Small Graph Classes and Bounded Expansion

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Abstract

A class of simple undirected graphs is *small* if it contains at most $n!\alpha^n$ graphs with n vertices, for some constant α. We prove that classes of graphs with expansion bounded by a function $f(r) = O(r^c)$ for any $0 \le c < \log_9 2 \approx 0.315$ are small.

We work with simple undirected graphs, without loops or parallel edges. A class of graphs is *small* if it contains at most $n!\alpha^n$ different (but not necessarily non-isomorphic) graphs on n vertices, for some constant α. For example, the class of all trees is small, as there are exactly $n^{n-2} < n!e^n$ trees on n vertices. Norine et al. [8] showed that all proper minor-closed classes of graphs are small, answering the question of Welsh [9]. This question was motivated by the results of McDiarmid et al. [2] regarding random planar graphs. These results in fact hold for any class of graphs that is small and addable[1]. Many naturally defined graph classes are addable (for example, proper minor-closed classes excluding a 2-connected minor), and this condition is usually easy to verify. The more substantial assumption thus is that the class is small. The aim of this paper is to prove that classes of graphs with expansion bounded by a slowly growing function ($f(r) = O(r^{0.315})$) are small. This generalizes the result of Norine et al. [8], as proper minor-closed classes have expansion bounded by a constant.

Let us now recall the notion of classes of graphs with bounded expansion, as defined by Nešetřil and Ossona de Mendez [6, 3, 4, 5]. The *grad*[2] *with rank*

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[1] A class $\mathcal{G}$ is *addable* if

- $G \in \mathcal{G}$ if and only if every component of G belongs to $\mathcal{G}$, and
- if $G_1, G_2 \in \mathcal{G}$, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then the graph obtained from the disjoint union of G_1 and G_2 by adding the edge $\{v_1, v_2\}$ belongs to $\mathcal{G}$.

[2] Greatest Reduced Average Density

r of a graph G is equal to the largest average density of a graph G' that can be obtained from G by removing some of the vertices (and possibly edges) and then contracting vertex-disjoint subgraphs of radius at most r to single vertices (arising parallel edges are suppressed). The grad with rank r of G is denoted by $\nabla_r(G)$. In particular, $2\nabla_0(G)$ is the maximum average degree of a subgraph of G. Given a function $f : N \to R^+$, a graph has *expansion bounded by* f if $\nabla_r(G) \leq f(r)$ for every integer r. A class $\mathcal{G}$ of graphs has *expansion bounded by* f if the expansion of every $G \in \mathcal{G}$ is bounded by f. Finally, we say that a class of graphs $\mathcal{G}$ has *bounded expansion* if there exists a function f such that the expansion of $\mathcal{G}$ is bounded by f.

The concept of classes of graphs with bounded expansion proves surprisingly powerful. Many classes of graphs have bounded expansion (proper minor-closed classes, classes of graphs with bounded maximum degree, classes of graphs excluding subdivision of a fixed graph, ...), and many results for proper minor-closed classes (existence of colorings, small separators, light subgraphs, ...) generalize to classes of graphs with bounded expansion (possibly with further natural assumptions). The classes of graphs with bounded expansion are also interesting from the algorithmic point of view, as the proofs of the mentioned results usually give simple and efficient algorithms. Furthermore, fast algorithms and data structures for problems like deciding whether a graph contains a fixed subgraph, or for determining the distance between a pair vertices (assuming that the distance is bounded by a fixed constant), have been derived. The reader is referred to [7] for a survey of the results regarding the bounded expansion.

1 Lower bound

In this section, we show that the class of graphs with expansion bounded by the function $f(r) = A + Br^c$ is small, for any $A, B > 0$ and $0 \leq c < \log_9 2$. Our proof is inspired by the proof of Norine et al. [8]. The main difference is that we avoid contracting the edges one by one (which would make us lose control over the expansion of the graph), instead contracting many edges at once—this way, the expansion of the graph grows, but slowly enough in comparison with the decreasing size of the graph, so that we can bound the number of graphs. Another difference is in the way that we deal with the case that the graph contains only a few light edges (i.e., edges connecting two vertices of degree at most d, for some bound d)—while the original approach would work, it would force us to use too large d in the definition of a light edge and result in an (exponentially) worse upper bound on the allowed expansion.

Lemma 1. *A graph G on n vertices contains at most $1 + 4^{\nabla_0(G)}n$ cliques, and the bound can be decreased by one if G has at least one edge.*

Proof. There exists an ordering $v_1, \ldots, v_n$ of vertices of G such that each vertex has at most $2\nabla_0(G)$ neighbors before it in this ordering, i.e., $|\{v_j : j < i, \{v_i, v_j\} \in E(G)\}| \leq 2\nabla_0(G)$ for each $i = 1, \ldots, n$. Suppose that G has k cliques, including the empty one. Let us count the number of pairs (C, v), where C is a nonempty clique in G and v is the last vertex of C in the ordering. On one hand, each nonempty clique has exactly one last vertex, thus there are $k-1$ such pairs. On the other hand, all the cliques in that v is the last vertex must be subgraphs of the graph induced by v and its neighbors that precede it in the ordering, thus there are at most $2^{2\nabla_0(G)}$ of them. It follows that $k \leq 1 + 4^{\nabla_0(G)}n$.

Suppose that G has at least one edge. As the density of the graph consisting of a single edge is $1/2$, $\nabla_0(G) \geq 1/2$. It follows that we have overestimated the contribution of v_1 by at least 1, hence $k \leq 4^{\nabla_0(G)}n$. $\square$

A graph F such that all of its components are stars is called a *star forest*. Let $C(F)$ denote the set of centers of the stars in F (in case that a component of F consists of a single edge, its center can be chosen arbitrarily). The vertices of F that do not belong to $C(F)$ are called *ray vertices*, and their set is denoted by $R(F)$. For an integer d and a vertex v of a graph, let $N_{>d}(v)$ be the set of neighbors of v whose degree is greater than d.

Lemma 2. *For every graph G on n vertices and an integer $d > 0$, there exists a set $S \subseteq V(G)$ of vertices of degree at most d and a star forest $F \subseteq G$ with $V(F) = V(G) \setminus S$, such that if H is the graph obtained from $G - S$ by contracting the edges of F, then*

- *$|C(F)| \leq \frac{2\nabla_0(G)}{d}n$, $|E(F)| \leq \frac{2\nabla_0(G)\nabla_1(G)}{d}n$, and*
- *for each $v \in S$, $N_{>d}(v) \subseteq C(F)$ and $N_{>d}(v)$ induces a clique in H, and*
- *the degree (in G) of each ray vertex of F is at most d.*

Proof. Let X be the set of vertices of G whose degree is greater than d, $|X| \leq \frac{2\nabla_0(G)n}{d}$. We say that a set $B \subseteq V(G) \setminus X$ is *extending* if to each vertex $v \in B$, we can assign two of its neighbors $a(v)$ and $b(v)$ in X such that

- $\{a(v), b(v)\} \notin E(G)$, and
- $\{a(v), b(v)\} \neq \{a(v'), b(v')\}$ for $v \neq v'$, $v, v' \in B$.

Let $Z \subseteq V(G) \setminus X$ be a maximal extending set, and $S = V(G) \setminus (X \cup Z)$. Let G_1 be the graph with $V(G_1) = X$ and $E(G_1) = E(G[X]) \cup \{\{a(v), b(v)\} : v \in Z\}$. Observe that G_1 is obtained from G by removing vertices and edges, and contracting edges of a star forest F with $C(F) = X$ and $R(F) = Z$ (some of the stars in F may be isolated vertices), thus $\nabla_0(G_1) \leq \nabla_1(G)$. Therefore, $|E(F)| = |Z| \leq |E(G_1)| \leq \nabla_1(G)|X| \leq \frac{2\nabla_0(G)\nabla_1(G)n}{d}$. By the definition of X, $N_{>d}(v) \subseteq X$ for each $v \in S$, and by the maximality of Z, $N_{>d}(v)$ induces a clique in $G_1 \subseteq H$. $\square$

Let $N(f, n)$ be the number of graphs on n vertices whose expansion is bounded by a non-decreasing positive function f, and $I(f, n) = N(f, n)/n!$. Let $f^\star$ be the function defined by $f^\star(x) = f(3x + 1)$. Note that if the expansion of a graph G is bounded by f, then the expansion of the graph obtained from G by contracting a star forest is bounded by $f^\star$. Let us recall the following well-known estimates: $k! \geq (k/e)^k$ and $\binom{k}{m} \leq (ek/m)^m$, valid for all integers $k \geq m > 0$.

Lemma 3. *Let $f : N \to R^+$ be a non-decreasing function with $f(0) \geq 1/2$ and $d > 0$, $n \geq 4$ integers. Let $\alpha = \frac{2f(0)(f(1)+1)}{d}$ and $k = \left\lfloor \frac{1+\alpha}{2} n \right\rfloor$. If $\alpha < 1/2$, then*

$$N(f, n) \leq I(f^\star, k) \left(42(ed^2)^{d+1} 4^{f(1)}\right)^n.$$

Proof. Let G be a graph with n vertices and expansion bounded by f. Consider $S \subseteq V(G)$ and a spanning forest $F \subseteq G - S$ given by Lemma 2, $|S| \geq (1 - \alpha)n$. Let L_1 be the set of vertices of S that do not have any neighbor of degree at most d, and $F_1' \subseteq G - L_1$ a star forest such that each vertex of F' has degree (in G) at most d, $S \setminus L_1 \subseteq V(F')$ and each component of F_1' contains at least one edge. Note that $|L_1| + |E(F_1')| \geq |S|/2$, hence $n - |L_1| - |E(F_1')| \leq k$. Consider $L \subseteq L_1$ and $F' \subseteq F_1'$ such that each component of F' contains at least one edge and $n - |L| - |E(F')| = k$. Let G' be the graph obtained from $G - L$ by contracting all edges of F'. Note that $|V(G')| = n - |L| - |E(F')| = k$, and the expansion of G' is bounded by $f^\star$. Furthermore, each vertex obtained by contracting a component of F' has degree at most $d(d - 1)$.

We now show that we can obtain a particular graph G' by this process only from a limited number of graphs. More precisely, given the graph G', the graph G is one of the graphs obtained in the following way:

1. Choose a subset Y of $[n]$ of size k ($\binom{n}{k}$ ways), and identify the set of vertices of G' with Y.

2. Choose the set $L \subseteq [n] \setminus Y$ (at most 2^n ways).

3. For each vertex $v \in [n] \setminus (L \cup Y)$ select a vertex $i(v) \in Y$. Let F' be the star forest with the edges $\{v, i(v)\}$ for all such vertices, with $C(F') = V(F') \cap Y$ (at most $k^{n-k-|L|} \leq n^{n-k-|L|}$ ways). We only consider the forests F' with maximum degree at most d.

4. Let us define $i(u) = u$ for each $u \in C(F')$. We only consider the forests F' such that the degree (in G') of each such vertex is at most $d(d-1)$. For each vertex $v \in V(F')$, let $P(v) = \{x : \{i(v), i(x)\} \in E(G')\} \cup \{x : i(v) = i(x)\} \setminus \{v\}$. We only consider the trees F' such that that $|P(v)| \leq d(d-1)(d+1) + (d+1) - 1 = d^3$. Choose a subset $I(v)$ of size at most d from $P(v)$ (at most $\left(d\binom{d^3}{d}\right)^n \leq \left(d(ed^2)^d\right)^n$ ways). We only consider the choices of I such that the neighborhood of v in F' is a subset of $I(v)$, and for each $u, v \in V(F')$, if $u \in I(v)$, then $v \in I(u)$. Let G'' be the graph with $V(G'') = [n]$, such that $G''[Y \setminus V(F')] = G'[Y \setminus V(F')]$, the neighborhood of each vertex $v \in V(F')$ consists of $I(v)$, and the vertices of L are isolated. Consider only the choices that lead to a graph G'' with expansion bounded by f.

5. Choose disjoint sets $X, Z \subseteq [n] \setminus L$, such that the degree of each vertex of Z in G'' is at most d (at most 3^n ways).

6. For each vertex of Z, choose an edge joining it with a vertex of X, and let the F be the star forest consisting of these edges (at most d^n ways).

7. Let H be the graph obtained from $G''[X \cup Z]$ by contracting the edges of F. By Lemma 1, H contains at most $4^{f(1)}n$ cliques—note that the estimate is valid even if H has no edges or $X = \emptyset$, as $f(1) \geq f(0) \geq 1/2$ and $n > 0$. For each vertex $v \in L$, choose a clique C in H, and add edges between v and the vertices of C (at most $\left(4^{f(1)}n\right)^{|L|} \leq 4^{f(1)n}n^{|L|}$ ways).

As there are $N(f^\star, k)$ graphs with expansion bounded by $f^\star$ on k vertices,

$$\begin{aligned} N(f,n) &\leq N(f^\star,k)\binom{n}{k}\left(6d^2(ed^2)^d4^{f(1)}\right)^n n^{n-k} \\ &= n!I(f^\star,k)\frac{n^{n-k}}{(n-k)!}\left(6d^2(ed^2)^d4^{f(1)}\right)^n \\ &\leq n!I(f^\star,k)\left(\frac{n}{n-k}\right)^{n-k}\left(6(ed^2)^{d+1}4^{f(1)}\right)^n. \end{aligned}$$

As $\alpha < 1/2$ and $n \geq 4$, $\frac{n}{n-k} \leq \frac{n}{n-\lceil 3n/4\rceil} \leq 7$. The claim of the lemma follows. □

Let us now show the main result:

Theorem 4. *For any $A, B > 0$ and $0 \leq c < \log_9 2$, the class of graphs whose expansion is bounded by $f(r) = A + B \cdot r^c$ is small.*

Proof. The claim is trivial if $f(0) < 1/2$, as the class of graphs bounded by such a function f contains only edgeless graphs, $N((,f),n) \leq 1$. Therefore, assume that $f(0) \geq 1/2$. Furthermore, f is non-decreasing, thus we may apply Lemma 3. Let us consider fixed α, $0 < \alpha < 1/3$, and set $d = \frac{6f^2(1))}{\alpha} \geq \frac{2f(0)(f(1)+1)}{\alpha}$. In particular, $\log d \geq 1$, $d \geq \log 42$, $d \geq \log 4 \cdot f(1)$, and $(d+1)(1+2\log d) \leq 6d\log d$, hence $42(ed^2)^{d+1}4^{f(1)} \leq e^{8d\log d}$. Let $q(x) = x^2\log(x+e)$ and $C = \frac{192+48\log(1/\alpha)}{\alpha}$. Note that

$$8d\log d = \frac{48}{\alpha}f^2(1)(\log 6 + 2\log f(1) - \log\alpha) \leq Cq(f(1)).$$

Lemma 3 implies that for any $n \geq 4$,

$$\begin{aligned} I(f,n) \leq I\left(f^\star, \left\lfloor\frac{1+\alpha}{2}n\right\rfloor\right)\left(42(ed^2)^{d+1}4^{f(1)}\right)^n \leq \\ \leq I\left(f^\star, \left\lfloor\frac{1+\alpha}{2}n\right\rfloor\right)e^{Cq(f(1))n}. \end{aligned}$$

For a real number $x \geq 4$, let $\overline{I}(f,x) = \max(8, \max_{4\leq n\leq x} I(f,n))$, and for $x < 4$, let $\overline{I}(f,x) = 8$ (the number of graphs on 3 vertices). Given $x \geq 4$ such that $\overline{I}(f,x) \neq 8$, there exists n $(4 \leq n \leq x)$ such that $\overline{I}(f,x) = I(f,n)$, in particular,

$$\overline{I}\,(f,x) \le I\left(f^\star, \left\lfloor \frac{1+\alpha}{2} n \right\rfloor\right) e^{Cq(f(1))n} \le \overline{I}\left(f^\star, \frac{1+\alpha}{2} x\right) e^{Cq(f(1))x}.$$

Note that in case that $x \ge 4$ and $\overline{I}\,(f,x) = 8$, this inequality holds as well. Let $f_0 = f$ and $f_{i+1} = f_i^\star$ for $i \ge 0$. Note that $f_i(d) = 3^i d + \frac{3^i-1}{2}$, in particular, $f_i(1) = \frac{3^{i+1}-1}{2}$. The recurrence can be expanded to

$$\overline{I}\,(f,x) \le 8 \prod_{i=0}^{t} e^{Cq(f_i(1))\left(\frac{1+\alpha}{2}\right)^i x} = 8 \prod_{i=0}^{t} e^{Cq\left(f\left(\frac{3^{i+1}-1}{2}\right)\right)\left(\frac{1+\alpha}{2}\right)^i x},$$

where t is the smallest integer such that $\left(\frac{1+\alpha}{2}\right)^{t+1} x < 4$. Taking the logarithm of this inequality, we obtain

$$\log \overline{I}\,(f,x) \le \log 8 + Cx \sum_{i=0}^{t} q\left(f\left(\frac{3^{i+1}-1}{2}\right)\right)\left(\frac{1+\alpha}{2}\right)^i,$$

thus the class of graphs with expansion bounded by f is small if

$$\sum_{i=0}^{\infty} q\left(f\left(\frac{3^{i+1}-1}{2}\right)\right)\left(\frac{1+\alpha}{2}\right)^i$$

is finite. If $f(r) = A + B \cdot r^c$ with $c < \log_9 2$, this is the case if we choose $\alpha < \frac{2}{9^c} - 1$. □

2 Upper bound

For any fixed $d > 2$, the results of Bender and Canfield [1] imply that the number of simple d-regular graphs on n vertices (with dn even) is $\Omega\left(\frac{(nd/2)!}{(d!)^n}\right)$. It follows that the class of 3-regular graphs (whose expansion is bounded by $f(n) = 3 \cdot 2^{n-1}$) is not small. We can improve this observation slightly in the following way: for a non-decreasing positive function $g : N \to N$, let $\mathcal{G}_g$ be the class of graphs such that $G \in \mathcal{G}_g$ if and only if there exists a 4-regular[3] graph H such that G is obtained from H by subdividing each edge of H by $g(|V(H)|)$ vertices. Let $N(g,n)$ be the number

[3]We could also use 3-regular graphs in the same construction, but the obtained bound would be similar and by using 4-regular graphs, we avoid the need to require that the number of vertices is even.

of graphs in $\mathcal{G}_g$ with n vertices, and $N_4(n)$ the number of 4-regular graphs with n vertices, $N_4(n) = \Omega\left(\frac{(2n)!}{24^n}\right) = \Omega\left(\frac{(n!)^2}{7^n}\right)$. If $n = k(1 + 2g(k))$, then $N(g, n) \geq \binom{n}{k} N_4(k)(n-k)!$ —we choose the vertices of a 4-regular graph H, order the remaining $n-k$ vertices arbitrarily, and distribute them to the edges of H according to some canonical ordering of $E(H)$. In particular, $N(g, n) \geq n!N_4(k)/k! = \Omega\left(\frac{n!k!}{7^k}\right)$. If $k \log k = \omega(n)$, this implies that $\mathcal{G}_g$ is not small. We can achieve this by choosing a function $g(x) = o(\log x)$.

Note that for any $t \geq 0$ and a graph H with $2k$ edges, $\nabla_t(H) \leq \sqrt{k}$ for any t—a minor with at most $2\sqrt{k}$ vertices has maximum degree at most $2\sqrt{k}$, while a minor with more than $2\sqrt{k}$ vertices has average degree at most $\frac{2\cdot 2k}{2\sqrt{k}} \leq 2\sqrt{k}$. Consider a graph $G \in \mathcal{G}_g$ with $n = k(1 + 2g(k))$ vertices, obtained from a 4-regular graph H on k vertices. A subgraph of G of radius r corresponds to a subgraph of H of radius at most $\left\lceil \frac{r}{g(k)+1} \right\rceil$, thus $\nabla_r(G) \leq \nabla_{\lceil \frac{r}{g(k)+1} \rceil}(H) \leq \min\left(\sqrt{k}, 6 \cdot 3^{\frac{r}{g(k)+1}}\right)$. Therefore, if h is the function that bounds the expansion of $\mathcal{G}_g$, then $h(r) \leq \sqrt{k_r}$, where k_r is the solution to the equation $6 \cdot 3^{\frac{r}{g(k_r)+1}} = \sqrt{k_r}$.

Theorem 5. *There exists a constant $c > 0$ such that the class of graphs with expansion bounded by the function $f(r) = c \cdot e^{\sqrt{r}\log(r+e)}$ is not small.*

Proof. Let $g(k) = \frac{\log k}{\log \log k} = o(\log k)$. As we observed, $\mathcal{G}_g$ is not small, thus it suffices to show that it has expansion bounded by f.

Consider first the function $f_1 = e^{\sqrt{r}\log(r+e)}$. We show that there exists r_0 such that for all $r > r_0$, $\sqrt{k_r} \leq f_1(r)$, hence $f_1(r) \geq \nabla_r(G)$ for all $r > r_0$ and $G \in \mathcal{G}$: the inequality $\sqrt{k_r} \leq f_1(r)$ is equivalent to $6 \cdot 3^{\frac{r}{g(f_1^2(r))+1}} \leq f_1(r)$. However,

$$6 \cdot 3^{\frac{r}{g(f_1^2(r))+1}} \leq 6 \cdot 3^{\frac{r \log\log f_1^2(r)}{\log f_1^2(r)}} \leq 6 \cdot 3^{\sqrt{r}\frac{\log\log f_1^2(r)}{\log(r+e)}} = 6 \cdot 3^{\sqrt{r}\log(r+e)\frac{\log\log f_1^2(r)}{\log^2(r+e)}}.$$

As $\lim_{r\to\infty} \frac{\log\log f_1^2(r)}{\log^2(r+e)} = 0$, there exists r_0 such that for $r > r_0$, $6 \cdot 3^{\frac{r}{g(f_1^2(r))+1}} \leq f_1(r)$.

Since the maximum degree of graphs in $\mathcal{G}_g$ is 4, $\nabla_r(G) \leq 4 \cdot 3^{r-1}$ for any $G \in \mathcal{G}_g$. It follows that the expansion of $\mathcal{G}_g$ is bounded by $c \cdot e^{\sqrt{r}\log(r+e)}$, where $c = 4 \cdot 3^{r_0}$. □

3 Conclusions

Significantly improving the constant $\log_9 2 \approx 0.315$ would require a new idea; ideally, we would like to decrease the size of the graph about $d = \nabla_0(G)$ times, instead of roughly twice, in each iteration of the recurrence. However, this appears hard to achieve if no or only a few vertices have degree approximately d. Nevertheless, we propose the following conjecture. A class of graphs has *subexponential expansion* if its expansion is bounded by a function f with $\frac{\log \log f(r)}{\log r} = o(1)$.

Conjecture 1. *Any class of graphs with subexponential expansion is small.*

This is motivated by the fact that classes of graphs with subexponential expansion have separators of sublinear size (Nešetřil and Ossona de Mendez [4]), indicating that some kind of structure appears in such classes.

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