting of two middle levels k and $k+1$ of Q_{2k+1} is Hamiltonian for all $k \geq 1$. This graph is a notorious example of a connected vertex transitive graph, all of which were conjectured by Lovász [11] to have Hamiltonian paths.

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Despite many attempts, the middle level problem remains open. The conjecture was verified for $k \leq 11$ by Moews and Reid in unpublished work in 1990. Then it was extended by Shields, Shields, and Savage [15, 16] also for $12 \leq k \leq 17$.

One possible relaxation of this problem is to show that G^k_{2k+1} at least contains long cycles. Savage and Winkler [13] showed that G^k_{2k+1} has a cycle of length at least $0.867|V(G^k_{2k+1})|$. The best lower bound by Johnson [9] shows that G^k_{2k+1} is "asymptotically" Hamiltonian: it contains a cycle of length $(1-o(1))|V(G^k_{2k+1})|$. On the other hand, Horák, Kaiser, Rosenfeld, and Ryjáček [7] showed that G^k_{2k+1} has a closed spanning walk in which every vertex appears at most twice, by proving that the prism over G^k_{2k+1} is Hamiltonian.

Our approach is to study this problem for the graph G^k_n where $0 \leq k \leq (n-1)/2$ is fixed. It is well-known that $G^0_n = Q_n$ is Hamiltonian for every $n \geq 2$. El-Hashash and Hassan [5], and independently (in a more general setting) Locke and Stong [12] proved that G^1_n is Hamiltonian for all odd $n \geq 3$. As a first step towards the general problem, in this paper we prove that also G^2_n is Hamiltonian for all odd $n \geq 5$. Now, it becomes naturally to conjecture:

Conjecture 1. *G^k_n is Hamiltonian for every $k \geq 0$ and every odd $n \geq 2k+1$.*

A different approach to generalize the middle level problem was proposed, as far as we know, independently by Dejter, Cedeno, and Jaurequi [2] and by Hurlbert [8] who studied Hamiltonian cycles in the graph H^k_n consisting of level k and $n-k$ of Q_n, and edges joining a vertex from level k with a vertex from level $n-k$ if their Hamming distance (distance in Q_n) is $n-2k$. In other words, H^k_n is the cover graph of the ordered set consisting of level k and $n-k$ of the Boolean lattice B_n with the order inherited from B_n. For other results on ordered sets obtained by removing selected levels of the Boolean lattice, called Boolean layer cakes, we refer to a survey of Schmidt [14].

2 Preliminaries

Let $[n]$ denote the set $\{1,\ldots,n\}$. For a binary vector $v \in \{0,1\}^n$ and $i \in [n]$ we denote by $v[i]$ the i-th coordinate of v. For vectors $u, v \in \{0,1\}^n$ let $u \oplus v$ denote the vector obtained by the coordinate-wise addition modulo 2 of u and v. The *n-dimensional hypercube* Q_n is a (bipartite) graph with all binary vectors of length n as vertices and with edges joining every two vertices that differ in exactly one coordinate, i.e.

$$V(Q_n) = \{0,1\}^n \text{ and } E(Q_n) = \{uv; |\Delta(u,v)| = 1\},$$

where $\Delta(u,v) = \{i \in [n]; u[i] \neq v[i]\}$. Thus the distance of vertices u and v is $d(u,v) = |\Delta(u,v)|$. The distance of two edges uv and xy is the minimum distance between a vertex of uv to a vertex of xy. A vertex v is said to be *even* (*odd*) if it has even (odd) weight. The *weight* of v is the number of 1's in v. Note that vertices of each parity form bipartite classes of Q_n. Consequently, u and v have the same parity if and only if $d(u,v)$ is even.

Let $\mathbf{0}, \mathbf{1} \in V(Q_n)$ be the vertices of all 0's and 1's, respectively. For $i \in [n]$ we denote by e_i the vertex containing 1 exactly in the i-th coordinate. Note that each vertex e_i is adjacent to $\mathbf{0}$. An i-th *level* of Q_n for $0 \leq i \leq n$ is the set of vertices of weight i. An *antipodal* vertex to a vertex $u \in V(Q_n)$ is the vertex denoted by $\overline{u}$ such that $\overline{u}[i] = \overline{u[i]}$ for all $i \in [n]$, that is $d(u, \overline{u}) = n$.

If adjacent vertices u and v of Q_n differ in the i-th coordinate, then $u \oplus v = e_i$ and we say that the edge $uv \in E(Q_n)$ has *direction* i. By removing all edges of a fixed direction $i \in [n]$, the hypercube Q_n is split into two (induced) subgraphs isomorphic to Q_{n-1}. We say that Q_n is split *along the direction* i into *subcubes* Q^0_{n-1} and Q^1_{n-1}. For $a \in \{0,1\}$ the subcube Q^a_{n-1} is induced by all vertices $u \in V(Q_n)$ with $u[i] = a$. Furthermore, by splitting Q^0_{n-1} and Q^1_{n-1} along another direction $j \in [n] \setminus \{i\}$ we obtain four subcubes Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2}, and Q^{11}_{n-2}. Note that for $a, b \in \{0,1\}$ the subcube Q^{ab}_{n-2} is induced by all vertices $u \in V(Q_n)$ with $u[i] = a$ and $u[j] = b$.

We consider a *path* P to be a nonempty sequence of distinct vertices such that every two consecutive vertices are adjacent. If a and b are the first and the last vertices of P, respectively, we say that P is an *ab-path* and a, b are its *endvertices*. Assume that an ab-path P and an cd-path R are (vertex) disjoint. If b and c are adjacent, then the concatenation of P and R is an ad-path. If P contains consecutive vertices x and y such that both x, c and y, d are adjacent, then by inserting R into P between x and y we obtain an ab-path containing vertices $P \cup R$. A *reversed* path of an ab-path P is the ba-path obtained by the reversed sequence.

It is well known that the hypercube Q_n for every $n \geq 2$ is Hamiltonian and also Hamiltonian-laceble; that is, there is a Hamiltonian path between every two vertices of opposite parity. We will also need several simple results on Hamiltonian cycles and paths in the hypercube with some removed

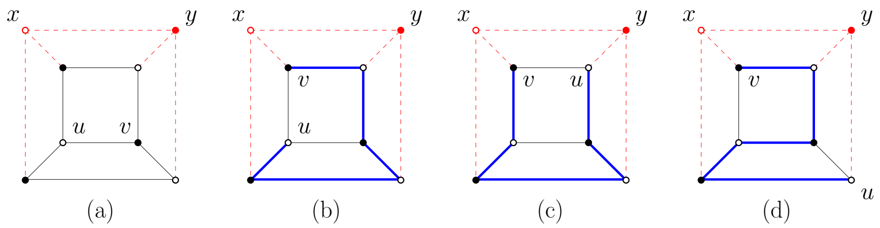

Figure 1: All configurations (up to isomorphism) in Proposition 2 for $n = 3$.

vertices. The case of one removed vertex was described by Lewinter and Widulski [10].

Proposition 1. *If distinct $u, v \in V(Q_n)$, where $n \geq 2$, have the same parity that is opposite to the parity of $x \in V(Q_n)$, then $Q_n - \{x\}$ has a Hamiltonian uv-path.*

A similar result holds, up to one exception, for the case of two removed vertices that are adjacent.

Proposition 2. *If $u, v \in V(Q_n) \setminus \{x, y\}$, where $xy \in E(Q_n)$ and $n \geq 2$, have the opposite parity, then $Q_n - \{x, y\}$ has a Hamiltonian uv-path unless:*

$$n = 3, \ u \oplus v = x \oplus y, \ \text{and} \ d(uv, xy) = 2. \tag{1}$$

Proof. The exceptional configuration (1) is depicted on Figure 1(a). We proceed by induction on the dimension n. For $n = 2$ the statement trivially holds. For $n = 3$, aside from the exceptional configuration (1), we have (up to isomorphism) another three configurations depicted on Figure 1(b)-(d). Observe that the statement holds for each of them.

For $n \geq 4$ we split Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} such that the edge xy belongs to Q^0_{n-1} or Q^1_{n-1}, and moreover, the vertex v is in the other subcube than the edge xy. Assume without loss of generality that $xy \in E(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Considering the position of the vertex u, we distinguish two cases.

Case 1: $u \in V(Q^1_{n-1})$. Let P_1 be a Hamiltonian uv-path of Q^1_{n-1}. We claim that P_1 contains consecutive vertices a and b such that their neighbors a' and b' in Q^0_{n-1} are distinct from both x and y, and the edge $a'b' \in E(Q^0_{n-1})$ does not form the exceptional configuration (1) in Q^0_{n-1}. Since $n \geq 4$, the path P_1 contains at least 7 edges. At most 4 of them contain a vertex whose neighbor in Q^0_{n-1} is x or y. In addition, at most one of them contains vertices whose neighbors in Q^0_{n-1} form the configuration (1). Hence P_1 contains at least 2 edges that satisfy the claim.

Applying induction we obtain a Hamiltonian $a'b'$-path P_0 of $Q^0_{n-1} - \{x, y\}$. By inserting P_0 into P_1 instead of the edge ab, we have the desired path.

Case 2: $u \in V(Q^0_{n-1})$. First we choose a neighbor $a \in V(Q^0_{n-1})$ of u such that the edge ua does not form the configuration (1). Amongst neighbors of u in Q^0_{n-1} at most one is x or y, and at most one forms the configuration (1). Thus, such neighbor a exists since u has at least 3 neighbors in Q^0_{n-1}. Applying induction we obtain a Hamiltonian ua-path P_0 of Q^0_{n-1}. Let a' be the neighbor of a in Q^1_{n-1}, and let P_1 be a Hamiltonian $a'v$-path of Q^1_{n-1}. By concatenating P_0 and P_1, we are finished. $\square$

Let us denote by $N_G(u)$ the set of neighbors of a vertex u in a subgraph G of Q_n. If $G = Q_n$, the subscript Q_n is omitted. Recall that $N(\mathbf{0}) = \{e_1, \dots, e_n\}$. For $n \geq 2$ and distinct $i, j \in [n]$, we define the set

$$A_{ij} = (N(\mathbf{0}) \setminus \{e_i, e_j\}) \cup (N(e_i) \setminus \{e_i \oplus e_j\}). \tag{2}$$

Note that A_{ij} contains $n-2$ odd vertices and $n-1$ even vertices of Q_n including the vertex $\mathbf{0}$. We continue with a result on Hamiltonicity of Q_n in case of $2n-3$ removed vertices of this set A_{ij}.

Proposition 3. *If $z \in V(Q_n) \setminus A_{ij}$ is odd, $n \geq 2$, and $z \neq e_i$ where $i, j \in [n]$ are distinct, then $Q_n - A_{ij}$ has a Hamiltonian $e_i z$-path.*

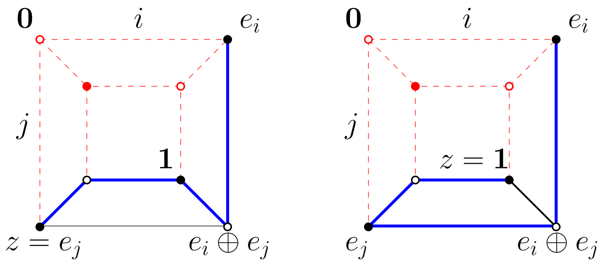

Figure 2: The Hamiltonian paths in Proposition 3 for $n = 3$.

Proof. We proceed by induction on the dimension n. For $n = 2$ the statement trivially holds. For $n = 3$ we have either $z = e_j$ or $z = \mathbf{1}$. The Hamiltonian paths for both cases are depicted on Figure 2.

For $n \geq 4$ we choose $k \in [n]$ distinct from i and j, and we split Q_n along the direction k into subcubes Q^0_{n-1} and Q^1_{n-1}. For $A_0 = A_{ij} \cap V(Q^0_{n-1})$ and $A_1 = A_{ij} \cap V(Q^1_{n-1})$ observe that A_0 restricted to $n-1$ directions of Q^0_{n-1} satisfies (2), and $A_1 = \{e_k, e_i \oplus e_k\}$. The idea is to apply induction in Q^0_{n-1} and Proposition 2 in Q^1_{n-1}. We distinguish the following two cases regarding z; see Figures 3 and 4 for an illustration.

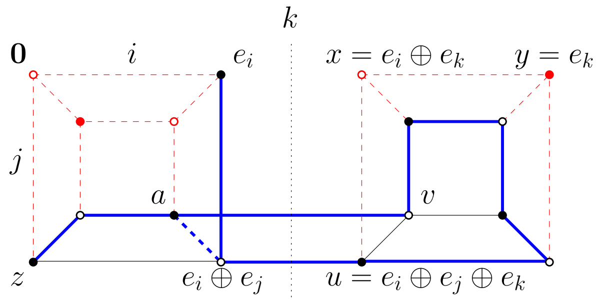

Figure 3: The case $z \in V(Q^0_{n-1})$ in Proposition 3 for $n = 4$.

Case 1: $z \in V(Q^0_{n-1})$. Applying induction we obtain a Hamiltonian e_iz-path P_0 of $Q^0_{n-1} - A_0$. Note that P_0 goes first from e_i to $e_i \oplus e_j$ since e_i has no other neighbors in $Q^0_{n-1} - A_0$. Let $a \neq e_i$ be the next vertex on P_0 after $e_i \oplus e_j$, and let u and v be the neighbors of $e_i \oplus e_j$ and a in Q^1_{n-1}, i.e. $u = e_i \oplus e_j \oplus e_k$ and $v = a \oplus e_k$. By Proposition 2 for $x = e_i \oplus e_k$ and $y = e_k$, we obtain a Hamiltonian uv-path P_1 of $Q^1_{n-1} - A_1$. Note that we avoid the exceptional configuration (1) since $d(u, x) = 1$. By inserting P_1 into P_0 instead of the edge between $e_i \oplus e_j$ and a, we construct the desired path.

Case 2: $z \in V(Q^1_{n-1})$. Applying induction we obtain a Hamiltonian path P_0 of $Q^0_{n-1} - A_0$ between e_i and $u = e_j$. By Proposition 2 for $x = e_i \oplus e_k$ and $y = e_k$ we obtain a Hamiltonian path P_1 of $Q^1_{n-1} - A_1$ between $v = e_j \oplus e_k$ and z. Note that we avoid the exceptional configuration (1) since $d(v, y) = 1$. It remains to concatenate P_0 and P_1, and we are done. $\square$

For the sake of simplicity, the set A_{ij} is defined and Proposition 3 is

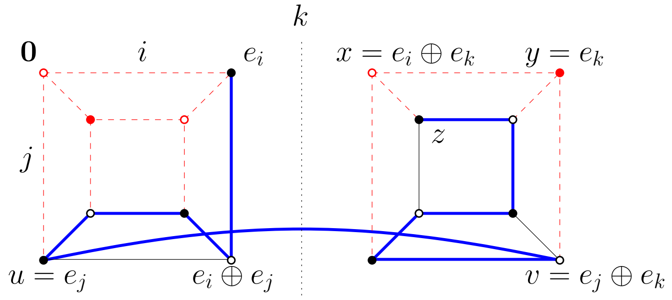

Figure 4: The case $z \in V(Q^1_{n-1})$ in Proposition 3 for $n = 4$.

stated with respect to the vertex $\mathbf{0}$. However, note that by the automorphism of Q_n, Proposition 3 could be stated more generally for the set $A'_{ij} = A_{ij} \oplus w$ and the endvertices $e'_i = e_i \oplus w$, $z' = z \oplus w$ for any $w \in V(Q_n)$. Typically, we will apply Proposition 3 in Lemma 1 for $w = \mathbf{1}$.

3 Path partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$

Assume that we have $2k$ distinct vertices $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of a subgraph G of Q_n. We say that G has an *a_ib_i-paths partition* if $V(G)$ can be partitioned into k vertex-disjoint paths of G between a_i and b_i. Note that this notion generalizes the problem of Hamiltonian paths for more paths with prescribed endvertices, and it was previously studied for hypercubes by Caha and Koubek [1], and by Dvořák and Gregor [4] and also in a variation of faulty vertices [3].

We proceed with a technical, but useful lemma on a_ib_i-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

Lemma 1. *Let $n \geq 3$ be odd, $k = n-1$, $\{a_1, \ldots, a_k\} \subseteq N(\mathbf{0})$, $\{b_1, \ldots, b_k\} \subseteq N(\mathbf{1})$ such that $a_1 = \overline{b_1}$ and $a_i \neq \overline{b_i}$ for every $1 < i \leq k$. Then $Q_n - \{\mathbf{0}, \mathbf{1}\}$ has an a_ib_i-paths partition.*

Proof. For $n = 3$ there is only one (up to isomorphism) configuration of sets $\{a_1, a_2\} \subseteq N(\mathbf{0})$ and $\{b_1, b_2\} \subseteq N(\mathbf{1})$ such that $a_1 = \overline{b_1}$ and $a_2 \neq \overline{b_2}$. This configuration with the a_ib_i-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$ is depicted on Figure 5.

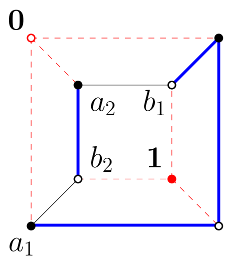

Figure 5: The only (up to isomorphism) configuration in Lemma 1 for $n = 3$.

Now we assume that $n \geq 5$. Let a_0 and b_0 denote the remaining neighbors of $\mathbf{0}$ and $\mathbf{1}$ that are not amongst $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively.

Claim 1. *The hypercube Q_n can be split along two distinct directions $d_1, d_2 \in [n]$ into four subcubes Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2}, and Q^{11}_{n-2} such that*

(i) $\{a_0, a_1\} \subseteq V(Q^{00}_{n-2})$, $\{b_0, b_1\} \subseteq V(Q^{11}_{n-2})$, *and*

(ii) $\{a_i, b_i\} \subseteq V(Q^{01}_{n-2})$ *or* $\{a_i, b_i\} \subseteq V(Q^{10}_{n-2})$ *for at most one* $i \in [k]$*;*

unless $n = 5$ and a_i's with b_i's comprise the configuration depicted on Figure 6.

Proof of Claim 1. To satisfy the condition (i), at most 3 directions from $[n]$ are forbidden for d_1 and d_2. More precisely, if $a_0 = e_p$, $\overline{b_0} = e_q$, and $a_1 = \overline{b_1} = e_r$, then we satisfy (i) if and only if we choose d_1 and d_2 from the set $D = [n] \setminus \{p, q, r\}$. Note that r is distinct from both p and q, but we may have $p = q$ in general.

In the first step, we choose d_1 arbitrarily from D, and we split Q_n into Q^0_{n-1} and Q^1_{n-1} along the direction d_1. Then we obtain $b_j \in V(Q^0_{n-1})$ and $a_l \in V(Q^1_{n-1})$ for exactly one j and exactly one l with $1 < j, l \leq k$. Observe that $j \neq l$ since $a_i \neq \overline{b_i}$ for every $1 < i \leq k$. By renaming the vertices we may assume that $j = 2$ and $l = 3$. Thus, we have

$$b_2[d_1] = a_2[d_1] = 0 \quad \text{and} \quad a_3[d_1] = b_3[d_1] = 1. \tag{3}$$

To satisfy also (ii), it suffices to choose $d_2 \in D \setminus \{d_1\}$ such that $a_2[d_2] \neq b_2[d_2]$ or $b_3[d_2] \neq a_3[d_2]$. Since a_i and b_i differ in exactly $n - 2$ directions for every $1 < i \leq k$, and by (3), such $d_2 \in D \setminus \{d_1\}$ exists if $n \geq 7$ or $p = q$.

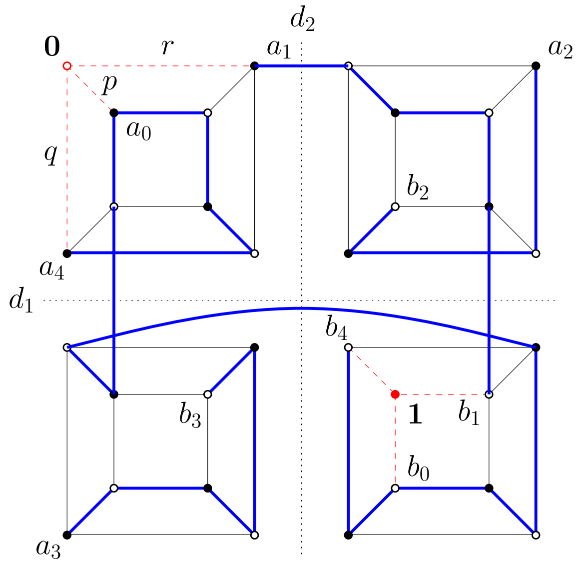

Figure 6: The only exceptional configuration which does not allow splitting satisfying conditions (i) and (ii) in Claim 1 for $n = 5$.

Now suppose that $n = 5$, $p \neq q$, and for the unique choice of $d_2 \in D \setminus \{d_1\}$ we have $a_2[d_2] = b_2[d_2]$ and $b_3[d_2] = a_3[d_2]$. Notice that it must be $a_2[d_2] = b_2[d_2] = 1$ and $b_3[d_2] = a_3[d_2] = 0$. If follows that

$$a_0 = \overline{b_4} = e_p,\ a_1 = \overline{b_1} = e_r,\ a_2 = \overline{b_3} = e_{d_2},\ a_3 = \overline{b_2} = e_{d_1}, \text{ and } a_4 = \overline{b_0} = e_q.$$

This is exactly the configuration which is depicted on Figure 6. Therefore the claim holds. $\square$

The a_ib_i-paths partition for this exceptional configuration is also depicted on Figure 6. So, now we assume that we have splitting of Q_n such that conditions (i) and (ii) hold. Furthermore, by renaming the vertices we may assume that $b_2 \in V(Q^{01}_{n-2})$ and $a_3 \in V(Q^{10}_{n-2})$. Moreover, by exchanging d_1 and d_2 we may assume that $\{a_i, b_i\} \subseteq V(Q^{01}_{n-2})$ for no $i \in [k]$, and therefore $a_2 \notin V(Q^{01}_{n-2})$. Thus, by renaming the vertices we have, say $a_4 \in V(Q^{01}_{n-2})$.

The idea of the rest of the proof is to apply induction in Q^{00}_{n-2}, Proposition 3 in Q^{10}_{n-2} and in Q^{11}_{n-2}, and Proposition 1 in Q^{01}_{n-2}, and then glue all the paths together in order to obtain an a_ib_i-paths partition of $Q_n - \{\mathbf{0}, \mathbf{1}\}$. To this end, we distinguish two cases regarding whether b_3 is in Q^{10}_{n-2}. But before, to avoid ambiguity, let us mention that below we write simply $\{i, j, \dots k\}$ also for $k \leq j$ to denote the set $\{i\} \cup ([k] \setminus [j-1])$.

Case 1: $b_3 \in V(Q^{10}_{n-2})$. We start with the construction of an a_4b_4-path in Q^{11}_{n-2} and Q^{01}_{n-2}. Note that most of the vertices of Q^{11}_{n-2} and Q^{01} are on this path. See Figure 7 for an illustration. Let $i, j \in [n] \setminus \{d_1, d_2\}$ be such that $b_4 = \overline{e_i}$ and $b_0 = \overline{e_j}$. Furthermore, let

$$B = \{b_l \mid l \in \{1, 5, \dots, k\}\} \text{ and } C = \{c_l = b_l \oplus e_i \mid l \in \{1, 5, \dots, k\}\}.$$

Note that

$$N_{Q^{11}_{n-2}}(\mathbf{1}) = B \cup \{b_0, b_4\} \text{ and } N_{Q^{11}_{n-2}}(b_4) = C \cup \{\mathbf{1}, b_4 \oplus e_j\}.$$

Thus, applying Proposition 3 for the set $A_{ij} = B \cup C \cup \{\mathbf{1}\}$, we obtain a Hamiltonian b_4b_0-path P_{11} of $Q^{11}_{n-2} - A_{ij}$.

Note that the vertex $w = b_0 \oplus e_{d_1}$ is adjacent to b_2 in Q^{01}_{n-2}, and therefore, w is distinct from a_4 but has the same parity as a_4 which is opposite to the parity of b_2. So, we may apply Proposition 1 to construct a Hamiltonian wa_4-path P_{01} of $Q^{01}_{n-2} - \{b_2\}$. By concatenating P_{11} and P_{01}, we obtain the reversed a_4b_4-path.

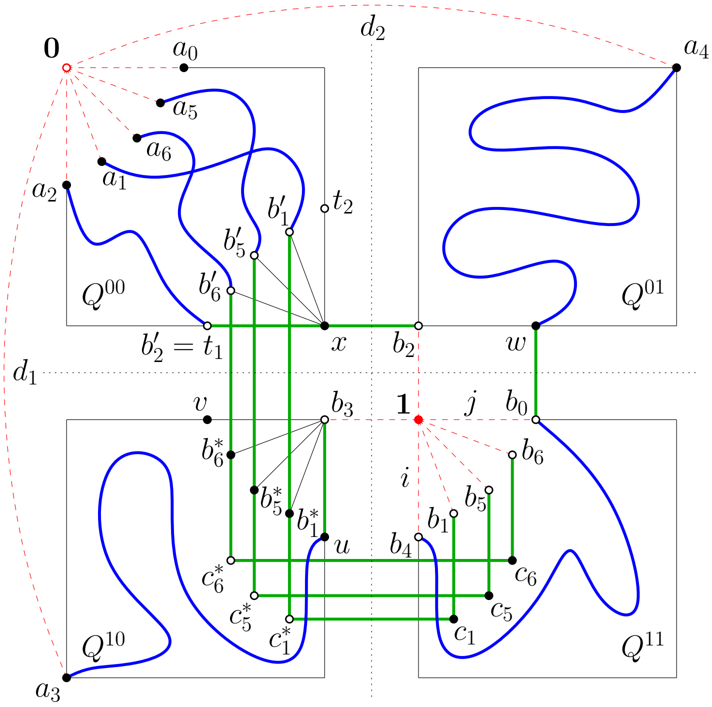

Figure 7: The construction of $a_i b_i$-path partition in Case 1 of Lemma 1.

Second, we construct an a_3b_3-path in Q^{10}_{n-2}. Note that most of the vertices of Q^{10}_{n-2} are on this path. Let

$$B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{1, 5, \dots, k\}\}$$

and

$$C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{1, 5, \dots, k\}\},$$

and let $u = b_3 \oplus e_i$ and $v = b_3 \oplus e_j$. Note that the vertices b_3, u, and v in Q^{10}_{n-2} correspond to the vertices $\mathbf{1}$, b_4, and b_0 in Q^{11}_{n-2}. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_3) = B^* \cup \{u, v\} \quad \text{and} \quad N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_3, u \oplus e_j\}.$$

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_3\}$, we obtain a Hamiltonian a_3u-path P_{10} of $Q^{10}_{n-2} - A^*_{ij}$. By prolonging this path from u to b_3, we have the a_3b_3-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 5, \dots, k\}\}$. Notice that all these vertices are in Q^{00}_{n-2}, x has the role of $\mathbf{1}$ in Q^{00}_{n-2} and is adjacent to all vertices of B'. Furthermore, let t_1, t_2 be the remaining two neighbors of x in Q^{00}_{n-2} that are not in B'; that is, $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$.

Observe that $d(a_1, b'_1) = n - 2$ since $d(a_1, b_1) = n$. So the vertices a_1 and b'_1 are complementary in Q^{00}_{n-2}. Similarly, $d(a_l, b'_l) = n - 4$ since $d(a_l, b_l) = n - 2$ for every $l \in \{5, \dots, k\}$. We choose $b'_2 \in \{t_1, t_2\}$ such that also $d(a_2, b'_2) = n - 4$. Applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2, 5, \dots, k\}$. Then, we prolong the $a_2b'_2$-path through x to b_2, and each $a_lb'_l$-path through b^*_l, c^*_l, and c_l to b_l for $l \in \{1, 5, \dots, k\}$. Thus, we obtain remaining a_lb_l-paths.

To conclude Case 1, observe (on Figure 7) that all a_ib_i-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

Case 2: $b_3 \notin V(Q^{10}_{n-2})$. The constructions in this case differ only in small details to the construction in the previous case. However, for the sake of completeness, we present here the entire argument. First, recall that b_0, b_1, b_2, and b_3 are not in Q^{10}_{n-2}. Moreover, b_4 cannot be in Q^{10}_{n-2} since a_4 is Q^{01}_{n-2} and $a_4 \neq \overline{b_4}$. Thus, by renaming the vertices we may assume that $b_5 \in V(Q^{10}_{n-2})$. Note that it follows that $n \geq 7$ in Case 2.

We start with the construction of an a_4b_4-path in Q^{11}_{n-2} and Q^{01}_{n-2} which is completely the same as above. Let $i, j \in [n] \setminus \{d_1, d_2\}$ be such that $b_4 = \overline{e_i}$

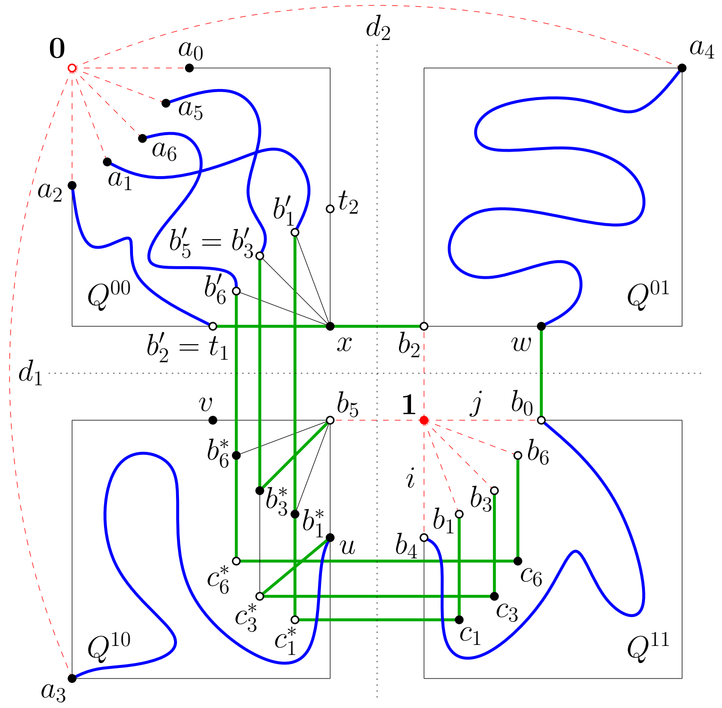

Figure 8: The construction of a_ib_i-path partition in Case 2.A of Lemma 1.

and $b_0 = \overline{e_j}$. Furthermore, let

$$B = \{b_l \mid l \in \{1, 3, 6, \dots, k\}\} \text{ and } C = \{c_l = b_l \oplus e_i \mid l \in \{1, 3, 6, \dots, k\}.$$

Note that

$$N_{Q^{11}_{n-2}}(\mathbf{1}) = B \cup \{b_0, b_4\} \text{ and } N_{Q^{11}_{n-2}}(b_4) = C \cup \{\mathbf{1}, b_4 \oplus e_j\}.$$

Thus, applying Proposition 3 for the set $A_{ij} = B \cup C \cup \{\mathbf{1}\}$, we obtain a Hamiltonian b_4b_0-path P_{11} of $Q^{11}_{n-2} - A_{ij}$. Second, we apply Proposition 1 to construct a Hamiltonian path P_{01} between vertices $w = b_0 \oplus e_{d_1}$ and a_4 in $Q^{01}_{n-2} - \{b_2\}$. By concatenating P_{11} and P_{01}, we obtain the reversed a_4b_4-path. Now we distinguish two subcases regarding $d(a_5, b_3)$.

Subcase 2.A: $d(a_5, b_3) = n - 2$. We construct an a_3b_3-path in Q^{10}_{n-2}

and Q^{11}_{n-2}. See Figure 8 for an illustration. Again, let

$$B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{1, 3, 6, \ldots, k\}\}$$

and

$$C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{1, 3, 6, \ldots, k\}\},$$

and let $u = b_5 \oplus e_i$ and $v = b_5 \oplus e_j$. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_5) = B^* \cup \{u, v\} \quad \text{and} \quad N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_5, u \oplus e_j\}.$$

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_5\}$, we obtain a Hamiltonian a_3u-path P_{10} of $Q^{10}_{n-2} - A^*_{ij}$. By prolonging this path from u through c^*_3 and c_3 to b_3, we have the a_3b_3-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 3, 6, \ldots, k\}\}$. Furthermore, let t_1, t_2 be the remaining two neighbors of x in Q^{00}_{n-2} that are not in B'; that is, $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$.

Observe that $d(a_1, b'_1) = n - 2$ since $d(a_1, b_1) = n$. So the vertices a_1 and b'_1 are complementary in Q^{00}_{n-2}. Similarly, $d(a_l, b'_l) = n - 4$ since $d(a_l, b_l) = n - 2$ for every $5 < l \leq k$. We put $b'_5 = b'_3$. Since $d(a_5, b_3) = n - 2$ in this subcase, we have that $d(a_5, b'_5) = n - 4$. Furthermore, we choose $b'_2 \in \{t_1, t_2\}$ such that also $d(a_2, b'_2) = n - 4$. Applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2\} \cup \{5, \ldots, k\}$. Then, we prolong the $a_2b'_2$-path through x to b_2, the $a_5b'_5$-path through b^*_3 to b_5, and the $a_lb'_l$-path through b^*_l, c^*_l, and c_l to b_l for every $l \in \{1, 6, \ldots, k\}$. Thus, we obtain the remaining a_lb_l-paths.

To conclude Subcase 2.B, observe (on Figure 8) that all a_ib_i-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$.

Subcase 2.B: $d(a_5, b_3) = n$. In this subcase we have to apply induction in Q^{00}_{n-2} in a different way than in the previous subcase. The reason is that we need $d(a_5, b'_5) = n - 4$, so now we cannot put $b'_5 = b'_3$ as we did before. As a consequence, we have to construct also an a_3b_3-path in Q^{10}_{n-2} and Q^{11}_{n-2} in a different way. See Figure 9 for an illustration.

Again, let $B^* = \{b^*_l = b_l \oplus e_{d_2} \mid l \in \{0, 1, 6, \ldots, k\}\}$ and $C^* = \{c^*_l = b^*_l \oplus e_i \mid l \in \{0, 1, 6, \ldots, k\}\}$, and let $u = b_5 \oplus e_i$, $b^*_3 = b_3 \oplus e_{d_2}$, and $c^*_3 = b^*_3 \oplus e_i$. Similarly as above, observe that

$$N_{Q^{10}_{n-2}}(b_5) = B^* \cup \{u, b^*_3\} \quad \text{and} \quad N_{Q^{10}_{n-2}}(u) = C^* \cup \{b_5, c^*_3\}.$$

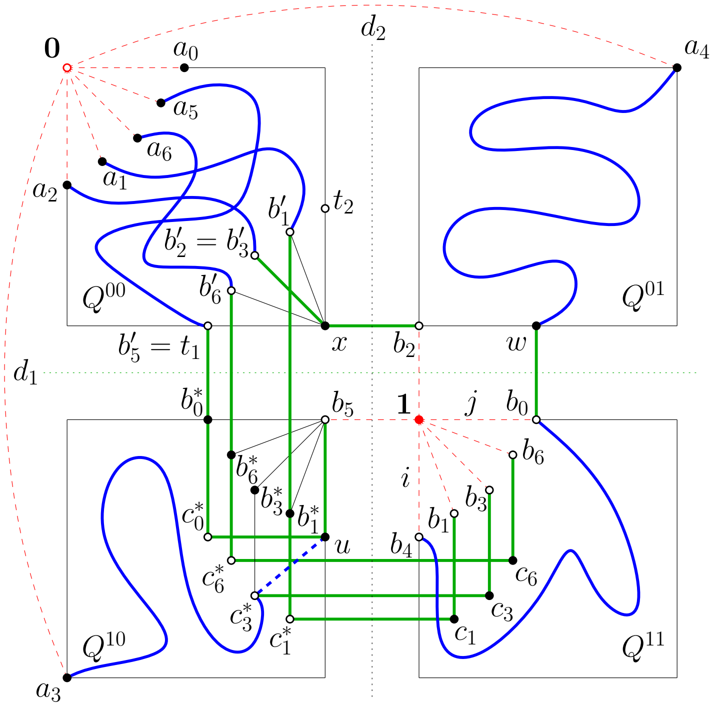

Figure 9: The construction of a_ib_i-path partition in Case 2.B of Lemma 1.

Hence, applying Proposition 3 for the set $A^*_{ij} = B^* \cup C^* \cup \{b_5\}$, we obtain a Hamiltonian a_3u-path P_{10} of $Q^{10}_{n-2} - A^*_{ij}$. Note that c^*_3 is the vertex previous to u on P_{10} since u has no other neighbor in $Q^{10}_{n-2} - A^*_{ij}$ than c^*_3. By deleting the vertex u from P_{10} and prolonging this path from c^*_3 through c_3 to b_3, we have the a_3b_3-path.

Finally, we construct the remaining paths. Let $x = b_2 \oplus e_{d_2}$, $B' = \{b'_l = b^*_l \oplus e_{d_1} \mid l \in \{1, 3, 6, \ldots, k\}\}$. Furthermore, let t_1, t_2 be the remaining two neighbors of x in Q^{00}_{n-2} that are $t_1 = x \oplus e_j = x \oplus \overline{b_0}$ and $t_2 = x \oplus e_i = x \oplus \overline{b_4}$. We put $b'_5 = t_1$, so $d(a_5, b'_5) = n - 4$ since $d(a_5, b'_3) = n - 2$. We also choose $b'_2 \in \{b'_3, t_2\}$ such that $d(a_2, b'_2) = n - 4$.

Again, observe that $d(a_1, b'_1) = n - 2$ since $d(a_1, b_1) = n$, and $d(a_l, b'_l) = n - 4$ for every $l \in \{1, 2, 5, \ldots, k\}$. Hence, applying induction we obtain an $a_lb'_l$-paths partition of $Q^{00}_{n-2} - \{\mathbf{0}, x\}$ where $l \in \{1, 2, 5, \ldots, k\}$. Then, we prolong the $a_2b'_2$-path through x to b_2, the $a_5b'_5$-path through b^*_0, c^*_0, and u to b_5, and each $a_lb'_l$-path through b^*_l, c^*_l, and c_l to b_l for $l \in \{1, 6, \ldots, k\}$. Therefore, we obtain the remaining a_lb_l-paths.

To conclude Subcase 2.B, observe (on Figure 9) that all a_ib_i-paths for $i \in [k]$ are vertex-disjoint and they cover all vertices of $Q_n - \{\mathbf{0}, \mathbf{1}\}$. □

4 Hamiltonicity of G^2_n

Recall that G^2_n is the graph induced on Q_n by levels from 2 to $n - 2$; that is,

$$G^2_n = Q_n - (\{\mathbf{0}, \mathbf{1}\} \cup N(\mathbf{0}) \cup N(\mathbf{1})).$$

For our convenience, we use the following notation of vertices in subcubes of Q_n. Assume that Q_n is split along two fixed directions $d_1, d_2 \in [n]$ into four subcubes Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2}, and Q^{11}_{n-2}, which are isomorphic to Q_{n-2}. For $w \in \{00, 01, 10, 11\}$ and $x \in V(Q_{n-2})$ we denote by x^w the copy of the vertex x in the subcube Q^w_{n-2}.

Theorem 1. *G^2_n is Hamiltonian for every odd $n \geq 5$.*

Proof. We proceed by induction on n. The statement holds for $n = 5$ since G^2_5 is the middle level graph which is known to be Hamiltonian [15]. For $n \geq 7$ we split Q_n along two arbitrary directions into four subcubes Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2}, and Q^{11}_{n-2}. For $w \in \{00, 11\}$ let H^w denote a copy of G^2_{n-2} in Q^w_{n-2}, that is

$$H^w = Q^w_{n-2} - (\{\mathbf{0}^w, \mathbf{1}^w\} \cup N_{Q^w_{n-2}}(\mathbf{0}^w) \cup N_{Q^w_{n-2}}(\mathbf{1}^w)).$$

Initially, applying induction we obtain Hamiltonian cycles C_1 and C_2 of H^{00} and H^{11}, respectively. Next, we construct a Hamiltonian cycle C_3 of $G_n^2 \setminus (H^{00} \cup H^{11})$ and finally, we interconnect C_3 with copies of C_1 and C_2 mapped by properly chosen automorphisms of H^{00} and H^{11}.

The construction of C_3 is as follows. First, we label the neighbors of $\mathbf{0}$ and $\mathbf{1}$ in Q_{n-2} in the following way. Let $p_i = e_i$ for every $i \in [n-2]$, and let

$$q_1 = \overline{e_1},\ q_2 = \overline{e_4},\ q_3 = \overline{e_2},\ q_{n-2} = \overline{e_3}, \text{ and } q_i = \overline{e_{i+1}} \quad \text{for every } 4 \leq i \leq n-3.$$

Note that $p_1 = \overline{q_1}$ and $p_i \neq \overline{q_i}$ for every $1 < i \leq n-3$. We use corresponding labelings p_i^w and q_i^w in Q_{n-2}^w for each $w \in \{00, 01, 10, 11\}$.

Hence, by Lemma 1, $Q_{n-2}^{01} - \{\mathbf{0}^{01}, \mathbf{1}^{01}\}$ can be partitioned into $n-3$ vertex-disjoint paths P_i between p_i^{01} and q_i^{01} for $i \in [n-3]$. Similarly, since $p_2 = \overline{q_3}$, $p_{n-2} \neq \overline{q_2}$ and $p_i \neq \overline{q_{i+1}}$ for $2 < i \leq n-3$, $Q_{n-2}^{10} - \{\mathbf{0}^{10}, \mathbf{1}^{10}\}$ can be partitioned into $n-3$ vertex-disjoint paths R_i between q_{i+1}^{10} and p_i^{10} for $2 \leq i \leq n-3$, and R_1 between q_2^{10} and p_{n-2}^{10}. See Figure 10 for an illustration.

To construct C_3, we interconnect paths R_i and P_i at Q_{n-2}^{11}, and paths P_i and R_{i-1} at Q_{n-2}^{00} as follows (here R_0 means R_{n-3}). The path R_1 continues from p_{n-2}^{10} through p_{n-2}^{11}, $\mathbf{0}^{11}$, p_1^{11} to the vertex p_1^{01} of P_1. The path R_i for $2 \leq i \leq n-3$ continues from p_i^{10} through p_i^{11} to the vertex p_i^{01} of P_i. The path P_1 continues from q_1^{01} through q_1^{00}, $\mathbf{1}^{00}$, q_{n-2}^{00} to the vertex q_{n-2}^{10} of R_{n-3}. The path P_i for $2 \leq i \leq n-3$ continues from q_i^{01} through q_i^{00} to the vertex q_i^{10} of R_{i-1}. These connections are presented with green color on Figure 10. The choices of endvertices of paths P_i's and R_i's allow these connections which assures that C_3 is a Hamiltonian cycle of $G_n^2 \setminus (H^{00} \cup H^{11})$.

To conclude the proof, we interconnect C_3 with copies of C_1 and C_2. For two adjacent vertices u and v of Q_n we say that u is a *light* neighbor of v if $w(u) < w(v)$, otherwise u is a *heavy* neighbor of v. Since $w(p_1) = 1$ and $w(q_1) = n-3 \geq 4$, the $p_1^{01}q_1^{01}$-path $P_1 \subset C_3$ contains a vertex x^{01} such that $w(x) = 2$ and x^{01} has a heavy neighbor on P_1. Note that the neighbor x^{00} of x^{01} in Q_{n-2}^{00} belongs to C_1.

Let y^{00} be one of two neighbors of x^{00} on C_1. Observe that y^{00} is a heavy neighbor of x^{00} since $w(x^{00}) = w(x) = 2$ and C_1 does not visit vertices of weight 1. Then, let z^{01} be a heavy neighbor of x^{01} on P_1. Let i and j be the directions of edges $x^{00}y^{00} \in E(C_1)$, $x^{01}z^{01} \in E(C_3)$, respectively. If $i = j$, then y^{00} and z^{01} are adjacent in G_n^2, so we can interconnect C_1 with C_3 directly by replacing the edges $x^{00}z^{00} \in E(C_1)$ and $x^{01}z^{01} \in E(C_3)$ with $x^{00}x^{01}, z^{00}z^{01} \in E(G_n^2)$.

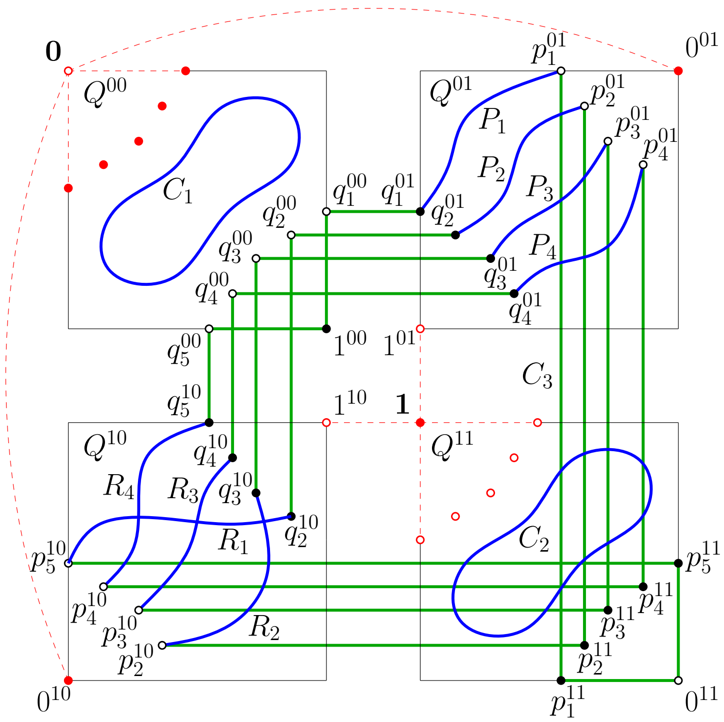

Figure 10: The construction in Theorem 1 with $n = 7$.

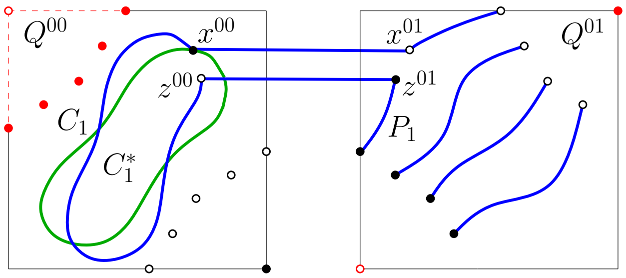

Figure 11: The interconnection of the path P_1 and a copy C_1^* of the cycle C_1.

Now we assume $i \neq j$. Thus, we have

- $y^{00}[i] = z^{01}[j] = 1$, and
- $x^{00}[i] = x^{01}[i] = z^{01}[i] = x^{01}[j] = x^{00}[j] = y^{00}[j] = 0$.

Hence, by switching the directions i and j we obtain a bijection π of $V(H^{00})$ such that $\pi(x^{00}) = x^{00}$ and $\pi(y^{00}) = z^{00}$ where z^{00} is the neighbor of z^{01} in Q^{00}_{n-2}. Moreover, π is an automorphism of H^{00}, and therefore, $C_1^* = \pi(C_1)$ is a Hamiltonian cycle of H^{00} containing the edge $x^{00}z^{00}$. Therefore, we can interconnect C_1^* with C_3 by replacing the edges $x^{00}z^{00} \in E(C_1^*)$ and $x^{01}z^{01} \in E(C_3)$ with $x^{00}x^{01}, z^{00}z^{01} \in E(G_n^2)$. See Figure 11 for an illustration, where C_1 is represented by green and C_1^* by blue color.

To interconnect C_3 also with a copy of C_2, we proceed similarly with the path $R_1 \subset C_3$ in Q^{10} as above. By a proper choice of a vertex x^{10} (with his neighbor z^{10}) on R_1 and an automorphism of H^{11} which maps an edge of C_2 to $x^{11}z^{11}$ we easily connect C_3 and a copy of C_2. Therefore, we obtain a Hamiltonian cycle of G_n^2. $\square$

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