Combinatorial map as multiplication of combinatorial knots

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Abstract

We show that geometrical map can be expressed as multiplication of combinatorial maps, i.e. map P is equal to multiplication of its knot, inner knot's square and trivial knot ($= \mu \cdot \nu^2 \cdot \pi_1$).

1 Introduction

We proceed with building combinatorial map theory that from different points of view and formulations is considered in from [1] to [35].

We multiply permutations from left to right. Geometrical *combinatorial map* is pair of permutations, *vertex and face rotations*, (P, Q) acting on set of elements C if $P \cdot Q^{-1} = \rho$ *edge rotation* or *inner edge rotation* $\pi = Q^{-1}P$ is involution without fixed elements. We consider set of maps with fixed π calling them *normalized maps*. Mostly we use one particular choice of π equal to (12) ... $(2k-1\ 2k)$, $k > 1$. If so, map may be characterized with one permutation, say vertex rotation P.

In [30] we saw that particular choice of ρ by fixed π induces partitioning of the set C into to subsets C_1 and C_2 [in general in several ways] so that the *knot* $\mu = \begin{cases} C_1 : & \pi \\ C_2 : & \rho \end{cases}$ is defined. Here, knot μ as permutation has 2^k choices if k is number of cycles in it. Rightly, changing direction of some cycle of μ we get another possible value for knot μ. Moreover, ρ with choice of particular μ partitions π into $\pi_1 \cdot \pi_2$, where we call π_1 *cut edges* and π_2 *cycle edges*, so that $P \cdot \pi_1 : C_1 \mapsto C_2$ and $P \cdot \pi_2 : C_1 \mapsto C_1$. In [31] was shown that by fixing μ map P may be expressed as multiplication $\gamma_1 \cdot \gamma_2 \cdot \pi_2$, where γ_1 acts within C_1 and γ_2 acts within C_2.

In [30] was shown that normalized map always may be expressed as $P = \mu \cdot \alpha$, where α is called *knotting* and it is selfconjugate map in sense that $\alpha^\pi = \alpha$. In [32] we got formulas for μ and α, i.e.,

$$\mu = \gamma_2 \pi \gamma_1^{-1}$$

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and

$$\alpha = \gamma_1 \gamma_1^\pi.$$

From [30] we know that α's form a group K_π with respect to multiplication of maps. Moreover, classes of maps with fixed ρ's, denoted as K_ρ, are cosets (left and right) of K_π.

2 Main part

We are going to regain main formulae from introduction.

Let us prove some theorems that leads us to the main result.

Theorem 1. *$\rho \cdot \pi$ [or $\pi \cdot \rho$] is equal to some combinatorial knot μ squared and one or other color cycles induced from this knot reversed.*

Proof. Let us write knot μ in the form $\left\{ \begin{array}{ll} C_1: & \pi \\ C_2: & \rho \end{array} \right.$. Then square of μ we would get applying $\pi \cdot \rho$ for one color corners and $\rho \cdot \pi$ for other color corners. □

Theorem 2. *By fixing the square of the knot it has 2^k knots in correspondence [in general for different maps] where k is the number of cycles in the knot.*

Proof. Two joined cycles of square of knot may be combined in the cycle of knot in two ways, and thus, k independent operations give 2^k results. □

Theorem 3. *[$\mu \cdot \pi$ is knot's half-square]*
1) $\mu \cdot \pi$ contains squared knot's cycles of only one color.
2) For vertex rotation $\mu \cdot \pi$ corresponding face rotation and knot are equal to μ, and knotting equal to π. $\gamma_1 = id$, $\gamma_2 = \mu \cdot \pi$, and $\pi_1 = \pi$, because all edges of this map are cut edges.

Proof. 1) μ expressing as $\left\{ \begin{array}{ll} C_1: & \pi \\ C_2: & \rho \end{array} \right.$, and multiplying by π, we obtain $\left\{ \begin{array}{ll} C_1: & \pi \cdot \pi \\ C_2: & \rho \cdot \pi \end{array} \right.$ and using Theorem 1 what was to be proved.
2) Corresponding graph to this map is set of star graphs as many as cycles in μ. Direct calculation gives what is stated by theorem. □

Theorem 4. *Map P can be expressed as $P\pi_1 = \gamma_1\gamma_2\pi = \gamma_2\gamma_1\pi$ with $\mu(P) = \gamma_2\pi\gamma_1^{-1}[= \gamma_1\rho\gamma_2^{-1}]$ and π_1 as inner cut edge rotation and π_2 inner cycle edge rotation.*

Proof. Let knot $\mu = \mu(P)$ be fixed. Then set of corners is partitioned into two sets C_1 and C_2. From form of $\mu(= \gamma_2 \cdot \pi \cdot \gamma_1^{-1})$ we directly judge that γ_1 belongs to, say, C_1 and γ_2 to C_2. Thus, γ_1 and γ_2 commute by multiplying. Let us choose vertex rotation with this fixed knot and $\pi_1 = id$, i.e., with all edges being cycle edges. Then vertex rotation is alternation of corners from C_1 and C_2 respectively, and face rotation's cycles are correspondingly of one color. Then form of $\mu = \gamma_2 \cdot \pi \cdot \gamma_1^{-1}$ shows directly that $P \cdot \pi_1$ must be equal to $\gamma_1 \cdot \gamma_2(= \gamma_2 \cdot \gamma_1)$. Finally, in general we get

$$\mu = \begin{cases} C_1 : & \pi \\ C_2 : & \rho \end{cases} = \begin{cases} C_2 : \gamma_2\pi\gamma_1^{-1} \\ C_1 : \gamma_1\rho\gamma_2^{-1} \end{cases} .$$

$\square$

Theorem 5. *Map $P\cdot\pi_1$ can be expressed as $\begin{cases} C_1 : & \beta_1 \\ C_2 : & \beta_2 \end{cases}$, where involutions β_1 and β_2 are equal to $\beta_1 = \pi^{-\gamma_1}$ and $\beta_2 = \pi^{-\gamma_2}$. Moreover, $\beta_1 = \gamma_1\gamma_2^{-1}\mu$ and $\beta_2 = \gamma_2\gamma_1^{-1}\mu$. Moreover, $\beta_1\beta_2$ is squared knot $\mu(..., \beta_1)$ with one color cycles reversed. See theorem 1. $\delta = \pi^{\gamma_1}$. $P = \gamma_1\gamma_2\pi_2$.*

Proof.

$$P\pi_1 = \begin{cases} C_1 : & \beta_1 \\ C_2 : & \beta_2 \end{cases} = \begin{cases} C_1 : & \gamma_1\pi\gamma_1^{-1} \\ C_2 : & \gamma_2\pi\gamma_2^{-1} \end{cases} = (\begin{cases} C_1 : & \gamma_1 \\ C_2 : & \gamma_2 \end{cases}) \cdot \pi = \gamma_1\gamma_2\pi.$$

$\square$

Corollary 6. *Map $P\pi_1$ is a knot for inner edge rotation β_1 and edge rotation β_2.*

Theorem 7. *Let for some fixed knot the map P be equal to $\mu\alpha$. Then α is equal to $\gamma_1\pi\gamma_1\pi_2$ or $\gamma_1\gamma_1^{\pi}\pi_1$.*

Proof. $\alpha = \mu^{-1}P\pi_1 = \gamma_1\pi\gamma_2^{-1}\gamma_2\gamma_1\pi = \gamma_1\pi\gamma_1\pi = \gamma_1\gamma_1^{\pi}$. This α is knotting for $P\pi_1$. For map P knotting is $\gamma_1\pi\gamma_1\pi_2$ or $\gamma_1\gamma_1^{\pi}\pi_1$. $\square$

Corollary 8. *$\alpha^{\pi} = \alpha$. Selfconjugate α's comprise group.*

Proof. $\alpha^{\pi} = (\gamma_1\gamma_1^{\pi}\pi_1)^{\pi} = \gamma_1^{\pi}\gamma^{\pi\pi}\pi_1^{\pi} = \gamma_2\gamma_1\pi_1 = \alpha.$ $\square$

Theorem 9. $\gamma_1\gamma_1^\pi$ *is some knot's square.*

Proof. Let us denote this knot by ν. Direct observation shows that theorem is correct. Then fixed knot μ induces α and it determines fixed ν such that $\nu^2 = \gamma_1\gamma_1^\pi$. □

Theorem 10. *Every combinatorial P can be expressed as multiplication of knots in the form*

$$P = \mu \cdot \nu^2 \cdot \pi_1.$$

Proof. It directly follows from previous theorems. Really, $P = \mu\alpha = \gamma_2\pi\gamma_1^{-1}\gamma_1\gamma_1^\pi\pi_1 = \gamma_2\gamma_1\pi_2 = \mu(\gamma_1\pi)^2\pi_1 = \mu\nu^2\pi_1$. □

It must be noted that π_1 is some knot too. We call this knot trivial knot. Let us call knot ν map's inner knot.

Corollary 11. *Map is multiplication of its knot with its inner knot's square and with its trivial knot.*

Theorem 12. *For $P\pi_1$ μ commutes with α, i.e.,*

$$P\pi_1 = \mu \cdot \alpha = \alpha \cdot \mu.$$

In general,

$$P = \mu\alpha = \alpha\mu^{\pi_1}.$$

Proof. For $P\pi_1$, $\mu^\alpha = (\gamma_2\pi\gamma_1^{-1})^{\gamma_1\pi\gamma_1\pi}$. Further, $\gamma_2^{\gamma_1\pi\gamma_1\pi} = \gamma_2^{\pi\cdot\pi} = \gamma_2$, because corners of γ_1 and γ_2 do not intersect. The same is true for the member γ_1^{-1}. Further, $\pi^{\gamma_1\pi\gamma_1\pi} = \pi^\alpha = \pi$. Thus, we get $\mu^\alpha = \mu$. □

Theorem 13. *For partial map $[P, \mu]$ its inner edge rotation is α.*

Proof. Direct observation. □

3 Conclusions

There are four types of permutations that are used to build "all" in combinatorial map theory, i.e., knot-type, knot-square-type, knot-square-with-reversed-cycles-type, two-color-involutions. Comprehensive algebra of all these types should be ground for combinatorial map theory.

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