Structural description for Kruskal-Friedman ideals of finite trees

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Abstract

In [13], Robertson, Seymour and Thomas have discovered a finite structural description for every Kruskal ideal $\mathcal{I}$ (a set $\mathcal{I}$ of finite trees quasi-ordered and also closed under the well known topological minor embedding $\leq_t$). In this note, we generalize their result to finite trees that are vertex labeled from an arbitrary well-quasi-ordered (wqo) set and edge labeled from an ordinal, assuming the stronger relation known as the Kruskal-Friedman gap-embedding, "$\leq_{KF}$ ". We do not assume the result in [13] and so this serves as a new proof as well. The proof in [13] uses theoretical arguments for existence, whereas here a recursive algorithm that computes the finite structure is given. More importantly, this note lays a background theory that leads to simpler proofs to some of the deep theorems of Nash-Williams in [5], [6], and [7]. A simple proof of [5] as a corollary of this result follows in [9].

1 Introduction

A broad account of why we are interested in structural descriptions can be found in [10]. However, at least two new questions arise naturally from this note: First, why is it interesting that the result in [13] is generalized

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to labeled trees? Next, why do the trees need to be ordered by a stronger relation instead of the usual topological minor relation "$\leq_t$"?

In [13], the finite structure found is a labeled tree. However, the classes for which the structure is found are classes of unlabeled trees. This difference makes us unable to iteratively find the structure of the structures. Unless we have a structure for classes of labeled trees as well, we can not work on iterated sets.

Note that Nash-Williams has proved in [5], [6], [7] classical theorems using "very complicated" arguments, as he admits in his papers. Perhaps, the authors in [13] had in mind the idea of finding simpler proofs of his theorems when working on the subject of structure. Their original work [13] is crucially important as it paved a new research area. What is found in this work (and the forthcoming papers which follow as corollaries to the result of this note) are continuation of that work toward attaining some of the intended goals. For one example, assuming the result in this note, a remarkably simple proof of [5] will be given in [9]. Further, assuming the result of Křiš, [2], we can easily extend the present result to trees with edge labels from an arbitrary ordinal and this in turn can provide alternate proof to results in [6], [7]. For simplicity, we refrain from detailed discussion of this subject in this note.

Another example showing why labeled trees are more interesting can be related to the graph minors theory of Robertson and Seymour, [12]. The tree-decomposition of graphs in [12] is a decomposition in to labeled trees, where the vertex labels are known as "bags", the edges are labeled from a finite ordinal and the order between two tree-decompositions is actually the "Kruskal-Friedman gap-embedding relation", which we denote by $\leq_{KF}$, (We shall define $\leq_{KF}$ shortly). Robertson and Seymour have shown [12] that for two tree-decompositions T and T' of two graphs G and G', respectively, the relation $T \leq_{KF} T'$ implies that G is a minor of G'. A better understanding of the structure of classes of tree-decompositions of graphs can follow from the result of this note. That is, suppose we have a sequence of classes of graphs $\mathcal{K}_1, \mathcal{K}_2, \ldots$. For each i, let $\mathcal{J}_i$ be the class of trees obtained by finding a tree-decomposition of each graph in $\mathcal{K}_i$. Hence, we have a sequence $\mathcal{J}_1, \mathcal{J}_2, ...$ of classes of trees. By the result in this note, we can find a sequence of structure-trees $\psi(\mathcal{J}_1)), \psi(\mathcal{J}_2)), \ldots$. We shall have several things to deduce from this sequence. Particularly, by applying a result in [11], (for equivalent definition of a concept known as "better-quasi-ordering"), this may lead to providing some clue about approaching a conjecture of Thomas (for the case of finite graphs), [14]. As this subject is quite subtle we will not further address this question in this note, but we find it worth mentioning.

Regarding trees, however, there is a clear and concrete reason why a stronger relation is used instead of the classical embedding $\leq_t$. The meaning out of the relation of two structure-trees related by the $\leq_t$ is generally useless. A restricting rule such as the "gap-condition", introduced by Friedman, [1], enables us to force a specific set of finite rules of construction to be used on a fixed class $\mathcal{K}$ of trees and *nowhere else*. This means at a fixed level on our structure-tree. Otherwise, if we allow our finite set of rules to be usable at any level (note that $\leq_t$ allows us to map a vertex to virtually any level) then the problem becomes quite intractable, our rules become meaningless and for any hope of finding a meaning in a $\leq_t$ relation, easily refuting counterexamples can be found etc. This will be clear as we go through this note and the paper that follows this. However, it is worth mention here that, we have a fortunate situation that the "$\leq_{KF}$" relation works as a natural solution to this problem. Of course this is good, because we know that due to Friedman's tree Theorem [1], trees ordered by $\leq_{KF}$ behave nicely.

Except for applying the classical Hall's perfect matching Theorem, and the tree theorem [1], the paper is self-contained. The reader need not be familiar with previous results. We start from basic definitions.

A relation $\leq$, in a set Q is a *quasi-order* if it is reflexive and transitive. We say Q is *well-founded* if Q contains no infinite strictly descending sequence $q_1 > q_2 > \cdots$. Let Q be a set quasi-ordered by $\leq$. Then Q is well-quasi-ordered (wqo) if it is well-founded and contains no infinite antichain. Equivalently, Q is wqo if and only if every infinite sequence in Q is "good", (A sequence $q_1, q_2, \dots$ of Q is *good* if there exist indexes i, j such that $i < j$ and $q_i \leq q_j$). A *lower set* of Q (or an *ideal*, for short) is a subset I of Q which is closed down under $\leq$. That is, for all $y \in I$ and all $x \in Q$, if $x \leq y$ then $x \in I$. The class LQ of ideals of Q is quasi-ordered by the subset relation. Let $I \subseteq Q$ be an ideal. The *obstruction set* of I, denoted by $O(I)$, is the set of all minimal elements of $Q - I$. It is easy to see the following:

Theorem 1 *Q is wqo if and only if LQ is well-founded if and only if $O(I)$ is finite for every ideal $I \in LQ$.*

Trees in this paper are finite, rooted and directed away from the root. We assume the vertex set $V(T)$ is labeled from a set Q, that is quasi-ordered by $\leq_Q$ and the edge set $E(T)$ of a tree T is labeled from a finite ordinal, say $[m] = \{0, 1, \dots, m\}$. (Our result can easily be generalized to trees with edge labels from any ordinal, but for simplicity finiteness suffices in this note). The root of T, denoted by $root(T)$, is a special vertex that has a *root-edge label* from $[m]$, in addition to its vertex label from Q. Hence, we assume every vertex $v \in V(T)$ has a pair of labels, $l(v) = (p, q)$, where $p \in [m], q \in Q$, and refer to p as $l_1(v)$ and to q as $l_2(v)$. If v is not a root, then we can treat $l_1(v)$ as labeling the unique edge $e = uv$ and will sometimes refer to it as the *edge-label of e*. Where no misunderstanding can occur we will write $l(e)$ instead of $l_1(v)$ for emphasis and clarity. Given trees T, T', we say that T is a *topological minor of* T' (and write $T \leq_t T'$) if a subdivision of T is isomorphic (without the labels) to a subtree T'' of T', such that $l_2(v) \leq_Q l_2(v'')$ for each $v \in V(T)$, where v'' is the isomorphism corresponding vertex in $V(T'')$. Note that the relation $T \leq_t T'$ is an injective mapping $f : V(T) \to V(T')$ and also that it ignores the edge label $l_1(v)$.

However, if $T \leq_t T'$, then f maps every edge e of T to a unique path $f(e)$ in T'. The map f is said to satisfy the *Kruskal-Friedman gap-condition* (written $T \leq_{KF} T'$) if in addition $l(e) \leq l(e')$ for every edge e' in $f(e)$. A stronger version of $\leq_{KF}$ has one more condition that $l_1(root(T)) \leq l_1(v')$ for every $v' \in V(T')$ that is on the path from $root(T')$ to $f(root(T))$. This stronger version is what we use throughout this paper.

We denote the set of all trees with vertex labels from a set Q by $\mathcal{F}(Q)$. Kruskal [3] proved in 1960 that $\mathcal{F}(Q)$ is wqo by $\leq_t$. Nash-Williams gave in [2] a short and elegant proof of this theorem.

Theorem 2 *(***J. Kruskal [3]***) If Q is wqo, then $\mathcal{F}(Q)$ is wqo by $\leq_t$.*

If the trees in $\mathcal{F}(Q)$ also have edge labels from $[m]$, then we denote the class of all trees by $\mathcal{F}(m, Q)$. Friedman [1] strengthened Kruskal's tree theorem, by using $\leq_{KF}$.

Theorem 3 *(***H. Friedman, [1]***) If Q wqo, then $\mathcal{F}(m, Q)$ is wqo by $\leq_{KF}$.*

The theorem in [13] is the following:

Theorem 4 *(***Robertson, Seymour, Thomas,[13]***) For every ideal $\mathcal{I} \subseteq \mathcal{F}$ of finite unlabeled trees, there is a labeled finite tree $\psi(\mathcal{I})$ that constructs precisely the elements of $\mathcal{I}$.*

The main result of this paper is the following:

Theorem 5 *For every ideal $\mathcal{I} \subseteq \mathcal{F}(m, Q), m \geq 0$ and every wqo Q, there is a finite tree $\psi(\mathcal{I})$ in $\mathcal{F}(2m + 4, Q')$ that constructs precisely the elements of $\mathcal{I}$, where Q' is a wqo set.*

In Section 2 we define a concept known as 'structure-tree', $\psi(\mathcal{I})$, of an ideal $\mathcal{I}$. In Section 3 we prove the existence and finiteness of $\psi(\mathcal{I})$ by computing it from $O(\mathcal{I})$. We conclude with discussion on future directions of this research.

2 Proper spanning sets and Structure-trees

We introduce the notion of a 'structure-tree', $\psi(\mathcal{I})$, for every ideal $\mathcal{I} \in L\mathcal{F}(m,Q)$. The idea roughly is as follows: $\psi(\mathcal{I})$ is a finitely branching tree, because using a particular finite set called "a proper spanning set", $\mathcal{I}$ is expressed in terms of a finite number of its proper subideals, $I_1, I_2, ...I_n$ and finite labels from some wqo label sets. Then each I_i is recursively expressed in the same manner. This finitely branching process is what gives us the structure-tree $\psi(\mathcal{I})$. The leaves of a structure-tree $\psi(\mathcal{I})$ correspond to the empty ideal $\emptyset$. Naturally, we terminate the recursion of finding proper subideals when we reach at the empty ideal $\emptyset$.

A beautiful part of this process (which we adopt from [13]) is proving that $\psi(\mathcal{I})$ is not only finitely branching but also finite. We introduce the following axiom on trees of $\mathcal{F}(m,Q)$: (For any wqo set Q, we may assume that there is a unique minimal element q_0, such that $q_0 \leq q$, for any $q \in Q$.)

Definition 6 (T1) *For any tree $T \in \mathcal{F}(m,Q)$ if L is a leaf of T but not its root, then $L = (0,q_0)$ (i.e. $L \leq T'$ for all $T' \in \mathcal{F}(m,Q)$). (Leaf Axiom)*

Note that (T1) can be achieved by gluing such a leaf to each leaf of each tree in $\mathcal{F}(m,Q)$, and by doing so no relation under $\leq_{KF}$ is lost. We emphasize that this assumption is analogous to our definition of a structure-tree $\psi(\mathcal{I})$ of an ideal $\mathcal{I}$, where the leaves of $\psi(\mathcal{I})$ correspond to the empty ideal $\emptyset$. Just as $\emptyset$ is a subset of any ideal, so do we assume that any leaf L is smaller than any tree. A detailed outline of how we obtain structure-trees of ideals follows:

First, we define a basic operation of constructing a new tree by "tree-summing" a finite number of trees in $\mathcal{F}(m,Q)$. The *null-tree*, Γ, is defined by $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, Γ is not a rooted tree. Let $T_1, T_2, \ldots, T_n, n \geq 0$, be pairwise vertex disjoint trees or Γ. For $p \in [m]$ and $q \in Q$ the *tree-sum* of $T_1, T_2, \ldots, T_n$ (each T_i is called a *summand*) is given by

$$T = Tree(T_1, T_2, \ldots, T_n)_{(p,q)},$$

where for all i the labels of T_i is preserved, t_0 is a new vertex (the root of T) labeled by (p, q), $V(T) = V(T_1) \cup V(T_2) \cup ... \cup V(T_n) \cup \{t_0\}$, and its set of edges is $E(T_1) \cup E(T_2) \cup ... \cup E(T_n) \cup \{e_i = (t_0, root(T_i)) : 1 \leq i \leq n, T_i \neq \Gamma\}$. (note that $Tree(T_1, T_2, \Gamma)_{(p,q)}$ is isomorphic to $Tree(T_1, T_2)_{(p,q)}$ but allowing the null-tree to appear in a tree-sum will be convenient later). Note also that each new edge e_i has the label of the root-edge label of T_i. Next, we define an "rst-cell" referring to the authors of [13] who invented "bits". The following is a slight modification of a bit:

$$H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0}$$

is an *rst-cell* where the parameters in H are defined as follows: $k \in \mathbb{N} \cup \{0, \infty\}$ is the *width* of H and is denoted by $k(H)$; we allow $k(H) = \infty$ only if $n = 0$ and $\mathcal{I}_\infty = \emptyset$; $\mathcal{I}_i$ for $i = \infty$ or $i = 0, 1, \ldots, n$ is an ideal of $\mathcal{F}(m, Q)$. If $1 \leq i \leq n$, then $\mathcal{I}_i \neq \emptyset$. $\mathcal{I}_0$ is the *root-ideal* of H such that every tree in $\mathcal{I}_0$ has height zero.

If $0 < i < \infty$, we say $\mathcal{I}_i$ is a *middle-component* of H. We say $\mathcal{I}_\infty$ is the *right-component* of H, and $\mathcal{I}_0$ is the *root-component* of H. Two rst-cells H and H' are assumed to be *equal* if they differ only by a permutation of their middle-components.

Let $\mathcal{H}$ be a set of rst-cells and $H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$. An ideal $\mathcal{I}$ is said to satisfy condition

(*) if $Tree(T_1, \ldots, T_k, T_{k+1}, \ldots, T_{k+n}, T_{k+n+1}, \ldots, T_{k+n+s})_{(p,q)}$ belongs to $\mathcal{I}$, where $s \geq 0$, $(p, q) \in \mathcal{I}_0$ is called the **root-part** ; the **left-part** for $a = 1, \ldots, k, T_a \in \mathcal{I} \cup \{\Gamma\}$; the **middle-part** for $a = k+1, \ldots, k+n$, $T_a \in \mathcal{I}_a \cup \{\Gamma\}$; **right-part** for $a = k+n+1, \ldots, s$, $T_a \in \mathcal{I}_\infty$.

We define the ideal $I(\mathcal{H})$ to be the intersection of all ideals satisfying (*). Let $\mathcal{I}$ be an ideal and let $\alpha(\mathcal{I}) \in [m]$ denote the maximum over all root-edge labels of trees in $\mathcal{I}$. Let T be a tree, $\mathcal{H}$ be a set of rst-cells and $H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$. Then, we say that T *conforms to* H *in* $I(\mathcal{H})$ if either $T \in \mathcal{I}_i, 0 \leq i \leq \infty$ such that $l_1(root(T)) \leq \alpha(\mathcal{I}_0)$ or T is a tree-sum as in (*) where the left-part summands are from $I(\mathcal{H}) \cup \{\Gamma\}$. The following is obvious by definition and by the fact that $I(\mathcal{H})$ is an ideal:

Theorem 7 *Let $\mathcal{H}$ be a set of rst-cells. Let T be a tree. Then, $T \in I(\mathcal{H})$ if and only if T conforms to H in $I(\mathcal{H})$ for some $H \in \mathcal{H}$.*

Let H be a width zero rst-cell. If $\mathcal{I}'$ is a component of H and if $I(\{H\}) \subseteq \mathcal{I}'$, then we assume H is of the form $(0; \mathcal{I}'; \emptyset)_{\mathcal{I}_0}$. Further, if $\alpha(\mathcal{I}_0) < \alpha(\mathcal{I}')$, then we call H a *trimmer-cell.* and denote it by a unique symbol H^*. Note that if $H^* = (0;\mathcal{I}';\emptyset)_{\mathcal{I}_0}$ is a trimmer-cell then $\mathcal{I}' \not\subseteq I(\{H^*\}) \subset \mathcal{I}'$. For example $H^* = (0;\mathcal{F}(m,Q);\emptyset)_{(0,Q)}$ is a trimmer-cell if $m > 0$, since $I(\{H^*\})$ consists of trees with root-edge label only zero. A set of rst-cells $\mathcal{H}$ need not have a trimmer-cell. So if $\mathcal{H}$ has no trimmer-cell we assume H^* is null. For convenience, if H^* is null then we assume $\{H^*, H_1, H_2 \ldots, H_t\} = \{H_1, H_2 \ldots, H_t\}$.

Definition 8 *Let $\mathcal{I}$ be an ideal such that $\alpha(\mathcal{I}) = m$. Then, a set of cells $\mathcal{H} = \{H^*, H_1, H_2 \ldots, H_t\}$ with at most one trimmer-cell H^* is a proper spanning set of $\mathcal{I}$ if the following three properties hold:*

(P1) *$I(\{H^*\}) \subset \mathcal{I}$ and if $\mathcal{I}'$ is a component of H in $\mathcal{H}$-$\{H^*\}$, then $\mathcal{I}' \subset \mathcal{I}$. (induction axiom)*
(P2) *$I(\mathcal{H}) = \mathcal{I}$. (spanning axiom)*
(P3) *for any component $\mathcal{I}'$ that is not a root-component of H^*, $\alpha(\mathcal{I}') = m$. (chain axiom)*

Remark. Note that (P3) allows us to deduce that if a tree T is in a component of a cell $H \in \mathcal{H} - \{H^*\}$, then it conforms to H in $I(\mathcal{H})$ regardless the label $l_1(root(T))$. Note also that a description of an ideal without induction, (P1), or without exact construction, (P2), is useless. However, (P3) is also critical because it allows us to find proper spans of components recursively, and construct a subset chain of ideals. The only component that breaks the subset chain is the one in a trimmer-cell. We offer a few simple examples of proper span of ideals below.

Example 9 *The set of all trees $\mathcal{F}(m,Q)$ has a spanning set*

$$\mathcal{H} = \{(\infty;;\emptyset)_{([m],Q)}\}.$$

For short, we use $\mathcal{F}$ for $\mathcal{F}(m,Q)$, when m and Q are obvious from the context. For any quasi-ordered set Q, and an ideal $\mathcal{Q} \subseteq Q$, let $\mathcal{Q}/\{q_1,\ldots,q_n\} = \{q \in \mathcal{Q} | q \not\geq q_i \text{ for all } i = 1,\ldots n\}$. If $n = 1$, we simply write $\mathcal{Q}/q_1$ for short. Recall that we denote the vertex tree by its label as (p,q) where $p \in [m]$ and $q \in Q$.

Example 10 *Let $\mathcal{I} \in L\mathcal{F}(m, \mathbb{N} \cup \{0\})$ such that $\mathcal{I} = \mathcal{F}/(1,4)$ Then, a spanning set of $\mathcal{I}$ is given by $\mathcal{H} = \{(0;\mathcal{F};\emptyset)_{(0,Q)} = H^*, (\infty;;\emptyset)_{([m],[3])}\}$. As an additional example, let $\mathcal{I}' = \mathcal{F}/T$ where $T = Tree((0,0),(0,0))_{(1,4)}$. Then, $\mathcal{H}' = \{(1;;\emptyset)_{([m],Q)}\} \cup \mathcal{H}$, is easily checked to be a spanning set of $\mathcal{I}'$.*

3 Existence

In this section we define and establish the existence of a finite structure-tree $\psi(\mathcal{I})$ for every ideal $\mathcal{I}$, via proof of existence of a proper spanning sets for every ideal. We compute a proper spanning set $\mathcal{H}(\mathcal{I})$ from the obstruction set $O(\mathcal{I})$. By Theorem1, we know that $|O(\mathcal{I})| < \infty$. Therefore, we use induction on $|O(\mathcal{I})|$ to compute $\mathcal{H}(\mathcal{I})$. We prove the case $|O(\mathcal{I})| = 1$ first. See Example 10, for the special case where $|O(\mathcal{I})| = 0$. We assume $\alpha(\mathcal{I}) = m$ when finding a proper span, and so we exclude the case of forbidding (p, q_0) in the next two theorems. We also exclude the case of forbidding $T_0 = Tree((0, q_0))_{(0,q_0)}$, because every ideal $\mathcal{I}$ forbidding T_0 consists of height zero trees and so is the form $([p], \mathcal{Q})$. Ideals of height zero trees actually appear as labels on the vertices of our structure-tree which we shall define shortly. Since we assume both $[m]$ and Q are wqo, the pairs $([p], \mathcal{Q})$ are also wqo, and hence we need not find proper description for them. They behave nicely.

If $H^* = (0; \mathcal{I}'; \emptyset)_{(p,\mathcal{Q})}$ and if $p < 0$, we assume H^* is *null.* Recall that if H^* is null then we assume $\{H^*, H_1, H_2 \ldots, H_t\} = \{H_1, H_2 \ldots, H_t\}$.

Theorem 11 *Let* $T = Tree(T_1, T_2, \ldots, T_n)_{(p,q)} \in \mathcal{F}(m, Q)$ *, and* $\mathcal{I} = \mathcal{F}/\{T\}$ *such that* $T \neq (p, q_0)$ *and* $T \neq Tree((0, q_0))_{(0,q_0)}$. *Then,* $\mathcal{H} = \mathcal{H}_{trim} \cup \mathcal{H}_\infty \cup \mathcal{H}_f$*, is a proper span of* $\mathcal{I}$*, where*

- $\mathcal{H}_{trim} = \{H^*\} = \{(0; \mathcal{F}; \emptyset)_{(p-1,Q)}\}$
- $\mathcal{H}_\infty = \{(\infty; ; \emptyset)_{(m,Q/q)}\}$
- $\mathcal{H}_f = \{(k; ; \mathcal{I}/S))_{([m],Q)}$: $0 \leq k \leq n-1$*, for every* $(k+1)$*-tuple* $S \subseteq \{T_1, T_2, \ldots, T_n\}$

Proof. By assumption we have $(p, q_0) \in \mathcal{I} - I(\{H^*\})$. Since $\alpha(I(\{H^*\})) = p - 1$, no tree in $I(\{H^*\})$ contains T, and so $\mathcal{H}_{trim}$ satisfies the induction axiom (P1). Next, observe that the root-component in $\mathcal{H}_\infty$ trivially satisfies (P1). Last, if T is of height zero, then note that $\mathcal{H}_f = \emptyset$. Otherwise, the root ideals consist of height zero trees and so satisfy (P1) because $Tree((0, q_0))_{(0,q_0)} \in \mathcal{I}$. Also, since $\emptyset \neq S \subseteq \mathcal{I}$ implies $\mathcal{I} \not\subseteq \mathcal{I}/S \subseteq \mathcal{I}$, we deduce that each rst-cell in $\mathcal{H}_f$ satisfies (P1).

Next, we show that (P2) holds. That is, $I(\mathcal{H}) = \mathcal{I}$. Suppose for a contradiction, there is a counterexample tree $T' = (T'_1, \ldots, T'_{n'})_{(p',q')}$ of minimal height h, $h \geq 0$, that exists in one but not in the other side of the equality.

We claim first that $(p,q) \leq (p',q')$, that is $p \leq p'$ and $q \leq q'$. This follows, because first if $T' \notin \mathcal{I}$ then $T \leq_{KF} T'$ and by the choice of h, $root(T)$ is mapped to $root(T')$, hence $(p,q) \leq (p',q')$. Next, if $T' \notin I(\mathcal{H})$, then unless $(p,q) \leq (p',q')$, by induction on h we have T' conforms to H^* or to $(\infty;;\emptyset)_{(m,Q/q)}$ in $I(\mathcal{H})$, a contradiction. The claim follows.

Now that we know the root of T can be mapped to the root of T' (and obviously we may assume $T' \notin \mathcal{I}/S$, for any $S \subseteq \{T_1,\ldots,T_n\}$), we see that $T' \in \mathcal{I}$ (that is, $T \not\leq_{KF} T'$) if and only if there is no perfect matching from $X = \{T_1,T_2,\ldots,T_n\}$ to $X' = \{T'_1,T'_2,\ldots,T'_{n'}\}$, where we can form a bipartite graph with edges from T_i to T'_j, if $T_i \leq_{KF} T'_j, 1 \leq i \leq n$, and $1 \leq j \leq n'$. By Hall's matching theorem, this occurs if and only if there is a $(k+1)$-subset S of $X, 0 \leq k \leq n-1$, such that the set S' of all trees containing some tree from S has size at most k. That is, by induction on h, if and only if T' conforms to $(k;;\mathcal{I}/S)_{([m],Q)}$ in $I(\mathcal{H})$, a contradiction and (P2) follows.

Trivially, (P3) holds for $\mathcal{H}_{trim} \cup \mathcal{H}_\infty$ and for all root-components. To see that (P3) holds for $H \in \mathcal{H}_f$, let $\mathcal{I}' = \mathcal{I}/S$. If $\alpha(\mathcal{I}') = p < m$, then S contains a height zero tree $T'' = (p+1,q_0)$, because $\mathcal{I}'$ is an ideal and $(p+1,q_0)$ is minimally not in $\mathcal{I}'$. But then T'' must be a leaf of T, since S consists of tree summands of T. Then by our tree axiom (T1), we have $T'' = (0,q_0)$ and so $\mathcal{I}' = \emptyset$. Hence, for every ideal forbidding at most one tree, we have a proper span with at most one trimmer-cell. $\square$

We may inductively assume that we have already computed a proper span $\mathcal{H}$ of an ideal $\mathcal{I} = \mathcal{F}(m,Q)/\{T_1,T_2,\ldots,T_r\}$ which has r obstructions, $r \geq 1$. Let $\mathcal{H}'$ be a proper span of $\mathcal{F}(m,Q)/\{T_{r+1}\}$ and let

$$\mathcal{I}'' = \mathcal{F}(m,Q)/\{T_1,\ldots,T_{r+1}\}.$$

Then $\mathcal{I}'' = I(\mathcal{H}) \cap I(\mathcal{H}') = \mathcal{I} \cap \mathcal{I}'$. We want to find $\mathcal{H}''$ that properly spans $\mathcal{I}''$.

Assume $l_1(root(T_i)) = p_i$ and that $p_1 \leq p_2 \ldots \leq p_r \leq p_{r+1}$. Inductively, assume $\mathcal{H}$ has at most one trimmer-cell

$$H^* = (0;\mathcal{F}/\{T_1,T_2,\ldots,T_{r-1}\};\emptyset)_{(p_r-1,Q)}.$$

Also recall that by Theorem 11, we have $\mathcal{H}'_{trim} = \{(0;\mathcal{F};\emptyset)_{(p_{r+1}-1,Q)}\}$.

As a notational ease, we use an integer coefficient $c \geq 1$ for a middle component $\mathcal{I}'$ of an rst-cell H as $c\mathcal{I}'$, which simply means that $\mathcal{I}'$ is repeated c times as a middle-component of H (see for instance, the first two middle components in Equation (4) below). The next definition outlines how we find the rst-cells of $\mathcal{H}''$ from $\mathcal{H}$ and $\mathcal{H}'$.

Definition 12 *Let $\mathcal{H}$ be a proper description of an ideal $\mathcal{I}$ with r obstructions, $T_1, \ldots, T_r, r \geq 1$. Let $T_{r+1} \in \mathcal{F}(m, Q)$ and let $\mathcal{H}'$ be a proper description of $\mathcal{I}' = \mathcal{F}(m, Q)/T_{r+1}$ given by Theorem 11. For every pair of rst-cells $(H, H') \in (\mathcal{H} - \{H^*\} \times \mathcal{H}' - \{H'^*\})$, where $H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0}$ and $H' = (k';; \mathcal{I}'_\infty)_{\mathcal{I}'_0}$, let $\mathcal{I}''_0 = \mathcal{I}_0 \cap \mathcal{I}'_0$, $\mathcal{I}''_\infty = \mathcal{I}_\infty \cap \mathcal{I}'_\infty$. The set $H \odot H'$, called the intertwine of H and H', is found as follows:*

- *If $k = k' = \infty$, let $H \odot H' = \{H''\}$, where*

$$H'' = (\infty;;\emptyset)_{\mathcal{I}''_0} \tag{1}$$

- *If $k = \infty$ and $k' < \infty$, let $H \odot H' = \{H''\}$, where*

$$H'' = (k';; \mathcal{I}'_\infty \cap \mathcal{I})_{\mathcal{I}''_0} \tag{2}$$

- *if $k' = \infty$ and $k < \infty$, let $H \odot H' = \{H''\}$, where*

$$H'' = (k; \mathcal{I}_1 \cap \mathcal{I}', \mathcal{I}_2 \cap \mathcal{I}', \ldots, \mathcal{I}_n \cap \mathcal{I}'; \mathcal{I}_\infty \cap \mathcal{I}')_{\mathcal{I}''_0} \tag{3}$$

- *finally, if both k and k' are finite let $k'' = \min(k, k')$ and $0 \leq j \leq k''$. For $0 \leq b \leq \min(n, k' - j)$, let $c = k - j, d = k' - j - b$;Let $H \odot H' = \{H''_{j,b} | j = 0, 1, \ldots k''$ and $b = 1, \ldots \min(n, k' - j)\}$, where $H_{j,b}$ (for every permutation of middle-components) is given by the following:*

$$(j; c(\mathcal{I} \cap \mathcal{I}'_\infty), \mathcal{I}_1 \cap \mathcal{I}', \ldots, \mathcal{I}_b \cap \mathcal{I}', d(\mathcal{I}_\infty \cap \mathcal{I}'), \mathcal{I}_{b+1} \cap \mathcal{I}'_\infty, \ldots, \mathcal{I}_n \cap \mathcal{I}'_\infty; \mathcal{I}''_\infty)_{\mathcal{I}''_0} \tag{4}$$

  An intertwine $\mathcal{H} \odot \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is a set of rst-cells obtained by taking the union of $H \odot H'$ for every pair $(H, H') \in (\mathcal{H} - \{H^*\}) \times (\mathcal{H}' - \{H'^*\})$.

Theorem 13 *For $r \geq 1$ and an ideal $\mathcal{I}''$ with $\alpha(\mathcal{I}'')$=m, let $O(\mathcal{I}'') = \{T_i | i = 1, 2, \ldots, r+1\}$. Let $\mathcal{H}$ and $\mathcal{H}'$ be proper spanning sets of $\bigcap_{l=1}^{r} \mathcal{F}/T_l = \mathcal{I}$ and $\mathcal{F}/T_{r+1} = \mathcal{I}'$, respectively. Let $\mathcal{H} \odot \mathcal{H}'$ as in Definition 12. Let $\mathcal{H}''_{trim} = \{H''^*\} = \{(0; \mathcal{I}; \emptyset)_{(p_{r+1}-1, Q}\}$. Then, $\mathcal{I}'' = \mathcal{I} \cap \mathcal{I}' = I(\mathcal{H}) \cap I(\mathcal{H}') = I(\mathcal{H} \odot \mathcal{H}' \cup \mathcal{H}''_{trim})$. Moreover, $\mathcal{H} \odot \mathcal{H}'$ can be refined to obtain a proper spanning set $\mathcal{H}''$ of $\mathcal{I}''$.*

Proof. By assumption we have $(p_{r+1}, q_0) \in \mathcal{I}'' - I(\{H''^*\})$. Also, no tree in $I(\{H''^*\})$ contains any of the $r+1$ obstructions, because the trees in $\mathcal{I}$ do not contain the first r obstructions and in $I(\{H''^*\})$ they all have root-edge label less that p_{r+1}. We deduce that $I(\{H''^*\}) \subset \mathcal{I}''$ and so $\mathcal{H}_{trim}$ satisfies the induction axiom (P1). Next, we refine $\mathcal{H} \odot \mathcal{H}'$ and obtain $\mathcal{H}''$ that satisfies the induction axiom (P1) as follows: Let $H'' \in \mathcal{H} \odot \mathcal{H}'$, so that $H'' \in H \odot H'$, for some $H \in \mathcal{H} - \{H^*\}$ and $H' \in \mathcal{H}' - \{H'^*\}$. By induction on r, H and H' satisfy (P1) in $\mathcal{H}$ and $\mathcal{H}'$, respectively. From Equations (1) - (4), we see that for every component $\mathcal{J}$ of $\mathcal{H} \odot \mathcal{H}'$, we have $\mathcal{J} \subseteq \mathcal{I}''$. However, it is possible to have a case $\mathcal{J} = \mathcal{I}''$, as a middle or right component.

Suppose first $\mathcal{I}''$ appears c times $c \geq 1$ as a middle component. Then we delete all c copies from the cell and increase the width of H'' by c. If it is the right component of H'', then we delete $\mathcal{J}$ to obtain a cell H''_1 of fewer components and we add a new cell H''_2 of width ∞ having the same root-ideal as H''. In this manner, we obtain (P1) for $\mathcal{H}''$. Note that the trees that conform to cells in $I(\mathcal{H}'')$ and to cells in $I(\mathcal{H} \odot \mathcal{H}' \cup \mathcal{H}''_{trim})$ are exactly the same, since the only difference between the two sets is that the former may have a component $\mathcal{J} = \mathcal{I}''$, whereas the later has larger width instead of $\mathcal{I}''$. Both cells (at the component $\mathcal{I}''$ or at the increased width) admit arbitrary trees from the ideal as summands with the same multiplicity, (whereas their remaining components are identical) and hence are equivalent. Next, we show that (P2) also holds for $\mathcal{H}''$.

To show that $I(\mathcal{H}'') \subseteq \mathcal{I}(\mathcal{H}) \cap \mathcal{I}(\mathcal{H}')$ is almost obvious: That is, a counterexample tree T of minimal height $h, h \geq 1$ conforms to H'' in $I(\mathcal{H}'')$. If $H'' = H''^*$, then $T \in I(\{H^*\}) \subset \mathcal{I}'' = \mathcal{I} \cap \mathcal{I}'$, a contradiction. Let H'' is as in Equation (1), (2), (3) or (4). By rearranging the components of H'' (if necessary), we see that H'' is only a weaker form of H (and also of H'), because some of the components of H (similarly of H') are intersected by $\mathcal{I}'$ (similarly by $\mathcal{I}$). So, we see that T conforms independently to each of the intertwined cells H and H' in $I(\mathcal{H})$ and $I(\mathcal{H}')$ respectively, a contradiction.

Conversely, let $T \in I(\mathcal{H}) \cap I(\mathcal{H}') - I(\mathcal{H}'')$ be a counterexample tree of minimal height $h, h \geq 1$. Then T conforms to H in $I(\mathcal{H})$ and to H' in $I(\mathcal{H}')$. Contrary to assumption, we show T conforms to H'' in $I(\mathcal{H}'')$, for some $H'' \in H \odot H' \cup \mathcal{H}''_{trim}$. If either H or H' is a trimmer-cell, then (since $p_r \leq p_{r+1}$) we have $I(\mathcal{H}_{trim}) \cap \mathcal{I}' \subseteq I(\mathcal{H}''_{trim})$ and $I(\mathcal{H}'_{trim}) \cap \mathcal{I} \subseteq I(\mathcal{H}''_{trim})$. Hence, $T \in I(\mathcal{H}''_{trim}) \subset I(\mathcal{H}'')$, a contradiction. Assume, $(H, H') \in (\mathcal{H} - \{H^*\} \times \mathcal{H}' - \{H'^*\})$. Now, if T is in a component of H' (or of H), then T is in a component of either Equation (2) or Equation (3) or Equation (4), a contradiction. Hence, T is a tree-sum as shown in (*) for both $I(\mathcal{H})$ and $I(\mathcal{H}')$. Now, if either $k(H)$ or $k(H')$ is ∞, then by the choice of h, we see that T conforms to a cell of type Equation (1) or (2) or (3) in $I(\mathcal{H}'')$. Hence, assume that $k(H) < \infty$ and $k(H') < \infty$.

Since T conforms to H in $I(\mathcal{H})$, we first write T as a tree-sum of three types of summands L, M, R, (for left, middle and right summands) as in (*). Recall that T is also conforms to H' in $I(\mathcal{H}')$ (which has only two: left and right-summands), and so we partition $L = L_{L'} \cup L_{R'}$, $M = M_{L'} \cup M_{R'}$, and $R = R_{L'} \cup R_{R'}$, such that $L_{R'}, M_{R'}, R_{R'}$ are as large as possible subject to the condition $L_{R'} \cup M_{R'} \cup R_{R'} \subseteq \mathcal{I}'_\infty$. Then $L_{L'} \cup M_{L'} \cup R_{L'} \subseteq I(\mathcal{H}') - \mathcal{I}'_\infty$.

In Equation (4), observe the following one-to-one correspondence of components, from left to right: $0 \leq |L_{L'}| = j$, $|L_{R'}| = k - j = c$, $|M_{L'}| = b$, $0 \leq b \leq \min(n, k' - j)$, (note that $b \leq k' - j$, for otherwise we have $|M_{L'}| + j > k'$ summands of $T \notin \mathcal{I}'_\infty$, contradicting that T conforms to H' in $I(\mathcal{H}')$), $|R_{L'}| \leq k' - j - b = d$, and $|M_{R'}| = n - b$, for $i = b+1, \ldots, n$. The remaining summands in $R_{R'}$ are from $\mathcal{I}_\infty \cap \mathcal{I}'_\infty = \mathcal{I}''_\infty$ of H''_j and we do not care how many summands should be in $\mathcal{I}''_\infty$. By induction on h, $L_{L'} \subseteq I(\mathcal{H}'')$, while $|L_{L'}| = j = k(H''_j)$ and so we deduce that T conforms to $H''_{j,b}$ in $I(\mathcal{H}'')$, contrary to the choice of T. This proves that the spanning axiom (P2) is satisfied by $\mathcal{H}''$.

Since $\alpha(\mathcal{J}_1 \cap \mathcal{J}_2) = m$, if $\alpha(\mathcal{J}_i) = m, i = 1, 2$, the chain axiom (P3) is attained easily by induction on r. $\square$

The remaining task in this section is to use proper spans of ideals iteratively and from this process obtain a finitely branching structure-tree $\psi(\mathcal{I})$ which shall not be infinite. A middle or a right-component $\mathcal{I}'$ in a proper span $\mathcal{H}$ of $\mathcal{I}$ is a *descendant of* $\mathcal{I}$. Recursively, any descendant $\mathcal{I}''$ of $\mathcal{I}'$ is also a descendant of $\mathcal{I}$. Call a sequence $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ a *descendant-sequence*, if for each $i \geq 1$, $\mathcal{I}_i$ is a middle or a right-component ideal in a proper span of $\mathcal{I}_{i-1}$. Let $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ be a descendant-sequence. For some $i \geq 1$, if $\mathcal{I}_{i+1}$ is a component of a trimmer-cell and $\mathcal{I}_{i+1} = \mathcal{I}_j$, for some $j < i$, then we say $\mathcal{I}_i$ is an *ancestor-dependent*. The difference $\delta = |i - j|$ is the *distance* of $\mathcal{I}_i$ from its ancestor $\mathcal{I}_j$. If $\mathcal{I}_j \neq \mathcal{I}_{i+1}$ for all $j, 0 \leq j < i$, then we say $\mathcal{I}_i$ is *ancestor-free*. The following example illustrates ancestor-dependency.

Example 14 *Let P_1 denote the path of height one such that its root is labeled by $(1, 0)$ and its leaf by $(0, 0)$. Let $T = Tree(P_1, P_1)_{(0,0)}$. Let $\mathcal{I} = \mathcal{F}/T$ and $\mathcal{I}' = \mathcal{F}/\{P_1, T\}$. Then a proper span of $\mathcal{I}$ is $\mathcal{H} = \{(1; ; \mathcal{I}')_{([m],Q)}\}$. It is* *also quite easy to see that* $\mathcal{I}'$ *is properly spanned by* $\mathcal{H}' = \{(0;;\emptyset)_{([m],Q)}, H^* = (0;\mathcal{I};\emptyset)_{(0,Q)}\}$.

In Example 14, although by repeatedly using (P1) we get $\mathcal{I} \supset \mathcal{I}' \supset I(\{H^*\})$, we have a descendant-sequence $(\mathcal{I}, \mathcal{I}', \mathcal{I})$. So, $\mathcal{I}'$ is ancestor-dependent. If we try to give a proper span for $\mathcal{I}$ again we end up with a circular definition. Our challenge is to encode the structure-tree which avoids circular definition without losing any essential information. A circular definition is obviously undesirable since it gives us an infinite tree. Accordingly, we define our structure-tree $\psi(\mathcal{I})$ for every ideal $\mathcal{I}$: Note that if $\alpha(\mathcal{I}) = p < m$, then $\mathcal{I}$ forbids the tree $(p+1, q_0)$. Hence there is a unique ideal $\mathcal{I}'$ that forbids all obstructions of $\mathcal{I}$ except $(p+1, q_0)$, such that $\alpha(\mathcal{I}') = m$. We use the proper span of $\mathcal{H}(\mathcal{I}')$ but differentiate between $\mathcal{I}$ and $\mathcal{I}'$ by their structure-tree root-edge label as shown below. Hence, regardless $\alpha(\mathcal{I})$ we may assume $\mathcal{H}(\mathcal{I})$ exists:

Definition 15 (Structure-tree) *Let* $\mathcal{I}$ *be an ideal in* $\mathcal{F}(m, Q)$, *where* $m \geq 0$ *is an integer and* Q *is a wqo set. Let* $\mathcal{H}(\mathcal{I}) = \{H^*, H_1, H_2, \ldots, H_t\}$ *be a proper span of* $\mathcal{I}$, *where* H^* *is the trimmer-cell of* $\mathcal{H}$ *or null. If* $\mathcal{I} = \emptyset$, *then set*

$$\psi(\mathcal{I}) = (0, q_0), \quad \textit{otherwise}, \quad \psi(\mathcal{I}) = Tree(R^*, R_1, R_2, \ldots, R_t)_{(\alpha(\mathcal{I}), \hat{q})}, \tag{5}$$

where $R^*(\mathcal{I})$ *corresponds to* $H^* \in \mathcal{H}(\mathcal{I})$ *and for* $i = 1, 2, \ldots t$, $R_i(\mathcal{I})$ *corresponds to* $H_i = (k_i; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n_i}; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$, *such that :*

(i) If $H^*(\mathcal{I})$ *is null, then* $R^*(\mathcal{I}) = \Gamma$. *Otherwise*

$$R^*(\mathcal{I}) = Tree(I(\{H^*\}))_{(2m+4, \mathcal{I}_0)}.$$

(ii) For each $R_i(\mathcal{I})$, *construct the summand* $R_i(\mathcal{I})$ *on the path* P_{n_i+3} *with vertices* $v_r, v_\infty, v_0, v_1, v_2, \ldots, v_{n_i}$, *rooted at* v_r, *set* $l(v_r(R_i)) = (2m+3, (k_i, \mathcal{I}_0))$, *set* $l(v_\infty(R_i)) = (2m+2, q_0)$ *and for* $0 \leq j \leq n_i$, *set* $l(v_j(R_i)) = (2m+1, q_0)$.

(iii)Join $(0, q_0)$ *to* $v_0(R_i)$ *by an edge. For each* $j, j = \infty$ *or* $1 \leq j \leq n_i$, *using a proper span* $\mathcal{H}(\mathcal{I}_j)$ *of* $\mathcal{I}_j$, *join the root of* $\hat{\psi}(\mathcal{I}_j)$ *to* v_j *by an edge, where* $\hat{\psi}(\mathcal{I}_j) = \psi(\mathcal{I}_j)$, *if* $\mathcal{I}_j$ *is ancestor-free, and otherwise (if* $H^*(\mathcal{I}_j) = (0; \mathcal{I}''; \emptyset)_{(p,\mathcal{Q})}$ *for some ancestor* $\mathcal{I}''$*), then* $\hat{\psi}(\mathcal{I}_j)$ *is obtained by setting* $H^*(\mathcal{I}_j)$ *to null and for* $\psi(\mathcal{I}_j)$ *in Equation(5) by changing its root label to* $(p+1+\alpha(\mathcal{I}_j), [\hat{q}, \delta])$, *and recursively proceeding to (i).*

Structure-trees for the ideals in Example 14 are depicted in Figure 1.

Figure 1: Structure-trees $\psi(\mathcal{I})$ (left) and $\psi(\mathcal{I}')$ (right) of Example 14. The shaded part $I(\{H^*\}))$ is identical to $\psi(\mathcal{I})$ except for its root label $(0, \hat{q})$

Proof. **(of Theorem 5)** Let $\mathcal{I}$ be any ideal. By Theorem13, we can find a proper span for $\mathcal{I}$ and recursively for every non-empty descendants of $\mathcal{I}$. Hence the tree $\psi(\mathcal{I})$ exists. By definition of an rst-cell, a proper span, and Definition15, we know that $\psi(\mathcal{I})$ is a finitely branching tree. Thus, by Köning's infinity lemma, to prove finiteness it suffices to show that $\psi(\mathcal{I})$ has no infinite path. Suppose $\psi(\mathcal{I})$ contains an infinite path P, then we find a corresponding infinite descendant sequence, $X = (\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ by walking along P, noting that for each encounter of a vertex labeled $\hat{q}$, there corresponds a $root(\hat{\psi}(\mathcal{I}')$, or a $root(\psi(I(\{(0;\mathcal{I}';\emptyset)_{\mathcal{I}_0}\}))$, for some descendant $\mathcal{I}'$ of $\mathcal{I}$.

Consider a recursively defined set $S = \{T | T \in O(\mathcal{I})$ or T is a summand of some tree in $S\}$. Since $O(\mathcal{I})$ is finite, so is S. The algorithm outlined in Theorem 11 and Definition 12 implies that for any descendant $\mathcal{I}'$ of $\mathcal{I}$, we have $O(\mathcal{I}') \subseteq S$. We deduce that the number of distinct descendants of $\mathcal{I}$ is at most $2^{|S|}$, because an ideal is uniquely identified by its obstruction set. It follows that our sequence X contains occurrence of trimmer-cell components infinitely many times, for otherwise there exists $N \geq \mathbb{N}$ such that the sequence $(\mathcal{I}_N, \mathcal{I}_{N+1}, \ldots)$ contains no descendants from a trimmer-cell. Then, by (P1) and the chain axiom (P3), we have an infinite descending chain, contradicting the well-founded property of ideals in Theorem1. Hence some descendant $\mathcal{I}'$ is repeated in a trimmer-cell $\mathcal{I}_i = \mathcal{I}_j = \mathcal{I}'$, for some $j < i - 1$. We deduce that $\mathcal{I}_{i-1}$ is ancestor-dependent. By definition, the branch $R^*(\mathcal{I}_{i-1}) = \Gamma$ and hence $root(\psi(I(\{(0;\mathcal{I}_i;\emptyset)_{\mathcal{I}_0}\}))$ could not be found on P, a contradiction. It follows that $\psi(\mathcal{I})$ has no infinite path and so is finite. The fact that the label set is wqo is obvious from Definition 15 and so the result follows. □

4 Conclusion

One of the main motivation of this research area is the strengthening of Kruskal's Theorem to infinite trees [6]. The other closely related question is that of the concept of better-quasi-ordering [7], introduced by Nash-Williams. Our finite structural description shades some light on these subjects as the proofs in these areas are quite subtle.

We would like to mention here that obtaining precise information about membership of a tree in an ideal $\mathcal{I}$ from its finite structure-tree $\psi(\mathcal{I})$ is a nice result by it self. However, a much more surprising power of $\psi(\mathcal{I})$ is exhibited when the $\leq_{KF}$ relation between two arbitrary structure-trees $\psi(\mathcal{I})$ and $\psi(\mathcal{I})$ allows us to decide on the relationship between the corresponding ideals $\mathcal{I}$ and $\mathcal{I}'$. This shall be the focus of [9] that follows shortly.

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