# 3-choosability of triangle-free planar graphs with constraint on 4-cycles*

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**Abstract**

A graph is $k$-choosable if it can be colored whenever every vertex has a list of at least $k$ available colors. A theorem by Grötzsch [2] asserts that every triangle-free planar graph is 3-colorable. On the other hand Voigt [10] gave such a graph which is not 3-choosable. We prove that every triangle-free planar graph such that 4-cycles do not share edges with other 4- and 5-cycles is 3-choosable. This strengthens the Thomassen's result [8] that every planar graph of girth at least 5 is 3-choosable. In addition, this implies that every triangle-free planar graph without 6- and 7-cycles is 3-choosable.

## 1 Introduction

All graphs considered in this paper are simple and finite. The concepts of list coloring and choosability were introduced by Vizing [9] and independently by Erdős et al. [1]. A *list assignment* of $G$ is a function $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. An *$L$-coloring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$

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and $\varphi(u) \neq \varphi(v)$ whenever $u$ and $v$ are adjacent vertices of $G$. If $G$ admits an $L$-coloring, then it is *$L$-colorable*. A graph $G$ is *$k$-choosable* if it is $L$-colorable for every list assignment $L$ such that $|L(v)| \geq k$ for all $v \in V(G)$. Cycles $C_1$ and $C_2$ in a graph are *adjacent* if they intersect in a single edge, i.e., if $V(C_1) \cap V(C_2) = \{u, v\}$ for an edge $uv$.

Thomassen [7, 8] proved that every planar graph is 5-choosable, and every planar graph of girth at least 5 is 3-choosable. Kratochvíl and Tuza [3] observed that every planar triangle-free graph is 4-choosable. On the other hand, Voigt [10, 11] found a planar graph that is not 4-choosable, and a triangle-free planar graph that is not 3-choosable. Numerous papers study additional conditions that force a triangle-free planar graph to be 3-choosable, see e.g. [4, 6, 12, 13, 14, 15].

In particular, let us point out the result of Li [5], strengthening the result of Thomassen [8]: every planar triangle-free graph such that no 4-cycle shares a **vertex** with another 4- or 5-cycle is 3-choosable. We further improve this result, only forbidding the 4-cycles sharing an **edge** with other 4- or 5-cycles:

**Theorem 1.** *Any planar triangle-free graph without 4-cycles adjacent to 4- and 5-cycles is 3-choosable.*

In particular, we obtain:

**Corollary 2.** *Any planar graph without 3-, 6- and 7-cycles is 3-choosable.*

This strengthens the results of Lidický [6] that planar graphs without 3-, 6-, 7- and 8-cycles are 3-choosable, and of Zhang and Xu [13] that planar graphs without 3-, 6-, 7- and 9-cycles are 3-choosable. Theorem 1 also implies the result of Lam et al. [4] that planar graphs without 3, 5 and 6-cycles are 3-choosable.

## 2 Proof of Theorem 1

A *path of length $k$* (or a *$k$-path*) is a path on $k+1$ vertices. Using the proof technique of precoloring extension developed by Thomassen [8], we show the following extension of Theorem 1:

**Theorem 3.** *Let $G$ be a triangle-free planar graph without 4-cycles adjacent to 4- and 5-cycles, with outer face $C$, and $P$ a path of length at most three such that $V(P) \subseteq V(C)$. The graph $G$ can be $L$-colored for any list assignment $L$ such that*

- $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$*;*
- $2 \leq |L(v)| \leq 3$ *for all* $v \in V(C) \setminus V(P)$*;*
- $|L(v)| = 1$ *for all* $v \in V(P)$*, and the colors in the lists give a proper coloring of the subgraph of* $G$ *induced by* $V(P)$*;*
- *the vertices with lists of size two form an independent set; and*
- *each vertex with lists of size two has at most one neighbor in* $P$*.*

Note that we view the single-element lists as a precoloring of the vertices of $P$. Also, $P$ does not have to be a part of the facial walk of $C$, as we only require $V(P) \subseteq V(C)$. Theorem 3 has the following easy consequence:

**Corollary 4.** *Let* $G$ *be a triangle-free planar graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles, with the outer face bounded by an induced cycle* $C$ *of length at most* 9*. Furthermore, assume that*

- *if* $\ell(C) = 8$*, then at least one edge of* $C$ *does not belong to a* 4*-cycle; and*
- *if* $\ell(C) = 9$*, then some two consecutive edges of* $C$ *do not belong to* 4*- and* 5*-cycles.*

*Let* $L$ *be an assignment of lists of size* 1 *to the vertices of* $C$ *and lists of size* 3 *to the other vertices of* $G$*. If* $L$ *prescribes a proper coloring of* $C$*, then* $G$ *can be* $L$*-colored.*

*Proof.* The claim follows from Theorem 3 for $\ell(C) = 4$. If $\ell(C) \in \{5, 6, 7\}$, then let $u_1w_1vw_2u_2$ be an arbitrary subpath of $C$. Let $L'$ be the list assignment obtained from $L$ by removing the color $L(v)$ from the lists of vertices adjacent to $v$. We also set the lists of $w_1$ and $w_2$ to 2-lists such that the precoloring of the other vertices of $C$ forces the prescribed color $L(w_1)$ on $w_1$ and $L(w_2)$ on $w_2$, i.e., $L'(w_1) = L(w_1) \cup L(u_1)$ and $L'(w_2) = L(w_2) \cup L(u_2)$. As all the vertices $x$ with $|L'(x)| = 2$ are neighbors of a single vertex $v$, the graph $G - v$ together with the list assignment $L'$ satisfies the assumptions of Theorem 3. It follows that we can $L'$-color $G - v$, giving an $L$-coloring of $G$.

Let us now consider the case that $\ell(C) = 8$, and let $C = w_1uvw_2r_1r_2r_3r_4$, where the edge $uv$ does not belong to a 4-cycle. Let us delete vertices $u$ and $v$ from $G$, remove the color in $L(u)$ from the lists of neighbors of $u$ and the color in $L(v)$ from the lists of neighbors of $v$, and change the list of $w_1$ to $L(w_1) \cup L(r_4)$ and the list of $w_2$ to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $P = r_1r_2r_3r_4$ forces the right colors on $w_1$ and $w_2$. As $uv$ does not belong to a 4-cycle, the vertices with lists of size two form an independent set. As $C$ is induced, both $w_1$ and $w_2$ have only one neighbor in the 3-path $P$. Let $x$ be a neighbor of $u$ other than $v$ and $w_1$. The vertex $x$ cannot be adjacent to both $r_1$ and $r_4$, as the 4-cycle $uxr_4w_1$ would be adjacent to a 5-cycle $xr_1r_2r_3r_4$. Similarly, $x$ cannot be adjacent to both $r_1$ and $r_3$ or both $r_2$ and $r_4$. As $G$ does not contain triangles, $x$ has at most one neighbor in $P$. By symmetry, this is also true for the neighbors if $v$. Therefore, the graph satisfies assumptions of Theorem 3, and can be colored from the prescribed lists.

Finally, suppose that $\ell(C) = 9$, and let $C = w_1uvww_2r_1r_2r_3r_4$, where the edges $uv$ and $vw$ are not incident with 4- and 5-cycles. We argue similarly as in the previous case. We delete vertices $u$, $v$ and $w$ from $G$ and remove their colors from the lists of their neighbors. We also set the list of $w_1$ to $L(w_1) \cup L(r_4)$ and the list of $w_2$ to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $r_1r_2r_3r_4$ forces the right colors on $w_1$ and $w_2$. Observe that the resulting graph satisfies assumptions of Theorem 3, hence it can be colored. □

Before we proceed with the proof of Theorem 3, let us describe the notation that we use in figures. We mark the precolored vertices of $P$ by full circles, the vertices with list of size three by empty circles, and the vertices with list of size two by empty squares. The vertices for that the size of the list is not uniquely determined in the situation demonstrated by the particular figure are marked by crosses.

*Proof of Theorem 3.* Suppose $G$ together with lists $L$ is the smallest counterexample, i.e., such that $|V(G)| + |E(G)|$ is minimal among all graphs that satisfy the assumptions of Theorem 3, but cannot be $L$-colored, and $\sum_{v \in V(G)} |L(v)|$ is minimal among all such graphs. Let $C$ be the outer face of $G$ and $P$ a path with $V(P) \subseteq V(C)$ as in the statement of the theorem. We first derive several properties of this counterexample. Note that each vertex $v$ of $G$ has degree at least $|L(v)|$.

**Lemma 5.** *The graph $G$ does not contain separating cycles of length at most seven. Every edge of each separating 8-cycle $K$ belongs to a 4-cycle lying inside $K$. And, at least one of every two consecutive edges of each separating 9-cycle $K$ belongs to a 4- or 5-cycle lying inside $K$.*

*Proof.* Let $K$ be the separating cycle. We may assume that $K$ is induced, as otherwise we could consider a shorter separating cycle of length at most 7. Let $G_1$ be the subgraph of $G$ induced by the exterior of $K$ (including $K$) and $G_2$ the subgraph of $G$ induced by the interior of $K$ (including $K$). By the minimality of $G$, Theorem 3 holds for $G_1$ and $G_2$ and their subgraphs. Therefore, there exists a coloring of $G_1$ from the prescribed lists, and this coloring can be extended to $G_2$ by Corollary 4. This is a contradiction, as $G$ cannot be colored from the lists. $\square$

A *chord* of a cycle $K$ is an edge in $G$ joining two distinct vertices of $K$ that are not adjacent in $K$. As $G$ does not have triangles and 4-cycles adjacent to 4- and 5-cycles, a cycle of length at most 7 does not have a chord. Therefore, Lemma 5 implies that every cycle of length at most 7 is a face. Similarly, a cycle $K$ of length 8 with an edge that does not belong to a 4-cycle in the interior of $K$ is either an 8-face, or it has a chord splitting it to a 4-face and a 6-face, or two 5-faces.

**Lemma 6.** *The graph $G$ is 2-connected.*

*Proof.* Obviously, $G$ is connected. Suppose now that $v$ is a cut vertex of $G$ and $G_1$ and $G_2$ are nontrivial induced subgraphs of $G$ such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. Both $G_1$ and $G_2$ satisfy the assumptions of Theorem 3. If $v$ is precolored, then by the minimality of $G$ there exist $L$-colorings of $G_1$ and $G_2$, and they combine to a proper $L$-coloring of $G$. If $v$ is not precolored, then we may assume that $P \subseteq G_1$. An $L$-coloring of $G_1$ assigns a color $c$ to $v$. We change the list of $v$ to $\{c\}$, color $G_2$ and combine the colorings to an $L$-coloring of $G$. $\square$

By Lemma 6, $C$ is a cycle. A *$k$-chord* of $C$ is a path $Q = q_0q_1 \ldots q_k$ of length $k$ joining two distinct vertices of $C$, such that $V(C) \cap V(Q) = \{q_0, q_k\}$ (e.g., 1-chord is just a chord).

**Lemma 7.** *The cycle $C$ has no chords.*

*Proof.* Suppose $e = uv$ is a chord of $C$, separating $G$ to two subgraphs $G_1$ and $G_2$ intersecting in $e$. If both $u$ and $v$ are precolored, then we $L$-color $G_1$ and $G_2$ by the minimality of $G$ and combine their colorings. Otherwise, by symmetry assume that $u \notin V(P)$, and that $|V(P) \cap V(G_1)| \geq |V(P) \cap V(G_2)|$. In particular, $|(V(P) \cap V(G_2)) \setminus \{u, v\}| \leq 1$. Furthermore, let us choose the chord in such a way that $G_2$ is as small as possible; in

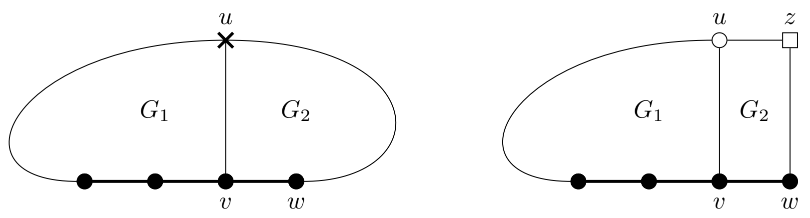


Figure 1: A chord of $C$

particular, the outer face of $G_2$ does not have a chord. Let us find an $L$-coloring of $G_1$ and change the lists of $u$ and $v$ to the colors assigned to them. If $G_2$ with these new lists satisfies assumptions of Theorem 3, then we find its coloring and combine the colorings to an $L$-coloring of $G$, hence assume that this is not the case.

Let $X = (V(P) \cap V(G_2)) \setminus \{u, v\}$. As $G_2$ does not satisfy assumptions of Theorem 3, there exists a vertex $z$ with list of size two adjacent to two precolored vertices. As $G$ is triangle-free, we conclude that $X$ is not empty, say $X = \{w\}$ (see Figure 1), and $z$ is adjacent to $u$ and $w$. As $G_2$ does not contain chords and separating 4-cycles and $z \in V(C)$, $G_2$ is equal to the cycle $uvwz$. Since $|L(z)| = 2$, it holds that $|L(u)| = 3$. Let $c_1$ be the color of $u$ in the coloring of $G_1$, and $c_2$ the single color in the list of $w$. If $L(z) \neq \{c_1, c_2\}$, then we can color $z$ and finish the coloring of $G$, hence assume that $L(z) = \{c_1, c_2\}$. Let $c$ be a color in $L(u) \setminus (\{c_1\} \cup L(v))$ (this set is nonempty, as $|L(v)| = 1$ and $|L(u)| = 3$).

Let us now color $z$ by $c_1$ and set the list of $u$ to $\{c\}$. If $G_1$ with this list at $u$ satisfies assumptions of Theorem 3, we can color $G_1$, and thus obtain an $L$-coloring of $G$. Since $G$ does not have such an $L$-coloring, the assumptions are violated, i.e., either $u$ is adjacent to a vertex of $P$ other than $v$, or $G_1$ contains a vertex (with list of size two) adjacent to both $u$ and a vertex of $P$. This is a contradiction, as $G$ would in both of these cases contain either a triangle, or a 4- or 5-cycle adjacent to the 4-cycle $uvwz$. $\square$

By the previous lemma, $P$ is a part of the facial walk of $C$, and $C$ is an induced cycle.

**Lemma 8.** *$\ell(C) \geq 8$.*

*Proof.* Suppose that $\ell(C) \leq 7$. If $V(C) \neq V(P)$, then color the vertices of $C$ properly from their lists. This can be done, as $C$ is chordless and

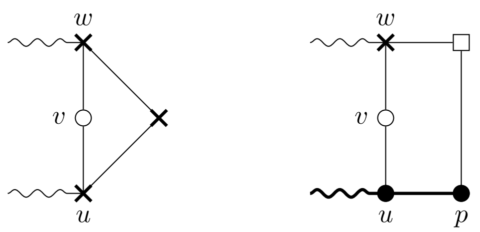


Figure 2: Possible 2-chords in $G$

contains at least one vertex with list of size three. If $5 \leq \ell(C) \leq 7$, then the claim follows from the proof of Corollary 4, as by the minimality of $G$, all subgraphs of $G$ satisfy Theorem 3. If $\ell(C) = 4$, then we delete one of the vertices of $C$ and remove its color from the lists of its neighbors. It is easy to verify that the resulting graph satisfies the assumptions of Theorem 3, hence it has a proper coloring by the minimality of $G$. This coloring extends to an $L$-coloring of $G$, which is a contradiction. $\square$

**Lemma 9.** *No* 4*-cycle shares an edge with another* 4*- or* 5*-cycle.*

*Proof.* Suppose that $C_1 = v_1v_2v_3v_4$ and $C_2 = v_1v_2u_3 \ldots u_t$ are cycles sharing the edge $v_1v_2$, $\ell(C_1) = 4$ and $t = \ell(C_2) \in \{4, 5\}$. Note that $C_1 \neq C$ and $C_2 \neq C$ by Lemma 8. By Lemma 5, both $C_1$ and $C_2$ bound a face. If $v_3 = u_3$, then $v_2$ would be a 2-vertex with list of size three. Thus, $v_3 \neq u_3$ and by symmetry, $v_4 \neq u_t$. As $G$ does not contain triangles, $v_3 \neq u_t$ and $v_4 \neq u_3$, and in case that $t = 5$, $v_3 \neq u_4$ and $v_4 \neq u_4$. Therefore, $C_1$ and $C_2$ are adjacent, contradicting the assumptions of Theorem 3. $\square$

Note that we can assume that $|V(P)| = 4$, as otherwise we can prescribe color for more of the vertices of $C$, without violating assumptions of Theorem 3. Let $P = p_1p_2p_3p_4$. We say that a $k$-chord $Q$ of $C$ *splits off* a face $F$ from $G$ if $F \neq C$ is a face of both $G$ and $C \cup Q$. See Figure 2 for an illustration of 2-chords splitting off a face.

**Lemma 10.** *Every* 2*-chord $uvw$ of $C$ splits off a $k$-face $F$ such that*

**(a)** $|V(F) \cap V(P)| \leq 2$ *and* $\{u, w\} \not\subseteq V(P)$,

**(b)** $k \leq 5$, *and*

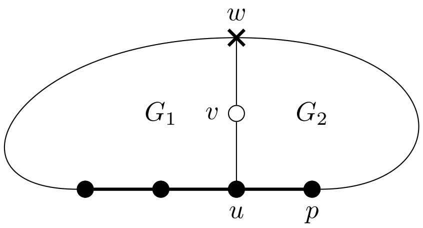


Figure 3: A 2-chord of $C$

**(c)** *if* $|V(F) \cap V(P)| \leq 1$, *then* $k = 4$.

*In particular, the cycle* $C$ *has no* 2*-chord with* $|L(w)| = 2$ *and* $u \neq p_2, p_3$.

*Proof.* Suppose first that $u, w \in V(P)$. By Lemma 5, the 2-chord $uvw$ together with a part of $P$ bounds a face $K$. Color $v$ by a color different from the colors of $u$ and $w$, and remove $V(K) \setminus \{u, v, w\}$ from $G$, obtaining a graph $G'$. Note that a path of length at most three is precolored in $G'$. Since $G$ cannot be $L$-colored, we may assume that $G'$ does not satisfy assumptions of Theorem 3, i.e., there exists $z$ with $|L(z)| = 2$ adjacent to both $v$ and a vertex $y \in V(P) \cap V(G')$. As $G$ is triangle-free, $y \notin \{u, w\}$. It follows that $yuvz$ or $ywvz$ is a 4-face. This is a contradiction, as $K$ would be an adjacent 4-face. Therefore, $\{u, w\} \not\subseteq V(P)$, and by symmetry we assume that $w \notin V(P)$.

The 2-chord $uvw$ splits $G$ to two subgraphs $G_1$ and $G_2$ intersecting in $uvw$. Let us choose $G_2$ such that $|V(P) \cap V(G_2)| \leq |V(P) \cap V(G_1)|$, see Figure 3. Note that $|V(P) \cap V(G_2)| \leq 2$. Let us consider the 2-chord $uvw$ such that $|V(P) \cap V(G_2)|$ is minimal, subject to the assumption that $G_2$ is not a face. By the minimality of $G$, there exists an $L$-coloring $\varphi$ of $G_1$. Let $L'$ be the list assignment for $G_2$ such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(w) = \{\varphi(w)\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus \{u, v, w\}$. Let $P'$ be the precolored path in $G_2$ (consisting of $u$, $v$, $w$, and possibly one other vertex $p$ of $P$ adjacent to $u$). As $C$ has no chords and $G_2$ is not a face, $P'$ is an induced subgraph. Since $G$ cannot be $L$-colored, we conclude that $G_2$ cannot be $L'$-colored, and thus $G_2$ with the list assignment $L'$ does not satisfy the assumptions of Theorem 3. Therefore, there exists a vertex $z$ with $|L(z)| = 2$, adjacent to two vertices of $P'$.

Since $G_2$ is not a face, Lemmas 5 and 7 imply that $z$ is not adjacent to both $w$ and $p$. Similarly, $z$ is not adjacent to both $u$ and $w$. It follows that $z$ is adjacent to $v$ and $p$, and thus $|V(P) \cap V(G_2)| = 2$. Since we have chosen the 2-chord $uvw$ so that $|V(P) \cap V(G_2)| = 2$ is minimal among the 2-chords for that $G_2$ is not a face, the 2-chord $wvz$ splits off a face $F'$ from $G$. Let $x$ be the neighbor of $z$ in $F'$ other than $v$. Since $|L(z)| = 2$, it holds that $|L(x)| = 3$. As $F'$ is a face, $\deg(x) = 2$, which is a contradiction. It follows that for every 2-chord, $G_2$ is a face. The choice of $G_2$ establishes (a).

Let $wvuv_4 \ldots v_k$ be the boundary of the face $G_2$. Note that $V(P) \cap V(G_2) \subseteq \{u, v_4\}$, and $v_4, \ldots, v_k$ have degree two. If $k > 5$, then at least one of $v_5$ and $v_6$ has list of size three, which is a contradiction, proving (b). Similarly, if $|V(F) \cap V(P)| \leq 1$ and $k = 5$, then at least one of $v_4$ and $v_5$ would be a 2-vertex with list of size three, proving (c).

Consider now a 2-chord $uvw$ such that $|L(w)| = 2$ and $u \notin \{p_2, p_3\}$, and let $x$ be the neighbor of $w$ in $G_2$ distinct from $v$. As $u \notin \{p_2, p_3\}$, no vertex of $V(P) \setminus \{u\}$ lies in $G_2$. Therefore, $|L(x)| = 3$ and $\deg(x) = 2$, a contradiction. We conclude that no such 2-chord exists. □

Let us now consider the 3-chords of $C$:

**Lemma 11.** *Every 3-chord $Q = uvwx$ of $C$ such that $u, x \notin \{p_2, p_3\}$ splits off a 4- or 5-face.*

*Proof.* Suppose that $Q$ splits $G$ into two subgraphs $G_1$ and $G_2$ intersecting in $uvwx$, such that $V(P) \cap V(G_2) \subseteq \{u, x\}$. Let us $L$-color $G_1$ and consider the vertices $u$, $v$, $w$ and $x$ of $G_2$ as precolored according to this coloring. If $ux$ were an edge, then $Q$ would split off a 4-face. It follows that $Q$ is an induced path thus this precoloring of $Q$ is proper. Similarly, as $Q$ does not split off a 5-face, $u$ and $x$ do not have a common neighbor with list of size two. Neither $v$ nor $w$ is adjacent to a vertex with list of size 2 by Lemma 10. Therefore, $G_2$ satisfies assumptions of Theorem 3, and the coloring can be extended to $G_2$, giving an $L$-coloring of $G$. This is a contradiction. □

Let $x_1x_2x_3x_4$ be the part of the facial walk of $C$ such that $x_1$ is adjacent to $p_4$ and $x_2 \neq p_4$. By Lemma 8, $\{x_1, x_2, x_3, x_4\} \cap V(P) = \emptyset$. Let us now show a few properties of the vertices $x_1$, $x_2$, $x_3$, $x_4$ and their neighbors.

**Lemma 12.** *Let $Q = v_0v_1 \ldots v_k$ be a $k$-chord starting and ending in vertices of $x_1x_2x_3x_4$, or a cycle intersecting $C$ in a single vertex $x \in \{x_1, x_2, x_3, x_4\}$. The following holds (for some $i \in \{1, 2, 3, 4\}$):*

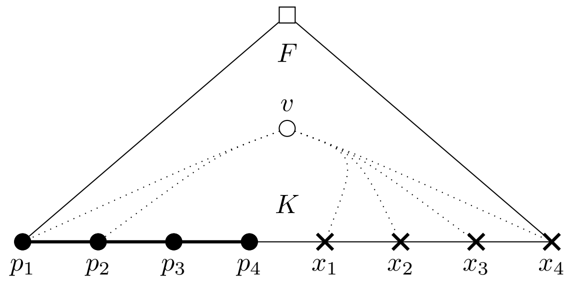


Figure 4: A 2-chord from $p_1$ or $p_2$ to $\{x_1, x_2, x_3, x_4\}$

- *If $\ell(Q) = 2$, then $Q = x_i v_1 x_{i+2}$ splits off a 4-face.*
- *If $\ell(Q) = 3$, then $Q$ splits off either a 4-face $x_i x_{i+1} v_1 v_2$, or a 5-face $x_i x_{i+1} x_{i+2} v_1 v_2$.*
- *If $\ell(Q) = 4$, then $Q$ forms a boundary of a 4-face $x_i v_1 v_2 v_3$, or splits off a 5-face $x_i x_{i+1} v_1 v_2 v_3$, or splits off a 6-face $x_i x_{i+1} x_{i+2} v_1 v_2 v_3$.*

*Proof.* By a simple case analysis. The details are left to the reader. $\square$

Note also that if $Q$ splits off a face of form $x_i x_{i+1} x_{i+2} v_1 \dots v_{k-1}$, then $\deg(x_{i+1}) = |L(x_{i+1})| = 2$.

**Lemma 13.** *If $Q$ is a $k$-chord with $k \leq 3$, starting in a vertex $x_i$ (where $1 \leq i \leq 4$) and ending in a vertex with list of size two, then $k = 3$ and $Q$ bounds a 4-face.*

*Proof.* Let $Q = q_0 q_1 \dots q_k$, where $q_0 \in \{x_1, x_2, x_3, x_4\}$ and $|L(q_k)| = 2$. By Lemmas 7 and 10, $k > 2$. If $k = 3$, then by Lemma 11, $Q$ splits off a 4- or 5-face. However, the latter is impossible, as $|L(q_3)| = 2$, so the remaining vertex of the 5-face, whose degree is two, would have a list of size three. $\square$

**Lemma 14.** *There is no 2-chord from $\{p_1, p_2\}$ to $\{x_1, x_2, x_3, x_4\}$.*

*Proof.* Suppose $Q = p_i v x_j$ is such a 2-chord, and let $K$ be the cycle formed by $Q$ and $p_i \dots p_4 x_1 \dots x_j$. Note that $\ell(K) \leq 9$. Let us choose $Q$ such that $\ell(K)$ is minimal. By Lemma 10, $Q$ splits off a face $F$ such that $\ell(F) \leq 5$. Furthermore, if $\ell(K) = 9$, then $i = 1$, and hence $|V(P) \cap V(F)| = 1$. In

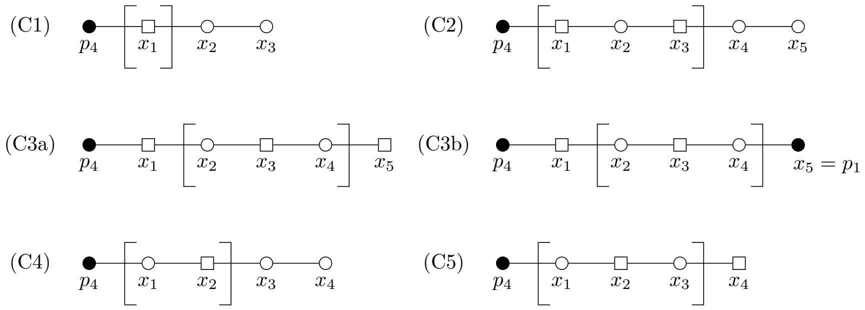


Figure 5: The construction of the set $X_1$

that case, the claim (c) of Lemma 10 implies $\ell(F) = 4$. See Figure 4 for illustration. It follows that the edges $p_i v$ and $vx_j$ are not incident with a 4-face inside $K$, and if $\ell(K) = 9$, then they are not incident with a 5-face. By Lemma 5, $K$ is not separating. If $\ell(K) \leq 7$, then $K$ is a face, and $\deg(v) = 2$, which is a contradiction. Similarly, if $\ell(K) > 7$, then $K$ has a chord incident with $v$. By the minimality of $\ell(K)$, $v$ is adjacent to $p_3$ or $p_4$. However, this contradicts Lemma 10(a). □

If both $x_1$ and $x_2$ have lists of size three, then we remove one color from $L(x_1)$ and find a coloring by the minimality of $L$ (note that $x_1$ is not adjacent to any vertex with list of size two, and has only one neighbor in $P$, as $C$ does not have chords). Therefore, exactly one of $x_1$ and $x_2$ has a list of size two. Let $x_5$ be the neighbor of $x_4$ in $C$ distinct from $x_3$. We now distinguish several cases depending on the lists of vertices in $\{x_1, x_2, x_3, x_4\}$, in order to choose a set $X_1 \subseteq \{x_1, x_2, x_3, x_4\}$ of vertices that we are going to color (and remove).

(C1) If $|L(x_1)| = 2$ and $|L(x_2)| = |L(x_3)| = 3$ (see Figure 5(1)), then we set $X_1 = \{x_1\}$.

(C2) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| = 3$ (see Figure 5(2)), then we set $X_1 = \{x_1, x_2, x_3\}$.

(C3) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| \leq 2$ (see Figure 5(3)), then we set $X_1 = \{x_2, x_3, x_4\}$.

(C4) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 3$ (see Figure 5(4)), then we set $X_1 = \{x_1, x_2\}$.

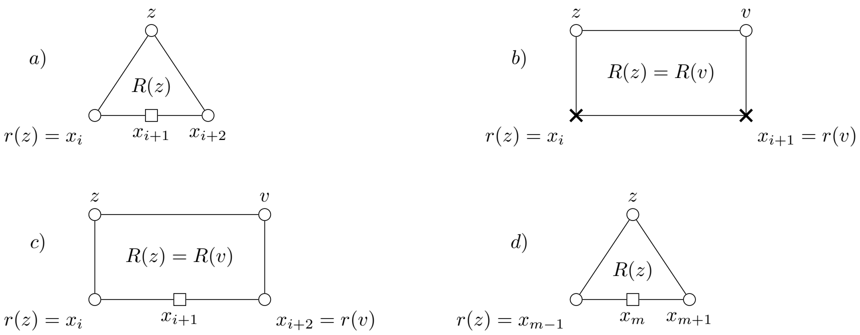


Figure 6: The construction of the set $X_2$

(C5) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 2$ (see Figure 5(5)), then we set $X_1 = \{x_1, x_2, x_3\}$.

Let $m = \max\{i : x_i \in X_1\}$. Note the following properties of the set $X_1$:

- $|X_1| \leq 3$.
- If $|L(x_m)| = 2$, then $m \leq 3$ and $|L(x_{m+1})| = |L(x_{m+2})| = 3$.
- If $|L(x_m)| = 3$, then $|L(x_{m+1})| \leq 2$.

Let $\mathcal{F}$ be the set of faces of $G$ incident with the edges of the path induced by $X_1$ ($\mathcal{F} = \emptyset$ in the case (C1)). We define a set $X_2 \subseteq V(G) \setminus V(C)$, together with functions $r : X_2 \to X_1$ and $R : X_2 \to \mathcal{F}$. A vertex $z \in V(G) \setminus V(C)$ belongs to $X_2$ if

- *$z$ is adjacent to two vertices in $X_1$* (see Figure 6(a) for an example). By Lemma 12, $z$ lies in a (uniquely determined) 4-face $F = x_i x_{i+1} x_{i+2} z$, where $x_i, x_{i+1}, x_{i+2} \in X_1$. We define $r(z) := x_i$ and $R(z) := F$. Or,
- *there exists a path $xzvy$ such that $x, y \in X_1$ and $v \notin \{p_1\} \cup X_1$* (see Figure 6(b), (c) and (d) for examples). If $v = x_{m+1}$, then by Lemma 10, the 2-chord $xzv$ splits off a 4-face $F$. Otherwise the 3-chord $xzvy$ splits off a 4- or 5-face $F$ by Lemma 12. We define $r(z) := x$ and $R(z) := F$. Note that $v \neq x_1$: otherwise, $x_1 \notin X_1$ and we are in case (C3), hence $|L(x_1)| = 2$ and the 2-chord $x_1 z x$ would contradict Lemma 10. It follows that $v$ also belongs to $X_2$, unless $v = x_{m+1}$.

Let us now show that $r(z)$ and $R(z)$ are well-defined. As a 4-face cannot be adjacent to a 4- or 5-face and $G$ is triangle-free, $z$ does not have another neighbor in $X_1$. Also, if there existed another path $xzv'y'$ with $y' \in X_1$ splitting off a face $F'$, then both $F$ and $F'$ would be 5-faces; however, that would imply $|X_1| \geq 5$, which is a contradiction. Therefore, $r$ and $R$ are defined uniquely. Furthermore, $v$ is the only neighbor of $z$ in $X_2$, and $R(v) = R(z)$ (assuming that $v \neq x_{m+1}$).

We now find an $L$-coloring of $X_1 \cup X_2$ that we aim to extend to a coloring of $G$.

**Lemma 15.** *Let $H = G[V(P) \cup X_1 \cup X_2]$ be the subgraph of $G$ induced by $V(P) \cup X_1 \cup X_2$. There exist an $L$-coloring $\varphi_1$ of $X_1$ and an $L$-coloring $\varphi_2$ of $X_2$ such that*

- *the coloring of $H$ given by $\varphi_1$, $\varphi_2$ and the precoloring of $P$ is proper,*
- *if $|L(x_{m+1})| \leq 2$, then $\varphi_1(x_m) \notin L(x_{m+1})$,*
- *if $x_1 \notin X_1$ (i.e., in the case (C3) of the definition of $X_1$), then $L(x_1) \neq L(p_4) \cup \{\varphi_1(x_2)\}$, and*
- *if $z \in X_2$ is adjacent to $x_{m+1}$, then $|L(x_{m+1}) \setminus \{\varphi_1(x_m), \varphi_2(z)\}| \geq 2$.*

*Proof.* Suppose first that there exists $z \in X_2$ adjacent to $x_{m+1}$. Note that $z$ is unique, $m \geq 2$ and $R(z) = x_{m-1}x_mx_{m+1}z$ is a 4-face. As $G$ does not contain a 2-vertex with list of size three, $|L(x_m)| = 2$ and $|L(x_{m-1})| = |L(x_{m+1})| = 3$. This happens only in the cases (C2) and (C4) of the definition of $X_1$, thus $x_1 \in X_1$ and $m \leq 3$. Furthermore, $x_{m-1}$ is the only neighbor of $z$ in $X_1$ and $z$ is not adjacent to any other vertex of $X_2$. As $R(z)$ is a 4-face and $G$ does not contain 4-cycles adjacent to 4- or 5-cycles, $z$ is not adjacent to $p_3$ and $p_4$. By Lemma 14, $z$ is not adjacent to $p_1$ and $p_2$, either, thus any choice of the color for $z$ is consistent with the precoloring of $P$. Let us distinguish the following cases:

- If $L(z) \cap L(x_m) \neq \emptyset$, then choose $c \in L(z) \cap L(x_m)$ and let $\varphi_1(x_m) = \varphi_2(z) = c$.
- If $L(z) \neq L(x_{m+1})$, then choose $\varphi_2(z) \in L(z) \setminus L(x_{m+1})$ and $\varphi_1(x_m) \in L(x_m)$ arbitrarily.

- Finally, consider the case that $L(z) \cap L(x_m) = \emptyset$ and $L(z) = L(x_{m+1})$, i.e., the lists of $x_m$ and $x_{m+1}$ are disjoint. We choose $\varphi_1(x_m) \in L(x_m)$ and $\varphi_2(z) \in L(z)$ arbitrarily.

On the other hand, suppose that no vertex of $X_2$ is adjacent to $x_{m+1}$. If $|L(x_{m+1})| = 2$, then choose $\varphi_1(x_m) \in L(x_m) \setminus L(x_{m+1})$. Otherwise, choose $\varphi_1(x_m) \in L(x_m)$ arbitrarily (in case that $m = 1$, choose a color different from the one in $L(p_4)$)

In both of these cases, the precoloring of $x_m$ (and possibly $z$) can be extended to a proper coloring $\psi$ of the subgraph induced by $\{x_1, \ldots, x_m, z\}$ consistent with the precoloring of $P$. We fix $\varphi_1$ as the restriction of $\psi$ to $X_1$.

Let us now construct (the rest of) the coloring $\varphi_2$. Consider a vertex $u \in X_2$ that is not adjacent to $x_{m+1}$. As $u \notin V(C)$, it holds that $|L(u)| = 3$. If $u$ has no neighbor in $X_2$, then it has two neighbors $r(u), x \in X_1$ and $R(u)$ is a 4-face. We claim that $u$ has no neighbor $p_i \in V(P)$. Otherwise, we obtain $i \geq 3$ by Lemma 14. By Lemma 10, the 2-chord $p_i u r(u)$ splits off a 4- or 5-face. This face shares an edge with $R(u)$, which is a contradiction. Therefore, any choice of $\varphi_2(u) \in L(u) \setminus \{\varphi_1(x), \varphi_1(r(u))\}$ is consistent with the precoloring of $P$.

Finally, suppose that $u$ has a neighbor $w \in X_2$. As we argued in the definition of $X_2$, each of $u$ and $w$ has exactly one neighbor in $X_1$, and $u$ and $w$ do not have any other neighbors in $X_2$. Also, $w$ is not adjacent to $x_{m+1}$, as otherwise $G$ would contain a triangle or two adjacent 4-cycles. By Lemma 10(a), each of $u$ and $w$ has at most one neighbor in $P$. If one of them does not have any such neighbor, then we can easily color $u$ and $w$, hence assume that $p_i u$ and $p_j w$ are edges. By Lemma 14, $i, j \geq 3$. Without loss on generality, $j = 3$ and $i = 4$. This is a contradiction, as the 4-face $p_3 p_4 u w$ shares an edge with $R(u)$. □

Consider the colorings $\varphi_1$ and $\varphi_2$ constructed in Lemma 15. Let $G' = G - (X_1 \cup X_2)$ and let $L'$ be the list assignment such that $L'(v)$ is obtained from $L(v)$ by removing the colors of the neighbors of $v$ in $X_1$ and $X_2$ for $v \neq x_1$, and $L'(x_1) = L(x_1)$ if $x_1 \notin X_1$. Suppose that $G'$ with the list assignment $L'$ satisfies assumptions of Theorem 3. Then there exists an $L'$-coloring $\varphi$ of $G'$, which together with $\varphi_1$ and $\varphi_2$ gives an $L$-coloring of $G$: this is obvious if $x_1 \in X_1$. If $x_1 \notin X_1$, then $|L(x_1)| = 2$, and $L(p_4) \subseteq L(x_1)$ by the minimality of $G$ (otherwise, we could remove the edge $p_4 x_1$). By the choice of $\varphi_1$, it holds that $\varphi_1(x_2) \neq \varphi(x_1)$. Since no other vertex of $X$

may be adjacent to $x_1$ by Lemmas 7 and 10, $\varphi$ together with $\varphi_1$ and $\varphi_2$ is a proper coloring of $G$. As $G$ is a counterexample to Theorem 3, it follows that $L'$ violates assumptions of Theorem 3, i.e.,

**(a)** a vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of $P$; or

**(b)** $|L'(v)| \leq 1$ for some $v \in V(G') \setminus V(P)$; or

**(c)** two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.

Let us now consider each of these possibilities separately.

**(a)** *A vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of $P$.* By Lemmas 7 and 10(a), this is not possible.

**(b)** *$|L'(v)| \leq 1$ for some $v \in V(G') \setminus V(P)$.* If $|L(x_{m+1})| = 2$, then $x_{m+1}$ does not have a neighbor in $X_2$ by Lemma 10 and hence $|L'(x_{m+1})| = 2$ by the choice of $\varphi_1$. If $|L(x_{m+1})| = 3$, then the choice of $\varphi_1$ and $\varphi_2$ according to Lemma 15 ensures $|L'(x_{m+1})| \geq 2$. Therefore, $v \neq x_{m+1}$.

Since $G$ has neither chords nor 2-chords starting in $X_1$ and ending in a vertex with list of size two, it holds that $|L(v)| = 3$. Therefore, $v$ has at least two neighbors $u_1, u_2 \in X_1 \cup X_2$. If at least one of $u_1$ and $u_2$ belonged to $X_1$, then $v$ would be included in $X_2$, hence we may assume that $u_1, u_2 \in X_2$.

Consider the path $x_i u_1 v u_2 x_j$, where $x_i = r(u_1)$ and $x_j = r(u_2)$. We may assume that $i \leq j$. The cycle $x_i \ldots x_j u_2 v u_1$ has length at most six, thus it bounds a face $F$. Note that $i = j$, as each of $R(u_1)$ and $R(u_2)$ shares at least one edge with the path induced by $X_1$ and $F \neq R(u_1) \neq R(u_2) \neq F$. Therefore, $F$ is a 4-face sharing an edge with 4-face $R(u_1)$ (and also with $R(u_2)$), which is a contradiction. Therefore, $|L'(v)| \geq 2$ for every $v \in V(G') \setminus V(P)$.

**(c)** *Two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.* As the vertices with lists of size two form an independent set in $G$, we may assume that $|L(u)| = 3$. Let $y_1$ be a neighbor of $u$ in $X_1 \cup X_2$.

Consider first the case that $|L(v)| = 2$. If $u \notin V(C)$, then by Lemma 10, $y_1 \notin V(C)$, and thus $y_1 \in X_2$ and $vuy_1r(y_1)$ is a 3-chord. By Lemma 13, this 3-chord splits off a 4-face $F$. Note that $F \neq R(y_1)$, as $u \notin X_2$. This is impossible, as the 4-face $F$ would share an edge with $R(y_1)$. Therefore, $u \in V(C)$, and hence $v \neq x_1$. If $y_1 \in X_2$, then

$uy_1r(y_1)$ is a 2-chord, and by Lemma 10, it splits off a 4-face adjacent to $R(y_1)$, which is again a contradiction. Assume now that $y_1 \in X_1$. As $C$ does not have chords, it follows that $y_1 = x_m$ and $u = x_{m+1}$. However, in that case $v = x_{m+2}$ and $|L(x_{m+2})| = 2$, which contradicts the choice of $X_1$.

Consider now the case that $|L(v)| = 3$. Let $y_2$ be a neighbor of $v$ in $X_1 \cup X_2$. As $u, v \notin X_2$, at least one of $y_1$ and $y_2$, say $y_1$, belongs to $X_2$. Let us consider the possibilities $y_2 \in X_1$ and $y_2 \in X_2$ separately:

- $y_2 \in X_1$: The cycle formed by $r(y_1)y_1uvy_2$ and a part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and $y_2$ has length at most six, thus it bounds a face $F$. Note that $R(y_1)$ shares an edge with $F$. Let $k_1$ and $k_2$ be the number of edges that $R(y_1)$ and $F$, respectively, share with the path induced by $X_1$, $k_1 \geq \ell(R(y_1)) - 3 \geq 1$ and $k_2 = \ell(F) - 4 \geq 0$. Since $|X_1| \leq 3$, it holds that $k_1 + k_2 \leq 2$. If $k_1 = 1$, then $R(y_1)$ is a 4-face. Since 4- and 5-faces cannot be adjacent to $R(y_1)$, we obtain $\ell(F) \geq 6$. It follows that $k_2 \geq 2$, which is a contradiction. Similarly, if $k_1 = 2$, then $F$ cannot be a 4-face, hence $\ell(F) \geq 5$ and thus $k_2 \geq 1$. This is again a contradiction.
- $y_2 \in X_2$: Let $F$ be the cycle bounded by $r(y_1)y_1uvy_2r(y_2)$ and the part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and $r(y_2)$. As $\ell(F) \leq 7$, $F$ bounds a face. Note that $R(y_1) \neq R(y_2)$ and $\ell(R(y_1)) = \ell(R(y_2)) = 4$, as each of $R(y_1)$ and $R(y_2)$ shares an edge with the path induced by $X_1$. Since $F$ shares edges with both $R(y_1)$ and $R(y_2)$, $\ell(F) \geq 6$. It follows that $F$ shares at least one edge with the path induced by $X_1$ as well. However, this is impossible, since $|X_1| \leq 3$.

Therefore, the assumptions of Theorem 3 are satisfied by $G'$ and $L'$. We conclude that we can find a proper coloring of $G$, which contradicts the choice of $G$ as a counterexample to Theorem 3. □

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