Tolerance analysis in linear programming

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Abstract

We consider a linear programming problem and suppose that we have an optimal solution. In practice it is often important to know how different optimality criteria (optimal solution, optimal basis, optimal partition, etc.) changes when we perturb the input data. Our aim is to compute tolerances (intervals) for the objective function and right-hand side coefficients such that these coefficients can independently and simultaneously vary inside their tolerances while preserving the optimality criterion. We put the tolerance analysis in a unified framework that is convenient for algorithmic processing. We survey the known results (pioneered by R. E. Wendell) and propose an improvement that is optimal is some sense (the resulting tolerances are maximal and they take into account proportionality). We apply our approach to several sensitivity invariancies: optimal basis, support set and optimal partition invariancy. Thus the approach is convenient not only for simplex method solvers, but also for the interior points methods. We also discuss the time complexity and show that it is NP-hard to determine the maximal tolerances.

Keywords: *Linear programming, tolerance analysis, sensitivity analysis, optimal partition.*

Notation

$A_{i\bullet}$	the i-th row of a matrix A
A_P	submatrix of A consisting of the columns indexed by P
$\lvert A\rvert$	absolute value of a matrix A, i.e., the matrix with components $\lvert A\rvert_{i,j} = \lvert A_{i,j}\rvert$
I	identity matrix (with convenient dimension)
e	a vector of all ones (with convenient dimension)

1 Introduction

Sensitivity analysis in mathematical programming is useful for many reasons: It characterizes impact of measurement errors an gives the user information about robustness of the model, among others. The traditional sensitivity analysis gives ranges of allowable variations for each parameter separately. Tolerance approach was developed to handle simultaneous and independent perturbations of the model parameters, and to easily interpret such perturbations.

Tolerance approach was established by Wendell [27, 28], who proposed the so called "symmetric tolerances". It provides the user with only one tolerance (absolute or relative) which is applicable for all required parameters. This is very easy and simple for the user to dealt with, but we loose a lot of information on the model. That is why non-symmetric extensions were proposed by Arsham & Oblak [3], Wondolowski [31] and Wendell [29].

Tolerances were studied in many mathematical programming problems. For instance, for network problems [24], facility location problems [23], transportation problems [1] and in multi-objective linear programming [17, 19, 20]. Tolerance analysis surveys can be found in [26, 30].

An attempt to a unification of different sensitivity analyses was done by Arsham [2], but his approach using implicit parameters is not so convenient for algorithmic processing. In this paper, we present a unified approach to tolerance analysis, which is easily programmable.

The paper proceeds as follows. In the next section we present our unified approach to tolerance analysis via a system of linear inequalities. We review the known results and propose another extension of tolerances being optimal from some point of view (Section 2.3). This general approach is illustrated on the ordinary linear programming issues (Section 3). We consider parameters in the objective function and in the right-hand sides. We apply tolerance analysis for the traditional optimal basis invariancy and also for the optimal partition invariancy. Furthermore, we discuss the complexity and show that computation of the largest tolerances is NP-hard problem (Section 3.1.3).

2 General approach to tolerance analysis

This section aims at finding a unified and universal approach to tolerance analysis of linear systems. Indeed, every such problem can be state as follows. Let $D \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^m$ and consider the linear system of inequalities

$$Dx \le d. \tag{1}$$

Given an interior point $x^* \in \mathbb{R}^n$, i.e., $Dx^* < d$, we want to find tolerances for the components of x^*. It means to find as largest as possible ranges for $x_1^*, \ldots, x_n^*$ such that all these components may simultaneously and independently perturb within these ranges while still preserving feasibility of the linear system. Formally, maximize the vector difference $x^+ - x^-$ over all $x^+, x^- \in \mathbb{R}^n$ such that $x^- \le x^* \le x^+$ and $Dx \le d$ is fulfilled for every x with $x^- \le x \le x^+$.

This is an multi-objective programming problem, and it is natural to seek for an efficient solution, that is, a feasible solution which cannot be improved in one objective without being worsen in another objective. Not every (efficient) solution is convenient for a decision maker, so we must take into account also readability, simplicity of interpretation and its basic purpose.

2.1 Wendell's approach

The tolerance approach was established by Wendell [26, 27, 28, 30]. We adapt his method for our linear system tolerance problem.

Let a non-negative vector $x^\Delta \in \mathbb{R}^n$ be given. We call it as a perturbation rates, as it scales the components of the vector x. The typical use of perturbation rates is as follows: we put $x_j^\Delta = 0$ if the j-th component of x is out of interest for the tolerance problem; we put $x_i^\Delta = 1$ for the absolute case and $x_j^\Delta = |x_j^*|$ for the relative (percentage) case.

Now, we want to determine maximal tolerance $\delta^* > 0$ such that $Dx \le d$ holds for every x with $|x_j - x_j^*| \le \delta^* x_j^\Delta$, $j = 1, \ldots, n$. Notice that this kind of tolerance is sometimes called *symmetric tolerance*, because it is the same for all perturbed parameters. Notice also that it may happen that $\delta^* = \infty$.

Theorem 1 (Wendell, 1984). *We have*

$$\delta^* = \inf_{i=1,\dots,m;\ |D_{i\bullet}|x^\Delta>0} \frac{d_i - D_{i\bullet}x^*}{|D_{i\bullet}|x^\Delta}.$$

Proof. Let $i \in \{1,\dots,m\}$. We show that $D_{i\bullet}x \le d_i$ for every x such that $|x_j - x_j^*| \le \delta^* x_j^\Delta$, $j = 1,\dots,n$. If $|D_{i\bullet}|x^\Delta > 0$ then

$$\begin{aligned} D_{i\bullet}x &= \sum_{j=1}^n d_{ij}x_j \le \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta^* x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta^* x_j^\Delta) \\ &= \sum_{j=1}^n d_{ij}x_j^* + \sum_{j=1}^n |d_{ij}|x_j^\Delta \delta^* \le D_{i\bullet}x^* + \sum_{j=1}^n |d_{ij}|x_j^\Delta \frac{d_i - D_{i\bullet}x^*}{|D_{i\bullet}|x^\Delta} \\ &\le D_{i\bullet}x^* + d_i - D_{i\bullet}x^* = d_i. \end{aligned}$$

Similarly, if $|D_{i\bullet}|x^\Delta = 0$ then

$$\begin{aligned} D_{i\bullet}x &= \sum_{j=1}^n d_{ij}x_j \le \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta^* x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta^* x_j^\Delta) \\ &= \sum_{j=1}^n d_{ij}x_j^* + \sum_{j=1}^n |d_{ij}|x_j^\Delta \delta^* = D_{i\bullet}x^* + 0 \le d_i. \end{aligned}$$

The maximality of the tolerance δ^* will be proven as follows. Clearly, if $\delta^* = \infty$ then it is maximal. Otherwise, define

$$k := \operatorname*{argmin}_{i=1,\dots,m;\ |D_{i\bullet}|x^\Delta>0} \frac{d_i - D_{i\bullet}x^*}{|D_{i\bullet}|x^\Delta}$$

and $\delta^\varepsilon := \delta^* + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Obviously, $|D_{k\bullet}|x^\Delta > 0$. We show that δ^ε cannot be admissible tolerance. Define the vector x^ε componentwise: $x_j^\varepsilon = x_j^* + \delta^\varepsilon x_j^\Delta$ if $d_{kj} > 0$, and $x_j^\varepsilon = x_j^* - \delta^\varepsilon x_j^\Delta$ if $d_{kj} < 0$. Now, we have for the k-th row of the inequality system

$$\begin{aligned}
D_{k\bullet}x^{\varepsilon} &= \sum_{j=1}^{n} d_{kj}x_j^{\varepsilon} = \sum_{j;\ d_{kj}>0} d_{kj}(x_j^* + \delta^{\varepsilon}x_j^{\Delta}) + \sum_{j;\ d_{kj}<0} d_{kj}(x_j^* - \delta^{\varepsilon}x_j^{\Delta}) \\
&= \sum_{j=1}^{n} d_{kj}x_j^* + \sum_{j=1}^{n} |d_{kj}|x_j^{\Delta}\delta^{\varepsilon} = D_{k\bullet}x^* + \sum_{j=1}^{n} |d_{kj}|x_j^{\Delta} \left(\frac{d_k - D_{k\bullet}x^*}{|D_{k\bullet}|x^{\Delta}} + \varepsilon \right) \\
&= D_{k\bullet}x^* + d_k - D_{k\bullet}x^* + \sum_{j=1}^{n} |d_{kj}|x_j^{\Delta}\varepsilon > d_k.
\end{aligned}$$

Thus x^{ε} is not a feasible point and δ^{ε} is not an admissible tolerance. $\square$

Wendell's tolerance method is nice and easy to interpret as it provides a decision maker with only one value applicable to all interested components of the vector x. However, this approach has also shortcomings. The serious one is that the tolerance is usually small, and for medium-sized and larger problems it is often zero [26, 30].

That is why extended tolerance ranges were developed. Instead of one tolerance usable for all parameters we calculate individual lower and upper tolerances for the particular parameters. It means that we determine (as large as possible) positive vectors $\delta^-, \delta^+ \in \mathbb{R}^n_+$ such that (1) is satisfied for very x with $x_j^* - \delta_j^- x_j^{\Delta} \le x_j \le x_j^* + \delta_j^+ x_j^{\Delta}$, $j = 1, \dots, n$.

2.2 Wondolowski's approach

Arsham & Oblak [3] and Wondolowski [31] proposed an extension of the Wendell's symmetric tolerances. The latter one was more involved and we present his original formulae adapted for our general case.

Theorem 2 (Wondolowski, 1991). *We have*

$$\begin{aligned}
\delta_j^+ &= \inf_{i=1,\dots,m;\ |D_{i\bullet}|x^{\Delta}>0,\ d_{ij}>0} \frac{d_i - D_{i\bullet}x^*}{|D_{i\bullet}|x^{\Delta}}, \\
\delta_j^- &= \inf_{i=1,\dots,m;\ |D_{i\bullet}|x^{\Delta}>0,\ d_{ij}<0} \frac{d_i - D_{i\bullet}x^*}{|D_{i\bullet}|x^{\Delta}}.
\end{aligned}$$

Proof. Let $i \in \{1, \dots, m\}$. We show that $D_{i\bullet}x \le d_i$ for every x such that

$x_j^* - \delta_j^- x_j^\Delta \leq x_j \leq x_j^* + \delta_j^+ x_j^\Delta$, $j = 1, \dots, n$. If $|D_{i\bullet}| x^\Delta > 0$ then

$$\begin{aligned}
D_{i\bullet} x &= \sum_{j=1}^{n} d_{ij} x_j \leq \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta_j^+ x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta_j^- x_j^\Delta) \\
&\leq \sum_{j=1}^{n} d_{ij} x_j^* + \sum_{j;\ d_{ij}>0} d_{ij} x_j^\Delta \frac{d_i - D_{i\bullet} x^*}{|D_{i\bullet}| x^\Delta} + \sum_{j;\ d_{ij}<0} -d_{ij} x_j^\Delta \frac{d_i - D_{i\bullet} x^*}{|D_{i\bullet}| x^\Delta} \\
&= D_{i\bullet} x^* + \sum_{j=1}^{n} |d_{ij}| x_j^\Delta \frac{d_i - D_{i\bullet} x^*}{|D_{i\bullet}| x^\Delta} = D_{i\bullet} x^* + d_i - D_{i\bullet} x^* = d_i.
\end{aligned}$$

Similarly, if $|D_{i\bullet}| x^\Delta = 0$ then $d_{ij} x_j^\Delta = 0$ for each $j = 1, \dots, n$, and hence

$$\begin{aligned}
D_{i\bullet} x &= \sum_{j=1}^{n} d_{ij} x_j \leq \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta^+ x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta^- x_j^\Delta) \\
&= \sum_{j=1}^{n} d_{ij} x_j^* + 0 \leq d_i.
\end{aligned}$$

□

The Wondolowski's method is a simple, but efficient generalization of the Wendell's method. However, the tolerance vectors δ^-, δ^+ are not efficient from the standpoint of the multi-objective programming problem stated at the beginning of Section 2. That is, the tolerance vectors δ^-, δ^+ are not maximal, and they can be sometimes enlarged.

They are different ways how to enlarge the individual tolerances such that they are maximal. We can enlarge them sequentially [30] or we can determine the tolerances such that the volume $\Pi_{j=1}^n (\delta_j^+ + \delta_j^-) x_j^\Delta$ of the corresponding box is maximal [30]. However, no such an approach takes into account the tolerance rates; we should try to enlarge all the individual tolerances proportionally. This motivates the following section.

2.3 Optimal individual tolerances approach

Herein, we propose a new method that extends the Wondolowski's approach in such a way that the resulting individual tolerances are "optimal" in some sense. By "optimality" we mean that no individual tolerance can be enlarged and the individual tolerances maximally respect the tolerance rates.

The basic idea is simple: We begin with the Wondolowski's individual tolerances. We fix all of them that cannot be enlarged, and repeat the process until all the tolerances are fixed. Algorithm 1 gives a formal description.

Note that the binary variable α_j^+ is zero iff δ_j^+ is fixed, and α_j^- is zero iff δ_j^- is fixed. The variable R_i denotes the quantity how much we can put to the left-hand side of the i-th inequality of (1) not to break feasibility.

The number of loops (while-cycles in Algorithm 1) is at most $\min(m, 2n)$. At each iteration we remove at least one element from the index set I; therefore it cannot exceed m cycles. At each iteration we fix at least one individual tolerance (at the first usually much more), from which the upper limit $2n$ follows.

Proposition 1. *Individual tolerances computed by Algorithm 1 are correct and maximal.*

Proof. We prove the correctness by induction with respect to iterations of the while-cycle. Consider an arbitrary iteration of the while-cycle and the i-th inequality of $Dx \leq d$. We show that $D_{i\bullet}x \leq d_i$ for every x such that $x_j^* - \delta_j^- x_j^\Delta \leq x_j \leq x_j^* + \delta_j^+ x_j^\Delta$, $j = 1, \ldots, n$. Denote

$$A := D_{i\bullet}x^* + \sum_{j;\; d_{ij}>0,\, \alpha_j^+=0} d_{ij}x_j^\Delta \delta_j^+ - \sum_{j;\; d_{ij}<0,\, \alpha_j^-=0} d_{ij}x_j^\Delta \delta_j^- .$$

If $S_i := \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}x_j^\Delta > 0$ then

$$\begin{aligned}
D_{i\bullet}x &= \sum_{j=1}^{n} d_{ij}x_j \leq \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta_j^+ x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta_j^- x_j^\Delta) \\
&= D_{i\bullet}x^* + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=0} d_{ij}\delta_j^+ x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=0} d_{ij}\delta_j^- x_j^\Delta \\
&\quad + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}\delta_j^+ x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}\delta_j^- x_j^\Delta \\
&= A + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}\delta_j^+ x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}\delta_j^- x_j^\Delta \\
&\leq A + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}x_j^\Delta \frac{d_i - A}{S_i} - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}x_j^\Delta \frac{d_i - A}{S_i} \\
&\leq A + \frac{d_i - A}{S_i}\left(\sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}x_j^\Delta \right) \\
&= A + d_i - A = d_i.
\end{aligned}$$

If $S_i = 0$ then $d_{ij}x_j^\Delta = 0$ for each $j \in \{1, \dots, n\}$ such that $d_{ij} > 0$ and $\alpha_j^+ = 1$, or $d_{ij} < 0$ and $\alpha_j^- = 1$. Thus

$$\begin{aligned}
D_{i\bullet}x &= \sum_{j=1}^{n} d_{ij}x_j \leq \sum_{j;\ d_{ij}>0} d_{ij}(x_j^* + \delta_j^+ x_j^\Delta) + \sum_{j;\ d_{ij}<0} d_{ij}(x_j^* - \delta_j^- x_j^\Delta) \\
&= D_{i\bullet}x^* + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=0} d_{ij}\delta_j^+ x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=0} d_{ij}\delta_j^- x_j^\Delta \\
&\quad + \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij}\delta_j^+ x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij}\delta_j^- x_j^\Delta \\
&= A + 0 \leq d_i.
\end{aligned}$$

The last inequality in this range of inequalities follows from the induction assumption.

The maximality of the resulting individual tolerances is a simple consequence of the proposed method; we exhaustively process through all inequalities of (1) and proportionally distribute the available amount to the non-fixed individual tolerances. We fix an individual tolerance only if its increase would yield breaking the feasibility of (1). □

Example 1. Consider the inequality system

$$\begin{aligned}-x_1 + x_2 &\le 2,\\ x_1 + 2x_2 &\le 12,\end{aligned}$$

and an initial feasible point $x^* = (2,3)^T$. Let $x^\Delta = |x^*|$. We compute tolerances according to all the presented approaches.

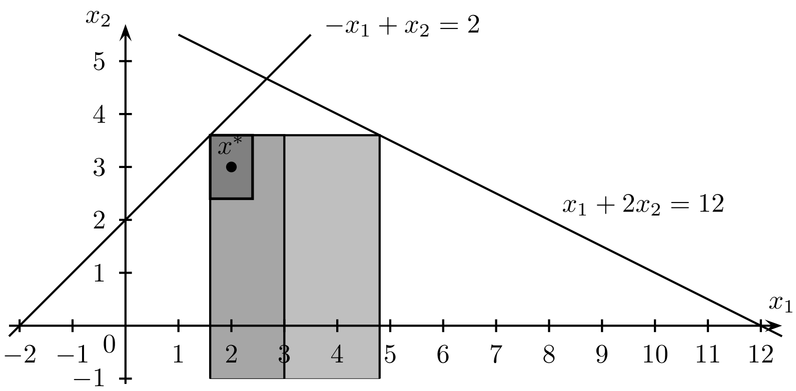

Figure 1: (Example 1) Tolerance regions: the dark grey is for the Wendell's region, grey for the Wondolowski's region and light grey for the ideal region.

The Wendell's tolerance is $\delta^* = \min(\frac{1}{5}, \frac{4}{8}) = \frac{1}{5}$. It means that all components of x^* can vary simultaneously and independently up to the 20% of their nominal values while the vector still remain feasible.

The Wondolowski's individual tolerances draw

$$\begin{aligned}\delta_1^- &= \frac{1}{5}, \qquad & \delta_1^+ &= \frac{1}{2},\\ \delta_2^- &= \infty, \qquad & \delta_2^+ &= \min\left(\frac{1}{5}, \frac{4}{8}\right) = \frac{1}{5}.\end{aligned}$$

The enlargement is significant: the first component x_1^* may perturb up to 50% upwards and x_2^* can even perturb downwards in any way.

Algorithm 1 proceeds as follows. At the first iteration we compute the Wondolowski's individual tolerances. Then we remove the first inequality, because it is filled when x_1^* is decreased and x_2^* is increased in 20%. We also have to fix tolerances δ_1^- and δ_2^+ by putting $\alpha_1^- := 0$ and $\alpha_2^+ := 0$.

At the second iteration we can rewrite the inequality $x_1 + 2x_2 \leq 12$ as $x \leq \frac{24}{5}$ by fixing x_2 at its highest value $2(3+3\frac{1}{5})$. Calling the Wondolowski's method to this inequality we improve the tolerance $\delta_1^+ := \frac{7}{5}$. The second inequality is filled and we return the individual tolerances:

$$\begin{aligned} \delta_1^- &= \frac{1}{5}, \qquad & \delta_1^+ &= \frac{7}{5}, \\ \delta_2^- &= \infty, \qquad & \delta_2^+ &= \frac{1}{5}. \end{aligned}$$

Contrary to the simple Wondolowski's method we achieved an improvement for the first component x_1^* which may vary upwards to 140%.

Remark 1. In most of the practical problems, the parameters $x_1, \ldots, x_n$ are not independent. Suppose we have a linear correlation structure [8, 26, 30]

$$x = Gy + g$$

where $g \in \mathbb{R}^n$, $G \in \mathbb{R}^{n \times l}$ are known and y is the l-dimensional vector of basic parameters. Here, we substitute into (1) and obtain

$$D(Gy + g) \leq d$$

or

$$DGy \leq d - Dg.$$

Eventually, we apply tolerance analysis to this inequality system. In the case of nonlinear correlations the problem is much more complex and we can generally only approximate.

Another frequent obstacle is that the parameters $x_1, \ldots, x_n$ are somehow bounded, i.e., they come from a certain set $\mathcal{S} \subseteq \mathbb{R}^n$. Typically, we are given upper or lower bounds on parameters [26]. Provided that $\mathcal{S}$ is convex polyhedral set described by inequality system

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid Hx \leq h\}$$

we can simply merge both systems and derive tolerances from inequalities

$$Dx \leq d, \quad Hx \leq h.$$

Remark 2. Sometimes it happens that the basic system consists of strict inequalities

$$Dx < d$$

instead of that in (1); see e.g. Section 3.1.2 and 3.2.2. Nevertheless, we can easily modify all the presented results in order to work for strict inequalities as well. Indeed, we can compute tolerances for the closed system (1) and then subtract from them an arbitrarily small positive number.

3 Linear programming issues

In this section we turn our attention to the particular linear programming issues which are the main source of applications of the general tolerance framework. We focus on the objective function coefficients and the right-hand side, and calculate tolerance ranges under which some optimality criteria are preserved.

Consider a linear programming problem in a standard form

$$\min\ c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0, \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let x^* be an optimal solution.

There are various sensitivity invariancies studied. Each of them has some advantages and drawbacks as well. For a comparison the differences see e.g. [22].

The traditional sensitivity analysis is based on the the optimal basis invariancy. Provided that x^* is an optimal basis solution with corresponding basis $B \subseteq \{1, \dots, n\}$ we ask how much can certain parameters perturb such that the basis B remain optimal.

Another sensitivity invariancies is the support set invariancy [6, 14, 15, 16]. The support set of a non-negative vector x is $\sigma(x) := \{i \mid x_i > 0\}$. In this case, we want to determine how much can certain parameters perturb such that there exists an optimal solution x^0 with $\sigma(x^*) = \sigma(x^0)$.

The last considered sensitivity invariancy is the optimal partition invariancy [4, 5, 11, 12, 13, 16, 21]. Let $\mathcal{P}^*$ be the optimal solution set to (2) and $\mathcal{D}^*$ the optimal solution set to its dual. Then the index set $\{1, \dots, n\}$ can be partitioned into two disjoint subsets

$$\begin{aligned} \mathcal{B} &:= \{i \mid x_i > 0 \text{ for some } x \in \mathcal{P}^*\},\\ \mathcal{N} &:= \{i \mid c_i - A_{\bullet i}^T y > 0 \text{ for some } y \in \mathcal{D}^*\}. \end{aligned}$$

This is known as the optimal partition. The question is how much can certain parameters perturb such that the optimal partition remain the same.

The parameters discussed are the objective function coefficients and the right-hand side components. This can be easily extend to simultaneous perturbation both of them [26]. Similar results can be obtained for parameters in any line (row or column) of the constraint matrix [8, 26].

3.1 Objective function coefficients

Herein we will be concerned with the tolerance analysis of the objective function coefficients. For that purpose we denote

$$\min\ \gamma^T x \ \text{ subject to } \ Ax = b,\ x \geq 0, \tag{3}$$

where $\gamma \in \mathbb{R}^n$ stands for a perturbed vector c.

3.1.1 Optimal basis invariancy

First we consider the well-known optimal basis invariancy. Let x^* be an optimal basic solution to (2) and $B \subseteq \{1, \dots, n\}$ the corresponding basis. The optimal basis invariancy region is the set of all objective function vectors $\gamma \in \mathbb{R}^n$ under which the basis B remain optimal to (3). It is described by the linear system [9, 10, 21, 26]

$$\gamma_N - (A_B^{-1} A_N)^T \gamma_B \geq 0, \tag{4}$$

where $N := \{1, \dots, n\} \setminus B$. Thus for the tolerance analysis of the objective function coefficients we simply apply the approach proposed in Section 2. In this case, we associate

$$D := \begin{pmatrix} -I & (A_B^{-1} A_N)^T \end{pmatrix}, \quad d := 0,$$

where I denotes the identity matrix (with convenient dimension) and 0 the zero matrix. The variables are $(\gamma_N, \gamma_B) \in \mathbb{R}^n$ and the initial value is $(\gamma_N, \gamma_B)^* := (c_N, c_B)$.

Optimal basis invariancy approach has several drawbacks. The simplex algorithm yields optimal basis solutions, but interior point methods generally result in non-basic solution. The second shortcoming is that the region described by (4) is not the maximal region of the objective function coefficients where the optimality of x^* is preserved. This happens when x^* is degenerate solution. In the following sections we will address these issues.

3.1.2 Support set and optimal partition invariancy

Recall that x^* stands for an optimal (not necessarily basic) solution of (2). Denote $Z := \{1, \ldots, n\} \setminus \sigma(x^*)$ and suppose that the lineality space $\{x \mid Ax = 0, x_Z = 0\}$ has zero dimension (otherwise the tolerances are zero due to dual degeneracy). According to [18, Theorem 1], support set invariancy region is characterized by the system

$$g_k^T \gamma \geq 0, \quad k \in K, \tag{5}$$

where g_k, $k \in K$, are all extremal directions of the convex polyhedral cone

$$\{x \mid Ax = 0, x_Z \geq 0\}. \tag{6}$$

Moreover, the support set invariancy region is exactly the set of all $\gamma \in \mathbb{R}^n$ for which x^* keeps its optimality. It is easy to see it via complementary slackness conditions. Let $\gamma \in \mathbb{R}^n$ and x be an optimal solution to (3) such that $\sigma(x) = \sigma(x^*)$. Then there is an optimal solution y to the corresponding dual problem and the complementary slackness condition $(A^T y - c)^T x = 0$ holds. Hence $(A^T y - c)^T x^* = 0$ is true as well, and therefore x^* is optimal to (3).

The system (5) consists of linear inequalities, and therefore we can directly apply the tolerance analysis as presented in Section 2. The main drawback of this approach is that we must enumerate all extremal directions of (6), which may be time consuming. We discuss this question in Section 3.1.3.

Under the same assumption optimal partition invariancy region is described by the system of strict inequalities [18, Theorem 3]

$$g_k^T \gamma > 0, \quad k \in K.$$

In this case we determine the tolerances along Remark 2.

3.1.3 Complexity

The support set invariancy region (5) is exactly the set of all objective function vectors for which the linear program (3) detains the optimal solution x^*. That is why it is so important in practice and why we discuss it more in detail.

Consider a linear programming problem (2) and let x^* be its optimal solution. We show that the problem to determine the maximal (Wendell's) symmetric tolerance δ^* under which x^* remains optimal is an NP-hard problem, even though we restrict ourselves to the simplest tolerance rates $x^\Delta := e$. We build on the following lemma; its proof is omitted since it is a slight modification of the proof of Theorem 2.3 from [7] where the NP-hardness of testing the solvability of a system $|Mx| \leq e,\ e^T|x| \geq 1$ was proved.

Lemma 1. *Let $M \in \mathbb{Q}^{n\times n}$ be a nonnegative positive definite matrix. Checking the solvability of the system*

$$|Mx| \leq e,\ e^T|x| > 1 \tag{7}$$

is an NP-hard problem.

Theorem 3. *Checking whether the maximal tolerance $\delta^* \leq 1$ is an NP-hard problem.*

Proof. We want to find maximum tolerance such that x^* remains optimal for any perturbation of c within this tolerance. Formally,

$$\max\ \delta\ \text{ subject to }\ \gamma^T x^* \leq \gamma^T x\ \ \forall x \in \mathcal{X}\ \forall \gamma : |\gamma - c| \leq \delta e.$$

This can be rewritten as

$$\inf\ \delta\ \text{ subject to }\ \gamma^T x^* > \gamma^T x,\ x \in \mathcal{X},\ |\gamma - c| \leq \delta e.$$

Substitute $z := x - x^*$. Then the problem reads

$$\inf\ \delta\ \text{ subject to }\ \gamma^T z < 0,\ z \in \mathcal{X}',\ |\gamma - c| \leq \delta e,$$

where

$$\mathcal{X}' := \{z \in \mathbb{R}^n \mid Az = b - Ax^*,\ z \geq -x^*\}.$$

The lowest possible value of $\gamma^T z$ over all γ such that $|\gamma - c| \leq \delta e$ is equal to $c^T z - \delta e^T|z|$, and it is achieved for that γ which satisfies $\gamma_i = c_i - \delta$ if $z_i \geq 1$ and $\gamma_i = c_i + \delta$ if $z_i < 1$. Now, the problem takes the form

$$\inf\ \delta\ \text{ subject to }\ c^T z - \delta e^T|z| < 0,\ z \in \mathcal{X}'. \tag{8}$$

Now, we construct a polynomial reduction from the NP-hard problem mentioned in Lemma 1 to our problem. Let $M \in \mathbb{Q}^{n\times n}$ be a nonnegative positive definite matrix and consider the system (7) rewritten in this form

$$Mx \leq e,\ -Mx \leq e,\ e^T|x| > 1. \tag{9}$$

We claim that it is solvable iff the system

$$Mx \leq ey,\ -Mx \leq ey,\ e^T|x| > y,\ y \geq 0 \tag{10}$$

is solvable with respect to variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. The proof is as follows: If x solves the former system then x and $y = 1$ solve the latter. Conversely, if x, y solve (10) then there are two possibilities. If $y > 0$ then $\frac{1}{y}x$ is a solution of (9). Otherwise, if $y = 0$ then we have $Mx = 0$, $e^T|x| > 0$, and the solution of (9) is e.g. $x' := \frac{2}{e^T|x|}x$ since

$$\begin{aligned} Mx' &= M\frac{2}{e^T|x|}x = 0 \leq e, \\ -Mx' &= -M\frac{2}{e^T|x|}x = 0 \leq e, \end{aligned}$$

and

$$e^T|x'| = e^T\left|\frac{2}{e^T|x|}x\right| = 2 > 1.$$

To finish the polynomial reduction we rewrite (10) as

$$\begin{aligned} Mx + u &= ey, \\ -Mx + v &= ey, \\ 2y + e^Tu + e^Tv - e^T|x| - |y| - e^T|u| - e^T|v| &< 0, \\ y, u, v &\geq 0, \end{aligned}$$

and associate $z := (x, y, u, v) \in \mathbb{R}^{3n+1}$. Thus the system takes the form

$$c^Tz - e^T|z| < 0,\ z \in \mathcal{X}', \tag{11}$$

where $c := \big(0^T, 2, e^T, e^T\big)^T$, $x^* := \big(-Ke^T, 0, 0^T, 0^T\big)^T$ and

$$A := \begin{pmatrix} M & -e & I & 0 \\ -M & -e & 0 & I \end{pmatrix}.$$

The constant K is chosen large enough, compare [25].

If we can calculate (8) in polynomial time then we can do so for solvability of (11): If the optimal value of (8) is $\delta^* < 1$ then (11) is solvable, and if $\delta^* > 1$ then (11) has no solution. It remains to discuss the pathological case when $\delta^* = 1$. In this case, (11) cannot be solvable. Otherwise, if z is its solution then z and $\delta = 1 - \varepsilon$ is a feasible point of (8) for sufficiently small $\varepsilon > 0$. It is a contradiction with optimality of $\delta^* = 1$. $\square$

Remark 3. NP-hardness results hold also true for linear programming problems in another forms. For instance, (2) is easily transformable to

$$\min\ c^T x \ \text{ subject to } \ x \in \mathcal{X}' := \{x \in \mathbb{R}^n \mid Fx \leq f\},$$

by putting

$$F := \begin{pmatrix} A \\ -A \\ -I \end{pmatrix}, \quad f := \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix},$$

and the objective function tolerances remain the same.

3.2 Right-hand side coefficients

Herein, we turn our attention to the right-hand side perturbation of (2).

3.2.1 Optimal basis invariancy

Let x^* be an optimal basic solution to (2) and $B \subseteq \{1, \ldots, n\}$ the corresponding basis. The set of all right-hand sides $\beta \in \mathbb{R}^m$ under which the basis B remain optimal is described by the linear system [10, 21, 26]

$$A_B^{-1}\beta \geq 0.$$

Now, tolerance analysis of the right-hand sides can be easily done by applying the approach proposed in Section 2. We simply put $D := -A_B^{-1}$ and $d := 0$; the variables are $\beta \in \mathbb{R}^m$ and the initial value is $\beta^* := b$.

3.2.2 Support set and optimal partition invariancy

Let x^* be an optimal solution of (2) and denote $P := \sigma(x^*)$. Suppose that the lineality space $\{y \mid A_P^T y = 0\}$ has zero dimension; otherwise the emerged degeneracy mostly causes that the tolerances are zero. Due to [18], both the support set and optimal partition invariancy regions are equal and described by

$$h_k^T \beta < 0, \quad k \in K',$$

where h_k, $k \in K'$, are all extremal directions of the convex polyhedral cone

$$\{y \mid A_P^T y \leq 0\}. \tag{12}$$

To compute tolerances we proceed along Remark 2.

The complexity results are similar to that derived in Section 3.1.3. We can simply consider the dual problem to (2) which has parameters in the objective function. The optimal partition does not depend on whether we consider the primal or dual problem. Thus the optimal partition invariancy region in the case of right-hand side parameters and primal problem is the same as optimal partition invariancy region for objective function coefficients and dual problem.

The following example not only illustrates the right-hand side tolerances, but also shows that the traditional optimal basis invariancy is not always convenient to use.

Example 2. Consider the linear programming problem [18]:

$$\begin{aligned} &\min \ 2x_2 - 3x_3 \\ &\quad \text{subject to} \ \ 3x_1 + x_2 = 3,\ 3x_1 - x_2 + 3x_3 = 3,\ x_1, x_2, x_3 \geq 0. \end{aligned}$$

The feasible set equals the optimal solution set and is represented by a segment with endpoints $x^1 = (1,0,0)^T$ and $x^2 = (0,3,2)^T$; see Figure 2. We set the tolerance rates as $\beta^\Delta = (3,3)^T$.

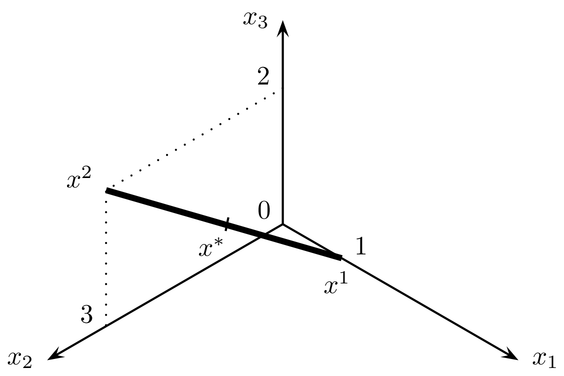

Figure 2: (Example 2) the bold segment illustrate the feasible set, which is identical to the optimal set.

First we consider an optimal basis solution, say x^1. The optimal basis invariancy region is described by

$$\beta_1 + \beta_2 \geq 0,\ \beta_1 - \beta_2 \geq 0,$$

and the support set (and optimal partition) invariancy region by

$$\beta_1 - \beta_2 = 0,\ -\beta_1 - \beta_2 < 0.$$

In both the cases, the Wendell's tolerance is zero; an arbitrarily small perturbation from the initial point $b = (3, 3)^T$ suffices to break its feasibility.

Nevertheless for nonbasic solutions better results are obtained. We take an analytic center $x^* = (\frac{1}{2}, \frac{3}{2}, 1)^T$ of the optimal solution set; interior point methods converge to this point. Herein, we cannot consider optimal basis invariancy, but the support set invariancy region has description

$$-\beta_1 < 0,\ -\beta_1 - \beta_2 < 0.$$

The simple Wendell's tolerance analysis yields $\delta^* < 1$. That is, the right-hand side parameters can vary up to the almost 100% while preserving the support set and optimal partition.

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Algorithm 1 (Optimal individual tolerances)

$I := \{1, \dots, m\}$;
$\alpha_j^+ := 1,\ \alpha_j^- := 1 \quad \forall j = 1, \dots, n$;
while $I \neq \emptyset$ and $\exists j : (\alpha_j^+ = 1$ or $\alpha_j^- = 1)$ **do**
  $R_i := d_i - D_{i\bullet} x^* - \sum_{j;\ d_{ij}>0,\ \alpha_j^+=0} d_{ik} x_j^\Delta \delta_j^+ + \sum_{j;\ d_{ij}<0,\ \alpha_j^+=0} d_{ik} x_j^\Delta \delta_j^-, \quad \forall i \in I$;
  $S_i := \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij} x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij} x_j^\Delta, \quad \forall i \in I$;
  for all $j \in \{1, \dots, n\}$ **do**
    if $\alpha_j^+ = 1$ **then**
      $\delta_j^+ := \inf_{i \in I;\ S_i>0,\ d_{ij}>0} \dfrac{R_i}{S_i}$;
    end if
    if $\alpha_j^- = 1$ **then**
      $\delta_j^- := \inf_{i \in I;\ S_i>0,\ d_{ij}<0} \dfrac{R_i}{S_i}$;
    end if
  end for
  for all $i \in I$ **do**
    if $D_{i\bullet} x^* + \sum_{k;\ d_{ik}>0} d_{ik} x_k^\Delta \delta_k^+ + \sum_{k;\ d_{ik}<0} d_{ik} x_k^\Delta \delta_k^- = d_i$ **then**
      $I := I \setminus \{i\}$;
      for all $j \in \{1, \dots, n\}$ **do**
        if $d_{ij} > 0$ **then**
          $\alpha_j^+ := 0$;
        else if $d_{ij} < 0$ **then**
          $\alpha_j^- := 0$;
        end if
      end for
    end if
  end for
end while