Markov bases of binary graph models of K_4-minor free graphs

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Abstract

Markov width of a graph is a graph invariant defined as the maximum degree of a Markov basis element for the corresponding graph model for binary contingency tables. We show that a graph has Markov width at most four if and only if it contains no K_4 as a minor, answering a question of Develin and Sullivant. We also present a lower bound of order $\Omega(n^{2-\varepsilon})$ on the Markov width of K_n.

1 Introduction

A *contingency table* is a $d_1 \times d_2 \times \cdots \times d_n$ array of non-negative integers. Contingency tables are used to record and analyze the relationship between n discrete random variables $X_1, X_2, \ldots, X_n$, where X_k takes values in the set $[d_k] = \{1, 2, \ldots, d_k\}$ for $k = 1, 2, \ldots, n$. In *hierarchial models* a simplicial complex Δ on $\{1, \ldots, n\}$ encodes interactions between the variables. One can estimate how well empirical data in the given table fits a hierarchial model by comparing statistics of this table with statistics of a random non-negative integral table with the same set of marginals. In [1] an algebraic

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approach for generating such a random table has been presented. This approach can be informally summarized as follows. A finite set of *moves*, such that any two tables with same set of marginals are connected by a sequence of such moves, is computed. Such a set of moves is called a *Markov basis*. Given a Markov basis, a random table is generated by performing a random walk using the moves in the basis.

Thus, description of Markov bases of a given model is of interest and has attracted attention of researchers in recent years. For a more detailed introduction see [2, 6]. In this paper, following [2], we concentrate our attention on binary graph models, i.e. hierarchial models of $2 \times \cdots \times 2$-contingency tables, where the simplicial complex Δ, which encodes the variable interactions, is a graph.

Let us now give a formal definition of a Markov basis of a binary graph model. For a finite set $X = \{x_1, x_2, \ldots, x_n\}$ let $\mathcal{Z}(X) \simeq \mathbb{Z}^{2^n}$ denote the integral lattice with an orthonormal basis $e_{a(x_1)a(x_2)\ldots a(x_n)}$ indexed by binary labeling $a : X \to \{0, 1\}$ of X. For a subset $Y \subset X$ there exists a natural projection map $\pi_{X\to Y} : \mathcal{Z}(X) \to \mathcal{Z}(Y)$ defined as the linear extension of the mapping $e_a \to e_{a|_Y}$. In most cases the original set X will be understood from context and we will frequently abbreviate $\pi_{X\to Y}$ to π_Y. Similarly, we will abbreviate $\pi_{\{a,b\}}$ to π_{ab}.

Given a graph G let

$$\pi_G : \mathcal{Z}(V(G)) \to \bigotimes_{e\in E(G)} \mathcal{Z}(V(e))$$

be a linear mapping obtained as the product of the maps π_{ij}, $ij \in E$. The mappings π_{ij} correspond to 2-way marginals of a $2 \times \cdots \times 2$ table. A finite subset $B \subseteq \mathrm{Ker}(\pi_G)$ is a *Markov basis* if for every non-negative integral vectors $v_1, v_2 \in \mathcal{Z}(V(G))$ with $\pi_G(v_1) = \pi_G(v_2)$, there exist $u_1, \ldots, u_\ell \in \pm B$ satisfying the following:

$$v_1 + \sum_{k=1}^{\ell} u_k = v_2 \qquad \text{and} \qquad v_1 + \sum_{k=1}^{\ell'} u_k \text{ is non-negative for all } 1 \leq \ell' \leq \ell.$$

The *Markov width* $\mu(G)$ of G is the smallest integer k such that there exists a Markov base B of G with $||v||_1 \leq 2k$ for every $v \in B$ where $||v||_1$ is the ℓ_1-norm of v. The motivation for considering Markov width as the measure of complexity of the set of Markov bases of a binary graph model comes from the fact that a binary model can be alternatively defined as a binomial ideal. In this setting Markov bases correspond to generating sets of the ideal, and Markov width equals to the degree of the largest minimal generator of the ideal. We omit the precise definitions, as we do not make use of this reinterpretation in our arguments, and refer the reader to [2, 9] for details.

It is known [3, 5, 11] that $\mu(G) = 2$ if and only G is a forest, and that $\mu(G) \geq 4$, otherwise. In this paper we characterize graphs with $\mu(G) = 4$, answering a question of Develin and Sullivant [2]. We show that $\mu(G) \leq 4$ if and only if G does not contain a subdivision of the complete graph K_4 as a subgraph. Additionally, we investigate the Markov width of complete graphs. Develin and Sullivant [2] showed that the Markov width $\mu(K_n)$ of the complete graph on n vertices is lower bounded by $2n-2$. We strengthen this lower bound, showing that $\mu(K_n) \geq \Omega(n^{2-\varepsilon})$ for every $\varepsilon > 0$.

2 Markov width of K_4-free graphs

We start by describing a standard construction used in inductive arguments on K_4-minor free graphs. *Series-parallel graphs* are graphs with two distinguished vertices called *poles*, obtained from elementary graphs by a recursive construction. The simplest series-parallel graph is an edge uv with the two poles being its end-vertices. If G_1 and G_2 are series-parallel graphs with poles u_1 and v_1, and u_2 and v_2, respectively, then the graph G obtained by identifying the vertex v_1 with u_2 is also a series-parallel graph and its two poles are the vertices u_1 and v_2. The graph G obtained in this way is called the *serial join* of G_1 and G_2. The *parallel join* of G_1 and G_2 is the graph obtained by identifying the vertex u_1 with u_2 and the vertex v_1 with v_2; the poles are the identified vertices. The *series-parallel graphs* are precisely those that can be obtained from edges by a series of serial and parallel joins. The sequence of such joins leading to a construction of a graph G is called a *series-parallel decomposition* of G. The series-parallel decomposition of G is not unique.

In our considerations, we will need the following (folklore) lemma.

Lemma 1. *Every 2-connected K_4-minor free graph is a series-parallel graph. If G is a 2-connected K_4-minor free graph that is not a cycle, then there exists a series-parallel decomposition of G such that G is obtained by a parallel join of at least three series-parallel graphs.*

Observe that if the last operation in a series-parallel decomposition of G is a parallel join, then G is 2-connected (the converse is also true). In particular, we can apply Lemma 1 to such a graph G.

Finally, following standard graph theory terminology, we define a $\{u,v\}$-*bridge* B of G to be a connected subgraph of G such that either $E(B) = \{uv\}$ or for some component C of $G \setminus \{u,v\}$ the set $E(B)$ consists of all edges of G with at least one end in $V(C)$. Note that a series parallel-graph G with poles u and v can be obtained from the set of its $\{u,v\}$-bridges by a sequence of parallel joins.

In this section we characterize graphs with Markov width equal to four. The cycles have this property, as shown in [2].

Lemma 2 ([2]). *If G is a cycle then $\mu(G) = 4$.*

Throughout the proof of the main theorem of this section, we will use the following straightforward observations repeatedly, and so we state them as lemmas. For convenience, let $\mathcal{Z}(\emptyset) = \mathbb{Z}$ and $\pi_{X\to\emptyset}(z) = ||z||_1$ for $z \in \mathcal{Z}(X)$.

Lemma 3. *Let X_1 and X_2 be finite sets, let $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$. Let $z \in \mathcal{Z}(X)$ and $\overline{z} \in \mathcal{Z}(X_1)$ such that $\pi_{X\to Y}(z) = \pi_{X_1\to Y}(\overline{z})$. Then there exists $z' \in \mathcal{Z}(X)$ such that $\pi_{X\to X_1}(z') = \overline{z}$, $\pi_{X\to X_2}(z') = \pi_{X\to X_2}(z)$ and $||z - z'||_1 = ||\pi_{X\to X_1}(z) - \overline{z}||_1$.*

Lemma 4. *Let X_1 and X_2 be finite sets and let $X = X_1 \cup X_2$. Let $z, z' \in \mathcal{Z}(X)$ be such that $\pi_{X\to X_i}(z) = \pi_{X\to X_i}(z')$ for $i = 1,2$. Then there exist vectors $z = z_0, z_1, z_2, \ldots, z_\ell = z' \in \mathcal{Z}(X)$ such that $||z_k - z_{k-1}||_1 = 4$ and $\pi_{X\to X_i}(z_{k-1}) = \pi_{X\to X_i}(z_k)$ for $i = 1,2$ and $k = 1,\ldots,\ell$. Moreover, if z and z' are non-negative, then $z_0, z_1, \ldots, z_\ell$ can be chosen to be non-negative.*

Lemma 5. *Let X_1 and X_2 be finite sets, let $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$. Let $z_1, z_1' \in \mathcal{Z}(X_1)$ and $z_2, z_2' \in \mathcal{Z}(X_2)$ such that $\pi_{X_1\to Y}(z_1) = \pi_{X_2\to Y}(z_2)$ and $\pi_{X_1\to Y}(z_1') = \pi_{X_2\to Y}(z_2')$. If*

$$||\pi_{X_1\to Y}(z_1) - \pi_{X_1\to Y}(z_1')||_1 = ||z_1 - z_1'||_1 = ||z_2 - z_2'||_1 \ ,$$

then there exist $z, z' \in \mathcal{Z}(X)$ such that $\pi_{X\to X_i}(z) = z_i$, $\pi_{X\to X_i}(z') = z_i'$ and $||z - z'||_1 = ||z_1 - z_1'||_1$.

We first show that the Markov width of every series-parallel graph is at most four. In fact we prove a slightly stronger and more technical result.

Theorem 6. *Let G be a series-parallel graph with a vertex set X and poles u and v. If $z, z' \in \mathcal{Z}(X)$ are two non-negative vectors with $\pi_G(z) = \pi_G(z')$, then there exist non-negative vectors $z_0, \ldots, z_\lambda \in \mathcal{Z}(X)$ such that*

1. $z_0 = z$, $z_\lambda = z'$,

2. $\pi_G(z) = \pi_G(z_k)$ *for every* $k = 0, \dots, \lambda$,

3. $||z_k - z_{k-1}||_1 \le 8$ *for every* $k = 1, \dots, \lambda$, *and*

4. *if* $\pi_{uv}(z_{k-1}) \ne \pi_{uv}(z_k)$, *then* $||z_k - z_{k-1}||_1 = 4$.

Proof. The proof proceeds by induction on the order of G, i.e., $|X|$. If $|X| = 2$, then the graph G is an edge and thus $z = z'$. The claim readily follows.

Assume now that G is a graph of order at least three obtained from graphs G_1 and G_2 by a serial or a parallel join. Let X_i be the vertex set of G_i, $i = 1, 2$. We distinguish four cases:

- **G is obtained by a serial join of G_1 and G_2.**
  Let w be the vertex shared by G_1 and G_2; by symmetry, we can assume that the pole u is contained in G_1 and v in G_2. We apply induction to G_i with $z^i = \pi_{X_i}(z)$ and $z'^i = \pi_{X_i}(z')$, for $i = 1, 2$. Let, for $i = 1, 2$, $z^i_0, \dots, z^i_\ell$ be the resulting sequences of vectors (note that by padding the sequences with $\pi_{X_i}(z')$ at the end, we can assume that the sequences have the same length).

  We construct the required sequence $z_0, \dots, z_\lambda$ as follows. We start by constructing a sequence of vectors We have now constructed a sequence of vectors $z_0, \dots, z_\ell$ such that $z_0 = z$, $\pi_{X_1}(z_k) = \pi_{X_1}(z^1_k)$ and $\pi_{X_2}(z_k) = \pi_{X_2}(z)$ for $k = 1, 2, \dots, \ell$, and Properties 2,3 and 4 from the lemma statement are satisfied. Let $k \in [1, \ell]$ and assume the vectors $z_0, \dots, z_{k-1}$ have been defined. If $\pi_{uw}(z^1_{k-1}) = \pi_{uw}(z^1_k)$, apply Lemma 3 with $X'_1 = X_1$, $X'_2 = X_2 \cup \{u\}$, $z = z_{k-1}$ and $\overline{z} = z^1_k$ and set z_k to be the resulting vector z'. Clearly, $||z_k - z_{k-1}||_1 = ||z^1_k - \pi_{X_1}(z_{k-1})||_1 = ||z^1_k - z^1_{k-1}||_1 \le 8$ and $\pi_{uvw}(z_{k-1}) = \pi_{uvw}(z_k)$. Since $\pi_{G_1}(z^1_{k-1}) = \pi_{G_1}(z^1_k)$, it follows $\pi_G(z_{k-1}) = \pi_G(z_k)$. In particular, Properties 2, 3 and 4 are satisfied in this step.

  If $\pi_{uw}(z^1_{k-1}) \ne \pi_{uw}(z^1_k)$, then $||z^1_k - z^1_{k-1}||_1 = 4$ by Property 4. By Lemma 3 applied with $z = z_{k-1}$ and $\overline{z} = z^1_k$, there exists $z_k = z'$ such that $\pi_{X_1}(z_k) = z^1_k$, $\pi_{X_2}(z_k) = \pi_{X_2}(z^2_{k-1}) = \pi_{X_2}(z)$ and $||z_k - z_{k-1}||_1 = 4$. Again, $\pi_{G_1}(z^1_{k-1}) = \pi_{G_1}(z^1_k)$ implies that $\pi_G(z_{k-1}) = \pi_G(z_k)$. Since $||z_k - z_{k-1}||_1 = 4$, Properties 2, 3 and 4 are also satisfied in this step.

  We have now constructed a sequence $z_0, \dots, z_\ell$ such that $z_0 = z$, $\pi_{X_1}(z_\ell) = \pi_{X_1}(z')$ and $\pi_{X_2}(z_\ell) = \pi_{X_2}(z)$. An analogous argument yields the existence of a sequence $z_\ell, \ldots, z_{2\ell}$ such that $\pi_{X_1}(z_{2\ell}) = \pi_{X_1}(z')$, $\pi_{X_2}(z_{2\ell}) = \pi_{X_2}(z')$ satisfying Properties 2, 3 and 4. By Lemma 4, the sequence $z_0, \ldots, z_{2\ell}$ can be completed to a sequence $z_0, \ldots, z_\lambda$ such that $z_\lambda = z'$ and $||z_k - z_{k-1}||_1 = 4$ for $k = \ell+1, \ldots, \lambda$. Clearly, the resulting sequence has Property 2. Since the ℓ_1-norm of the vectors $z_k - z_{k-1}$, $k > 2\ell$, is four, the sequence $z_{2\ell}, \ldots, z_\lambda$ has also Properties 3 and 4.

- **G is obtained by a parallel join of G_1 and G_2, uv is an edge of G, and G has at least three $\{u, v\}$-bridges.**
  By permuting the order of the parallel joins in the series-parallel decomposition of G, we can assume that neither G_1 nor G_2 is an edge. By symmetry, G_1 contains the edge uv. Let G'_1 be G_1 and G'_2 be G_2 with the edge uv added.

  We apply induction to G'_i, $z^i = \pi_{X_i}(z)$ and $z'^i = \pi_{X_i}(z')$, $i = 1, 2$. Let $z^i_0, \ldots, z^i_\ell$ be the resulting sequence of vectors (note that by padding the sequences with $\pi_{X_i}(z')$ at the end, we can assume the sequences to have the same length). By Lemma 3, there exist a sequence of vectors $z_0, \ldots, z_{2\ell}$ such that $\pi_{X_1}(z_i) = z^1_i$ and $\pi_{X_2}(z_i) = z^2_0$ for $i = 0, \ldots, \ell$, $\pi_{X_1}(z_i) = z^1_\ell$ and $\pi_{X_2}(z_i) = z^2_{i-\ell}$ for $i = \ell+1, \ldots, 2\ell$ and $||z_i - z_{i-1}|| \leq 8$ for $i = 1, \ldots, 2\ell$. Clearly, this sequence has Properties 2 and 3.

  Lemma 4 yields that there is a sequence $z_{2\ell}, \ldots, z_\lambda$ such that $z_\lambda = z'$, $\pi_{X_i}(z_{2\ell}) = \cdots = \pi_{X_i}(z')$ and $||z_k - z_{k-1}||_1 \leq 4$ for $k = 2\ell + 1, \ldots, \lambda$. The presence of the edge uv in G implies that $\pi_{uv}(z_0) = \cdots = \pi_{uv}(z_\lambda)$ which yields Property 4.

- **G is obtained by a parallel join of G_1 and G_2 and uv is not an edge of G.**
  If $\pi_{uv}(z) = \pi_{uv}(z')$, add the edge uv to G and proceed as in the previous case. Hence, $\pi_{uv}(z) \neq \pi_{uv}(z')$. Observe that there exists a (unique) sequence of vectors $w_0, \ldots, w_m \in \mathbb{Z}(\{u, v\})$ such that $w_0 = \pi_{uv}(z)$, $w_m = \pi_{uv}(z')$, $||w_r - w_{r-1}||_1 = 4$ for $r = 1, \ldots, m$ and $||\pi_{uv}(z') - \pi_{uv}(z)||_1 = 4m$.

  We now apply induction for G_i, $z^i = \pi_{X_i}(z)$ and $z'^i = \pi_{X_i}(z')$, $i = 1, 2$. Let $z^i_0, \ldots, z^i_{\ell_i}$ be the resulting sequence of vectors. By Property 4, there exist indices k^i_r such that $\pi_{uv}(z^i_{k^i_r - 1}) = w_{r-1}$ and $\pi_{uv}(z^i_{k^i_r}) = w_r$ for $r = 1, \ldots, m$ and $i = 1, 2$. Since it is possible to prolong the sequence by repeating some of the vectors several times, we can assume that $k^1_r = k^2_r$; let k_r be their common value in the rest.

By Lemma 5, there exist vectors $z_1, \dots, z_m$ and $z'_0, \dots, z'_{m-1}$ such that $\pi_{X_i}(z_r) = z^i_{k_r}$ for $i = 1, 2$ and $r = 1, \dots, m$, $\pi_{X_i}(z'_{r-1}) = z^i_{k_r-1}$ for $i = 1, 2$ and $r = 1, \dots, m$, and $||z_r - z'_{r-1}||_1 = 4$ for every $r = 1, \dots, m$. For convenience, set $z_0 = z$ and $z'_m = z'$.

Let G' be the graph obtained from G by adding the edge uv. The choice of the indices k_r implies that $\pi_{G'}(z_r) = \pi_{G'}(z'_r)$ for $r = 0, \dots, m$. In particular, we can apply the argument used in the previous case for G' with z_0 and z'_0, G' with z_1 and z'_1, $\dots$, G' with z_r and z'_r and concatenate the obtained sequences of vectors. Clearly, the final sequence has Properties 1, 2 and 3. Because of the presence of the edge uv in G', the mapping π_{uv} is constant inside each of the r sequences in the concatenation. Since $||z_r - z'_{r-1}||_1 = 4$ for $r = 1, \dots, m$, the resulting sequence also has Property 4.

- **G is obtained by a parallel join of G_1 and G_2, uv is an edge of G and G has only two $\{u, v\}$-bridges.**
  Clearly, one of the $\{u, v\}$-bridges is the edge uv. If G is a cycle, then there exist vectors $z_0, \dots, z_\lambda$ satisfying Properties 1, 2 and 3 by Lemma 2. Since uv is an edge, $\pi_{uv}(z_0) = \cdots = \pi_{uv}(z_\ell)$ which implies that the sequence also satisfies Property 4.

  On the other hand, if G is not a cycle, then Lemma 1 implies that G has another series parallel decomposition, say with poles u' and v', such that G has at least three $\{u', v'\}$-bridges. Based on whether G contains the edge $u'v'$, we apply the arguments presented in the second case or the third case to obtain a sequence of vectors $z_0, \dots, z_\lambda$ satisfying Properties 1, 2, 3 and 4 with respect to u' and v'. Since uv is an edge of G and thus $\pi_{uv}(z_0) = \cdots = \pi_{uv}(z_\lambda)$, Property 4 also holds with respect to u and v.

$\square$

It is now easy to derive the main result of this section:

Theorem 7. *The Markov width $\mu(G)$ of a graph G is at most four if and only if G does not contain K_4 as a minor. In particular, $\mu(G) = 4$ if and only if G is not a forest and has no K_4 as a minor.*

Proof. It is shown in [2] that $\mu(K_4) = 6$, and $\mu(H) \leq \mu(H')$ if H is obtained from H' by a sequence of edge contractions and vertex deletions. Therrefore, the Markov width $\mu(G)$ of every graph G containing K_4 as a minor is at least six. Moreover, if G is not a forest, then $\mu(G) \geq 4$ by Lemma 2.

On the other hand, it is easy to see that, the Markov width of a graph is the maximum of the Markov widths of its blocks (maximal 2-connected subgraphs). As every 2-connected K_4-minor-free graph is series-parallel (in case of a forest, the blocks are single edges), the Markov width of every graph with no K_4 minor is at most four by Theorem 6. $\square$

3 Lower bound for complete graphs

We will derive a lower bound on the Markov width of a complete graph from bounds on maximum density of clean triangulations of surfaces. A triangulation T of a surface is *clean* if every triangle of T is a face triangle. A triangulation T is 2-*face-colorable* if its faces can be colored with two colors in such a way that every two adjacent faces receive distinct colors, i.e., the dual graph of T is bipartite. For other, more standard, definitions related to triangulations of surfaces, we refer the reader to the monograph [7].

Lemma 8. *If there exists a clean 2-face-colorable triangulation T with n vertices and m edges, then the Markov width $\mu(K_n)$ of the complete graph K_n is at least $m/3$.*

Proof. Assume that the faces of T are colored with red and blue and let $G = K_n$ be the complete graph on the same vertex set as T, which we identify with the set $[n]$. Remember that $\mathcal{Z}([n])$ has a basis e_a, where a ranges over all functions from $[n]$ into $\{0,1\}$, which could be considered as indicator functions of subsets of $[n]$. Let $z^r \in \mathcal{Z}([n])$ be defined as the sum of vectors $e_{\chi(F)}$ over all red faces F of T. In particular, $||z^r||_1 = m/3$. The vector $z^b \in \mathcal{Z}([n])$ is defined analogously with respect to blue faces. It is not hard to see that $\pi_G(z^r) = \pi_G(z^b)$. Let $\Pi_G(z^r) = \{z \in \mathcal{Z}([n]) \mid \pi_G(z) = \pi_G(z_r)\}$ be the fiber of π_G containing z^r. We will show that $\Pi_G(z^r) = \{z^r, z^b\}$ which will imply the statement of the lemma by the definition of the Markov width.

Consider $z \in \Pi_G(z^r)$ and let $A_1, \ldots, A_{m/3}$ be (not necessarily distinct) subsets of $[n]$ such that

$$z_{\chi(A)} = \sum_{i=1}^{m/3} e_{\chi(A_i)}.$$

By considering $\pi_{ij}(z^r) = \pi_{ij}(z)$ for $i, j \in [n]$ we see that if $ij \in E(T)$ then $\{i,j\}$ belong to exactly one member of $(A_1, \ldots, A_{m/3})$ and, if $ij \notin E(T)$

then $\{i, j\}$ belongs to no member of this family. It follows that no two sets A_1 share more than one elements, and that every A_i forms a vertex set of a complete subgraph of T. Since T is a 2-face-colorable clean triangulation, T contains no subgraph isomorphic to K_4 and consequently $|A_i| \leq 3$. On the other hand, the sum of the sizes of $A_1, \ldots, A_{m/3}$ is independent on the choice of $z \in \Pi_G(z^r)$ and we have

$$\sum_{i=1}^{m/3} |A_i| = m.$$

This implies that $|A_1| = |A_2| = \ldots = |A_{m/3}| = 3$. Hence, each A_i corresponds to a face of T. Therefore $A_1, \ldots, A_{m/3}$ is a collection of the vertex sets pairwise non-adjacent faces of T, such that every edge of T is contained in some face of the collection. There are only two such collections, namely the vertex sets of the red faces and of the blue faces of T. In particular, $z \in \{z^r, z^b\}$, as claimed. □

We now apply Lemma 8 to obtain a lower bound on $\mu(K_n)$. A simple triangulation satisfying the assumptions of Lemma 8 is a double-wheel drawn on the sphere: It has $N + 2$ vertices and $3N$ edges. Hence, Lemma 8 gives us that the Markov width of K_n is at least $n - 2$. To obtain a superlinear bound, we need denser clean triangulations. Such triangulations of essentially optimal density were constructed by Seress and Szabó in [8]. For every $\varepsilon > 0$ and for every sufficiently large integer n, they construct a clean triangulation $T_{n,\varepsilon}$ of some surface Σ with n vertices and $n^{2-\varepsilon}$ edges. The constructed triangulations are 3-vertex-colorable. If the corresponding surface Σ is orientable then the triangulation is 2-face-colorable, as the clockwise orders of the vertex colors around adjacent faces are different. If Σ is non-orientable, it is possible to obtain a clean triangulation with $2n$ vertices and $2n^{2-\varepsilon}$ edges which is 3-vertex-colorable by considering an orientable 2-cover of Σ and a corresponding clean triangulation.

We can now infer from Lemma 8 the following:

Theorem 9. *For every $\varepsilon > 0$, there exists n_0 such that $\mu(K_n) \geq n^{2-\varepsilon}$ for all $n \geq n_0$.*

4 Final Remarks

1. Our definition of binary graph models differs from the standard one. We do not consider 1-way marginals corresponding to the vertices of

the graph. This is a very minor distinction, as for graphs of minimum degree one the fibers of π_G remain unchanged, and a binary graph model of a disconnected graph can be considered as a toric fiber product (see [10]) of the models corresponding to its components.

2. In [9] Sturmfels and Sullivant consider cut ideals of graphs. Those are binomial ideals which are closely related to binary graph models as pointed out in [9]. In particular, the Markov width $\mu(G)$ of a graph G equals to the maximum degree of a binomial appearing in a minimal generating set of the cut ideal $I_{\hat{G}}$, where $\hat{G}$ is a graph obtained from G by adding a universal vertex, that is a new vertex joined by an edge to every vertex of G.

   Sturmfels and Sullivant conjecture that I_G can be generated in degree four if and only if G has no K_5 minor. Our result can be interpreted as a partial result towards this conjecture, verifying it for all graphs containing a universal vertex. Also let us mention that Engström [4] recently proved another conjecture from [9] showing that a cut ideal I_G can be generated in degree two if and only if G has no K_4 minor.

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