A combinatorial proof of Rayleigh monotonicity for graphs

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Abstract

We give an elementary, self-contained, and purely combinatorial proof of the Rayleigh monotonicity property of graphs.

Consider a (linear, resistive) electrical network – this is a connected graph $G = (V, E)$ and a set of positive real numbers $\mathbf{y} = \{y_e : \ e \in E\}$ indexed by E. In this paper we allow graphs to have loops and/or multiple edges. The value of y_e is interpreted as the electrical conductance of a wire joining the vertices incident with e. For any edge $e \in E$, there is a simple formula for the effective conductance $\mathcal{Y}_e(G; \mathbf{y})$ of the rest of the graph $G \smallsetminus \{e\}$, measured between the ends of e. This is due to Kirchhoff [11] and is also known as Maxwell's Rule [12]. For a subset $S \subseteq E$, let

$$\mathbf{y}^S = \prod_{c \in S} y_c.$$

Spanning subgraphs of G will be identified naturally with their edge-sets.

[1] Research of JC and JH supported by the DIMACS-DIMATIA REU grant (NSF CNS 0138973)
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Let $\mathcal{T}(G)$ be the set of all spanning trees of G, and let

$$T(G;\mathbf{y}) = \sum_{T\in\mathcal{T}(G)} \mathbf{y}^T$$

be the *tree-generating polynomial* of G. Kirchhoff's Formula for the effective conductance of an electrical network is

$$\mathcal{Y}_e(G;\mathbf{y}) = \frac{T(G\smallsetminus\{e\};\mathbf{y})}{T(G/\{e\};\mathbf{y})}.$$

It is physically intuitive that if all edge-conductances are positive and we increase the conductance of an edge f, then the effective conductance $\mathcal{Y}_e(G;\mathbf{y})$ does not decrease. That is,

$$\frac{\partial}{\partial y_f}\frac{T(G\smallsetminus\{e\};\mathbf{y})}{T(G/\{e\};\mathbf{y})} \geq 0$$

provided that $y_c > 0$ for all $c \in E$. This property is known as *Rayleigh monotonicity*. To keep formulas readable, we often suppress the variables $\mathbf{y}$ (and sometimes the graph G) from the notation unless they require particular attention. A further shorthand is to write

$$T(G) = T^e(G) + y_e T_e(G),$$

in which $T^e(G) = T(G\smallsetminus\{e\})$ and $T_e(G) = T(G/\{e\})$. Applying the quotient rule and a little algebra, Rayleigh monotonicity is seen to be equivalent to the inequality

$$T_e^f(G)T_f^e(G) - T_{ef}(G)T^{ef}(G) \geq 0$$

whenever $y_c > 0$ for all $c \in E$. We also use the notation $\mathcal{T}_e^f(G) = \mathcal{T}((G\smallsetminus\{f\})/\{e\})$ for the set of spanning trees of $(G\smallsetminus\{f\})/\{e\}$, *et cetera*.

In fact, the Rayleigh monotonicity property of graphs follows from a more precise – and rather surprising – combinatorial identity. Fix two distinct edges $e, f \in E$, and orient them arbitrarily. Let $\mathcal{X} = \mathcal{X}(G;e,f)$ denote the set of spanning forests F of G such that both $F\cup\{e\}$ and $F\cup\{f\}$ are spanning trees of G. Thus, $F\cup\{e,f\}$ contains a unique cycle C, and C contains both e and f. Let $\mathcal{X}^+ = \mathcal{X}^+(G;e,f)$ denote the set of those $F \in \mathcal{X}$ for which the edges e and f are oriented in the same direction around the corresponding cycle C. Let $\mathcal{X}^- = \mathcal{X}^-(G;e,f)$ denote the set of those $F \in \mathcal{X}$

for which the edges e and f are oriented in opposite directions around the corresponding cycle C. Define

$$X^+(G;e,f) = \sum_{F \in \mathcal{X}^+(G;e,f)} \mathbf{y}^F$$

and

$$X^-(G;e,f) = \sum_{F \in \mathcal{X}^-(G;e,f)} \mathbf{y}^F.$$

Theorem 1 *Let $G = (V,E)$ be a connected graph, and let $\mathbf{y} = \{y_c : c \in E\}$ be indeterminates. With the notation given above, for distinct $e, f \in E$,*

$$T_e^f(G)T_f^e(G) - T_{ef}(G)T^{ef}(G) = \left[X^+(G;e,f) - X^-(G;e,f)\right]^2.$$

The case of Theorem 1 with all $\mathbf{y} \equiv \mathbf{1}$ appears as equation (2.34) of Brooks, Smith, Stone, and Tutte [2], and the generalization of this to regular matroids is Theorem 2.1 of Feder and Mihail [10]. The case of general conductances $\mathbf{y} > \mathbf{0}$ on graphs can be found in Section 3.8 of Balabanian and Bickart [1]. Choe [4, 5, 6] gives two proofs of Theorem 1. One – based on Tutte's theory of chain groups as in [10] – generalizes to all sixth-root-of-unity matroids but gives a less precise description of the right-hand side. The other proof (as in [1, 2]) uses the All-Minors Matrix-Tree Theorem [3] and Jacobi's theorem on complementary minors of inverse matrices, together with substantial and elaborate algebraic manipulations. Neither of these proofs is completely combinatorial. Here we give a proof of Theorem 1 that is elementary, self-contained, and purely combinatorial. This is not a direct bijective proof, however: we proceed by induction on the number of edges of the graph, and in the induction step we resort to natural 2:2 or 2:1 correspondences (as well as 1:1 correspondences) and in one case we employ a sign-reversing involution. Similar versions of this proof were found independently – by JC and JH, and by MAL and DGW – at about the same time. An earlier description is given by Cibulka and Hladký [9].

Proof of Theorem 1.

Consider a connected graph $G = (V,E)$ and conductances $\mathbf{y} = \{y_c : c \in E\}$ and a pair of edges $e, f \in E$. For short, let us write T_e^f for $T_e^f(G;\mathbf{y})$ and X^+ for $X^+(G;e,f;\mathbf{y})$, and so on. We establish Theorem 1 in the form

$$T_e^f T_f^e + 2(X^+)(X^-) \quad = \quad T_{ef}T^{ef} + (X^+)^2 + (X^-)^2 \qquad (1)$$

by induction on the number of edges of G. To establish the polynomial equation (1), we consider an arbitrary monomial $\mathbf{y}^\alpha$ and show that the coefficients of $\mathbf{y}^\alpha$ on each side of the equation are equal. A general term appearing in equation (1) is indexed by a pair of sets (A, B) with $A, B \subseteq E$, and it contributes the monomial $\mathbf{y}^A \mathbf{y}^B$. The pairs in the set

$$(\mathcal{T}_e^f \times \mathcal{T}_f^e) \cup (\mathcal{X}^+ \times \mathcal{X}^-) \cup (\mathcal{X}^- \times \mathcal{X}^+)$$

contribute to the left-hand side of (1), and the pairs in the set

$$(\mathcal{T}_{ef} \times \mathcal{T}^{ef}) \cup (\mathcal{X}^+ \times \mathcal{X}^+) \cup (\mathcal{X}^- \times \mathcal{X}^-)$$

contribute to the right-hand side of (1). The *type* of a pair (A, B) that contributes to equation (1) is defined to be that one of the six expressions

$$\mathcal{T}_e^f \mathcal{T}_f^e,\ \mathcal{X}^+\mathcal{X}^-,\ \mathcal{X}^-\mathcal{X}^+,\ \mathcal{T}_{ef}\mathcal{T}^{ef},\ \mathcal{X}^+\mathcal{X}^+,\ \mathcal{X}^-\mathcal{X}^-$$

which describes the set which contains it.

Notice that T and X^+ and X^- are *multiaffine* – each indeterminate y_c occurs at most to the first power. Therefore, equation (1) is at most quadratic in each variable. Thus, we need only consider monomials $\mathbf{y}^\alpha$ such that $\alpha : E \to \{0, 1, 2\}$. Notice also that neither y_e nor y_f occur in equation (1), so we need only consider monomials $\mathbf{y}^\alpha$ such that $\alpha(e) = \alpha(f) = 0$. Furthermore, if $g \in E$ is a loop in G then y_g does not occur in equation (1). Thus, it suffices to prove Theorem 1 for loopless graphs G. We even have the following more substantial reduction.

Lemma 2 *Let G be a connected graph that is the union of subgraphs H and J which have exactly one vertex (and no edges) in common. If (1) holds for H and for J, for every choice of edges e, f, then (1) holds for G, for every choice of edges e, f.*

Proof. The key observation is that $T(G) = T(H)T(J)$. Consider any two distinct edges e, f of G. Up to symmetry, there are two cases: either e, f are both in H, or e is in H and f is in J.

If e, f are both in H, then $X^+(G) = X^+(H)T(J)$ and $X^-(G) = X^-(H)T(J)$, so that equation (1) for (G, e, f) is just $T(J)$ times equation (1) for (H, e, f). By the hypothesis, this equation holds.

If e is in H and f is in J, then $X^+(G) = X^-(G) = 0$ and $T_e^f(G) = T_e(H)T^f(J)$, $T_f^e(G) = T^e(H)T_f(J)$, $T_{ef}(G) = T_e(H)T_f(J)$, and $T^{ef}(G) = T^e(H)T^f(J)$. In this case, equation (1) for (G, e, f) can be verified directly.

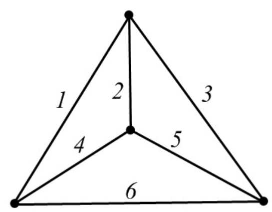

Figure 1: The basis of induction K_4.

Basis of Induction.

The basis of induction consists of three cases.

Case 1: $G = C_2$. The graph consists of two edges e, f in parallel. Orient them to form a directed 2-cycle. We have $T_e^f = T_f^e = 1$, $T_{ef} = T^{ef} = 0$, $X^+ = 1$ and $X^- = 0$. Equation (1) states that

$$1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0,$$

which is clearly true.

Case 2: $G = P_3$. The graph consists of two edges e, f incident at one common vertex. Direct them arbitrarily. We have

$$T_e^f = T_f^e = T^{ef} = X^+ = X^- = 0 \text{ and } T_{ef} = 1$$

so that both sides of equation (1) are zero.

Case 3: $G = K_4$. As will be seen, our induction step does not apply to K_4. (The induction step in [9] does, however.) Thus, we verify equation (1) for K_4 directly. Consider K_4 with edges labelled as in Figure 1. The edges e and f are either adjacent or not, so (up to automorphism) we have the following two subcases.

- $\{e, f\} = \{1, 2\}$. Direct edges 1 and 2 towards their common vertex. Then

$$\begin{aligned}
T_1^2 &= y_3y_4 + y_3y_5 + y_4y_5 + y_4y_6 + y_5y_6 \\
T_2^1 &= y_3y_4 + y_3y_6 + y_4y_5 + y_4y_6 + y_5y_6 \\
T_{12} &= y_3 + y_5 + y_6 \\
T^{12} &= y_3(y_4y_5 + y_4y_6 + y_5y_6) \\
X^+ &= 0 \\
X^- &= y_3y_4 + y_4y_5 + y_4y_6 + y_5y_6
\end{aligned}$$

• $\{e,f\} = \{1,5\}$. Direct edges 1 and 5 towards the vertices they share with the edge 2. Then

$$\begin{aligned}
T_1^5 &= (y_2+y_4)(y_3+y_6) \\
T_5^1 &= (y_2+y_3)(y_4+y_6) \\
T_{15} &= y_2+y_3+y_4+y_6 \\
T^{15} &= y_2y_3y_4+y_2y_3y_6+y_2y_4y_6+y_3y_4y_6 \\
X^+ &= y_3y_4 \\
X^- &= y_2y_6
\end{aligned}$$

In each subcase equation (1) is easily verified.

Induction Step.

Consider a loopless connected graph $G=(V,E)$ with n vertices and $m \geq 3$ edges, let $\mathbf{y} = \{y_c : e \in E\}$, and fix distinct edges $e,f \in E$. Assume that equation (1) holds for any connected graph with at most $m-1$ edges. By Lemma 2 we can assume that G has no cut-vertices, and hence no cut-edges.

Consider any monomial $\mathbf{y}^\alpha$ such that $\alpha : E \to \{0,1,2\}$ and $\alpha(e) = \alpha(f) = 0$. We need to show that

$$[\mathbf{y}^\alpha](T_e^f T_f^e + 2(X^+)(X^-)) = [\mathbf{y}^\alpha](T_{ef}T^{ef} + (X^+)^2 + (X^-)^2) \tag{2}$$

in which the $[\mathbf{y}^\alpha]P(\mathbf{y})$ denotes the coefficient of the monomial $\mathbf{y}^\alpha$ in the polynomial $P(\mathbf{y})$.

Now, two easy reductions. If $\alpha(g) = 0$ for some $g \in E \smallsetminus \{e,f\}$, then we can use the fact that a pair (A,B) contributes $\mathbf{y}^A\mathbf{y}^B = \mathbf{y}^\alpha$ to one side of equation (1) for G if and only if it contributes $\mathbf{y}^A\mathbf{y}^B = \mathbf{y}^\alpha$ to the same side of equation (1) for $G \smallsetminus \{g\}$. By induction we can assume that (1) holds for $G \smallsetminus \{g\}$ (since g is not a cut-edge), and we conclude that (2) holds for G and α.

Similarly, if $\alpha(g) = 2$ for some $g \in E \smallsetminus \{e,f\}$, then we can use the fact that a pair (A,B) contributes $\mathbf{y}^A\mathbf{y}^B = \mathbf{y}^\alpha$ to one side of equation (1) for G if and only if $(A \smallsetminus \{g\}, B \smallsetminus \{g\})$ contributes $y_g^{-2}\mathbf{y}^\alpha$ to the same side of equation (1) for G/g. By induction we can assume that (1) holds for G/g, and we conclude that (2) holds for G and α.

For the rest of the induction step we need only consider the monomial $\mathbf{y}^\gamma$ such that $\gamma(e) = \gamma(f) = 0$ and $\gamma(g) = 1$ for all $g \in E \smallsetminus \{e,f\}$. It will be convenient to have the notation

$$\begin{aligned} \text{LHS}(G) &= [\mathbf{y}^\gamma](T_e^f T_f^e + 2(X^+)(X^-)) \\ \text{and} \quad \text{RHS}(G) &= [\mathbf{y}^\gamma](T_{ef}T^{ef} + (X^+)^2 + (X^-)^2) \end{aligned}$$

(since the edges e and f remain fixed throughout, we suppress them from the notation). To complete the induction step it suffices to prove that

$$\text{LHS}(G) = \text{RHS}(G). \tag{3}$$

The pairs (A, B) contributing to equation (3) are ordered partitions of the edge-set $E \smallsetminus \{e, f\}$ into two (possibly empty) subsets.

Lemma 3 *Let $G = (V, E)$ be a graph, let $e, f \in E$, and let K be an edge-cut of G disjoint from $\{e, f\}$. For any pair (A, B) contributing to equation (3), both $A \cap K \neq \varnothing$ and $B \cap K \neq \varnothing$.*

Proof. For any pair (A, B) contributing to equation (3), the spanning subgraphs $A \cup \{e, f\}$ and $B \cup \{e, f\}$ of G are both connected. Thus, if $K \cap \{e, f\} = \varnothing$ then both $A \cap K \neq \varnothing$ and $B \cap K \neq \varnothing$.

The polynomial equation (1) is homogeneous of degree $2(n-2)$. The monomial $\mathbf{y}^\gamma$ has degree $m-2$. Thus, equation (3) is trivial except in the case that $m = 2n-2$, and so we reduce to this case. Since the sum of the degrees of the vertices of G is $2m = 4n-4$, it follows that G must contain a vertex of degree at most three. The following cases are indexed by the degree of a vertex at which we perform a reduction of the graph G, in order to apply the induction hypothesis.

Cases 0 and 1.

A vertex of degree zero in G is not possible since in the induction hypothesis we assume that the graph is connected and loopless with $m \geq 3$ edges. If G had a vertex of degree one then, since $m \geq 3$ and G is loopless, G would contain a cut-vertex. Since we have reduced to the case that G has no cut-vertices, a vertex of degree one in G is also impossible.

Case 2.

If v_* is a vertex of degree 2 in G, then there are three subcases:

(i) v_* is incident with neither e nor f;

(ii) v_* is incident with e but not f;

(iii) v_* is incident with both e and f.

(By symmetry, (ii) also covers the case that v_* is incident with f but not e.)

- In subcase (i) let v_* be incident with g and h. Then

$$\mathrm{LHS}(G) = 2 \cdot \mathrm{LHS}(G \smallsetminus \{v_*\})$$

since each pair (A, B) contributing to $\mathrm{LHS}(G \smallsetminus \{v_*\})$ gives rise to two pairs $(A \cup \{g\}, B \cup \{h\})$ and $(A \cup \{h\}, B \cup \{g\})$ contributing to $\mathrm{LHS}(G)$. (By applying Lemma 3 with $K = \{g, h\}$ one sees that these are the only possibilities.) Similarly,

$$\mathrm{RHS}(G) = 2 \cdot \mathrm{RHS}(G \smallsetminus \{v_*\}).$$

Note that $G \smallsetminus \{v_*\}$ is connected, since G has no cut-vertices. By the induction hypothesis applied to $G \smallsetminus \{v_*\}$, this suffices to establish equation (3) in this subcase.

- In subcase (ii) let v_* be incident with e and g. Every forest F contributing $\mathbf{y}^F$ to $X^+(G)$ or $X^-(G)$ must use the edge g. Therefore,

$$[\mathbf{y}^\gamma](X^+)(X^-) = [\mathbf{y}^\gamma](X^+)^2 = [\mathbf{y}^\gamma](X^-)^2 = 0.$$

Also, if $T \in \mathcal{T}^e(G)$ then $g \in T$. Therefore,

$$\begin{aligned}
[\mathbf{y}^\gamma] T_e^f(G) T_f^e(G) &= [\mathbf{y}^\gamma] y_g T_e^{fg}(G) T_{fg}^e(G) \\
&= [\mathbf{y}^\gamma] y_g T_{ef}^g(G) T_g^{ef}(G) = [\mathbf{y}^\gamma] T_{ef}(G) T^{ef}(G),
\end{aligned}$$

in which the second equality is a consequence of the 1:1 correspondence $(A, B) \leftrightarrow (B \cup \{e\} \smallsetminus \{g\}, A \cup \{g\} \smallsetminus \{e\})$. This proves equation (3) directly in this subcase.

- In subcase (iii) vertex v_* is incident with e and f. Re-orienting e and/or f, if necessary, we may assume that in the definition of X^+ and X^- both e and f are directed towards v_*. Notice that $T_e^f(G) = T_f^e(G) = T(G \smallsetminus \{v_*\})$ and $T^{ef}(G) = 0$, and $X^+(G) = 0$ and $X^-(G) = T(G \smallsetminus \{v_*\})$. Thus, in this subcase, equation (3) reduces to

$$\mathrm{LHS}(G) = [\mathbf{y}^\gamma] T(G \smallsetminus \{v_*\})^2 = \mathrm{RHS}(G),$$

completing the analysis of Case 2.

For the rest of the induction step we need only consider a two-connected graph G with n vertices and $m = 2n - 2 \geq 3$ edges, and with minimum

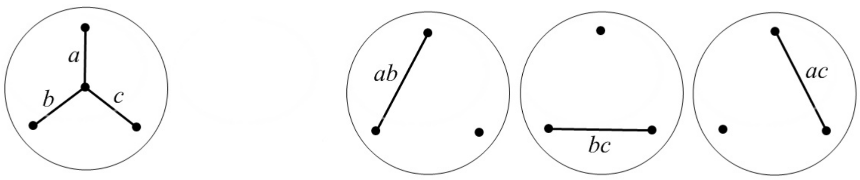

Figure 2: Reduction at a 3-valent vertex.

$(A \cap K, B \cap K)$	H	A'	B'
$(\{a, b\}, \{c\})$	$H(ab)$	$(A \smallsetminus \{v_*\}) \cup \{ab\}$	$B \smallsetminus \{v_*\}$
$(\{c\}, \{a, b\})$	$H(ab)$	$A \smallsetminus \{v_*\}$	$(B \smallsetminus \{v_*\}) \cup \{ab\}$
$(\{a, c\}, \{b\})$	$H(ac)$	$(A \smallsetminus \{v_*\}) \cup \{ac\}$	$B \smallsetminus \{v_*\}$
$(\{b\}, \{a, c\})$	$H(ac)$	$A \smallsetminus \{v_*\}$	$(B \smallsetminus \{v_*\}) \cup \{ac\}$
$(\{b, c\}, \{a\})$	$H(bc)$	$(A \smallsetminus \{v_*\}) \cup \{bc\}$	$B \smallsetminus \{v_*\}$
$(\{a\}, \{b, c\})$	$H(bc)$	$A \smallsetminus \{v_*\}$	$(B \smallsetminus \{v_*\}) \cup \{bc\}$

Table 1: 1:1 in Case 3.

degree three. Such a graph must have at least four vertices of degree three. In fact, one of the two following cases must occur:
Case 3: there is a 3-valent vertex v_* incident with neither e nor f;
Case 4: e and f are not adjacent, their four ends are 3-valent, and every other vertex of G is 4-valent.

Case 3.

Let v_* be a vertex of degree three in G that is not incident with either edge e or edge f. Let $a = \{v_*, a_*\}$, $b = \{v_*, b_*\}$, and $c = \{v_*, c_*\}$ be the edges incident with v_* in G. Consider three new edges (not in G) with ends given by $ab = \{a_*, b_*\}$ and $bc = \{b_*, c_*\}$ and $ac = \{a_*, c_*\}$. Form the graphs $H(ab) = (G \smallsetminus \{v_*\}) \cup \{ab\}$, $H(bc) = (G \smallsetminus \{v_*\}) \cup \{bc\}$, and $H(ac) = (G \smallsetminus \{v_*\}) \cup \{ac\}$, as depicted in Figure 2. (Note that $a_* = b_*$ is possible, for example, in which case ab is a loop.) Since v_* is not a cut-vertex of G, it follows that all of the graphs $H(ab)$, $H(ac)$, and $H(bc)$ are connected.

We claim that

$$\begin{aligned}&\mathrm{LHS}(G) = \\ &\mathrm{LHS}(H(ab)) + \mathrm{LHS}(H(ac)) + \mathrm{LHS}(H(bc))\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\mathrm{RHS}(G) = \\ &\mathrm{RHS}(H(ab)) + \mathrm{RHS}(H(ac)) + \mathrm{RHS}(H(bc)).\end{aligned} \tag{5}$$

By the induction hypothesis, equation (3) holds for each of the graphs $H(ab)$, $H(ac)$, and $H(bc)$, so that (4) and (5) suffice to prove equation (3) for G.

Consider any pair (A, B) contributing to equation (3) for G. By Lemma 3 applied with $K = \{a, b, c\}$, the induced ordered partition $(A \cap K, B \cap K)$ of $\{a, b, c\}$ is one of the six cases presented in the first column of Table 1. The corresponding entry of the second column indicates the graph H to which it is assigned. The corresponding entries of the third and fourth columns describe the pair (A', B') associated with (A, B) that contributes to equation (3) for H. It is easy to see that the pairs (A, B) and (A', B') have the same type. Thus, the construction described in Table 1 gives a 1:1 correspondence $(A, B) \leftrightarrow (A', B')$ that proves equations (4) and (5). Therefore, we have verified equation (3) for G in Case 3.

Case 4.

If $G = K_4$ then we have already verified the conclusion as part of the induction hypothesis. Otherwise, let v_* be a vertex of G that is not incident with either edge e or edge f, and note that v_* has degree 4 and is not a cut-vertex. Let $a = \{v_*, a_*\}$, $b = \{v_*, b_*\}$, $c = \{v_*, c_*\}$, and $d = \{v_*, d_*\}$ be the edges incident with v_* in G. Consider six new edges (not in G) with ends given by $ab = \{a_*, b_*\}$, $ac = \{a_*, c_*\}$, $ad = \{a_*, d_*\}$, $bc = \{b_*, c_*\}$, $bd = \{b_*, d_*\}$, and $cd = \{c_*, d_*\}$. Form the graphs $H(ab|cd) = (G \smallsetminus \{v_*\}) \cup \{ab, cd\}$, $H(ac|bd) = (G \smallsetminus \{v_*\}) \cup \{ac, bd\}$, and $H(ad|bc) = (G \smallsetminus \{v_*\}) \cup \{ad, bc\}$, as depicted in Figure 3. (Note that $a_* = b_*$ is possible, for example, in which case ab is a loop.) Since v_* is not a cut-vertex, all three of $H(ab|cd)$, $H(ac|bd)$, and $H(ad|bc)$ are connected. By induction we can assume that equation (3) holds for each of these graphs, which we will call H *graphs*.

The analogues of (4) and (5) do not hold in this case. Instead, we define numbers L_0 and R_0 by

$$\begin{aligned}&\mathrm{LHS}(G) + \mathrm{L}_0 = \\ &\mathrm{LHS}(H(ab|cd)) + \mathrm{LHS}(H(ac|bd)) + \mathrm{LHS}(H(ad|bc))\end{aligned} \tag{6}$$

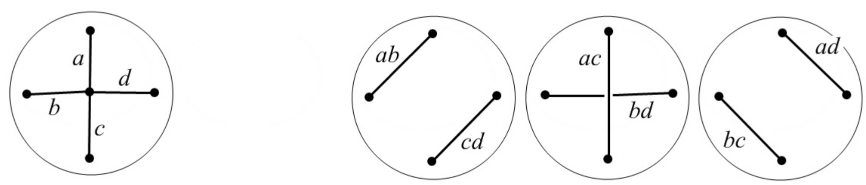

Figure 3: Reduction at a 4-valent vertex.

and

$$\begin{aligned}&\mathrm{RHS}(G) + \mathrm{R}_0 = \\ &\mathrm{RHS}(H(ab|cd)) + \mathrm{RHS}(H(ac|bd)) + \mathrm{RHS}(H(ad|bc))\end{aligned} \tag{7}$$

and complete the proof by showing that

$$\mathrm{L}_0 = \mathrm{R}_0. \tag{8}$$

The induction hypotheses (3) for the three H graphs, together with equations (6), (7), and (8), suffice to prove equation (3) for G.

To prove equation (8) we compare the set of pairs (A, B) contributing to equation (3) for G with the set of pairs (A', B') contributing to equation (3) for the three H graphs. This results in a combinatorial description of L_0 and R_0, from which equation (8) follows easily. To make this comparison, consider any pair (A, B) contributing to equation (3) for G. By Lemma 3 applied with $K = \{a, b, c, d\}$, the induced unordered partition $\{A \cap K, B \cap K\}$ of $\{a, b, c, d\}$ falls into one of the following two subcases: either
(i) $\{A \cap K, B \cap K\}$ has two blocks of size two, or
(ii) $\{A \cap K, B \cap K\}$ has one block of size three and one block of size one.

In subcase (i), if $(A \cap K, B \cap K) = (\{h, i\}, \{j, k\})$ for some labelling $\{h, i, j, k\} = \{a, b, c, d\}$, then let

$$(A', B') = ((A \smallsetminus \{v_*\}) \cup \{hi\}, (B \smallsetminus \{v_*\}) \cup \{jk\})$$

as depicted in Figure 4. This gives a 1:1 correspondence $(A, B) \leftrightarrow (A', B')$ between the pairs (A, B) for G in subcase (i) and the pairs (A', B') for some $H(hi|jk)$ with edges hi and jk in different sets from the pair (A', B'). It is clear that the pairs (A, B) and (A', B') are of the same type.

In subcase (ii), we describe in detail the situation in which $A \cap K = \{h, i, j\}$ and $B \cap K = \{k\}$ for some labelling $\{h, i, j, k\} = \{a, b, c, d\}$. (The

Figure 4: 1:1 in Case 4(i).

other situation is analogous under exchange of A with B.) Note that the vertices h_*, i_*, and j_* are distinct, since A is a forest. We distinguish among a number of possibilities, exactly one of which occurs: either
(ii-a) both A and B are spanning trees, or
(ii-b) both A and B are forests with two components.
In case (ii-b), both $A \cup \{e\}$ and $A \cup \{f\}$ are spanning trees. Let $A[e]$ be the unique path in $A \cup \{e\}$ from v_* to k_*, and let $A[f]$ be the unique path in $A \cup \{f\}$ from v_* to k_*. Let $\widetilde{e}$ be the edge incident with v_* in $A[e]$, and let $\widetilde{f}$ be the edge incident with v_* in $A[f]$. We further divide case (ii-b) as follows: either
(ii-b1) the edges $\widetilde{e}$ and $\widetilde{f}$ are equal, or
(ii-b2) the edges $\widetilde{e}$ and $\widetilde{f}$ are not equal.

It remains to associate to each such pair (A, B), a pair (A', B') that contributes to equation (3) for one of the H graphs. In the situation we are describing in detail ($A \cap K = \{h, i, j\}$ and $B \cap K = \{k\}$) we always put $B' = B \smallsetminus \{v_*\}$.

In case (ii-a), let h_* be the vertex adjacent to v_* on the unique path from v_* to k_* in A, and put $\widetilde{e} = \widetilde{f} = h$. We associate two different pairs (A', B') and (A'', B') with (A, B) in this case, by putting

$$A' = (A \smallsetminus \{v_*\}) \cup \{hi, jk\} \quad \text{and} \quad A'' = (A \smallsetminus \{v_*\}) \cup \{hj, ik\}.$$

Figure 5 illustrates this construction (with $k = d$ and $h = b$). Note that the pairs (A, B), (A', B'), and (A'', B') have the same type. Consider the pair of sets (P, Q) with $P = A \cup \{k\} \smallsetminus \{h\}$ and $Q = B \cup \{h\} \smallsetminus \{k\}$. This pair contributes to equation (3) for G, and is also in case (ii-a). Moreover, the two pairs (P', Q') and (P'', Q') associated to it are in fact the same as (A', B') and (A'', B').

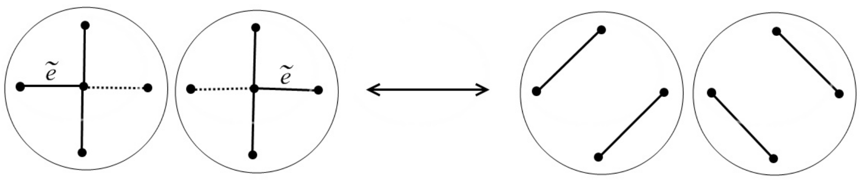

Figure 5: 2:2 in Cases 4(ii-a) and 4(ii-b1).

Figure 6: 1:1 in Case 4(ii-b2).

In case (ii-b1), let h_* be the neighbour of v_* that is incident with $\widetilde{e} = \widetilde{f}$. In this case we apply the same construction as in case (ii-a). Note that the pairs (A, B), (A', B'), and (A'', B') have the same type. The pair of sets (P, Q) with $P = A \cup \{k\} \smallsetminus \{h\}$ and $Q = B \cup \{h\} \smallsetminus \{k\}$ contributes to equation (3) for G, and is also in case (ii-b1), and the two pairs (P', Q') and (P'', Q') associated to it are the same as (A', B') and (A'', B').

In case (ii-b2) let h_* be the neighbour of v_* that is incident with $\widetilde{e}$, and let i_* be the neighbour of v_* that is incident with $\widetilde{f}$. In this case we associate a single pair (A', B') with (A, B) by putting

$$A' = (A \smallsetminus \{v_*\}) \cup \{hi, jk\}$$

as depicted in Figure 6 (with $k = d$, $h = a$ and $i = c$). Note that the pairs (A, B) and (A', B') have the same type.

Table 2 summarizes these correspondences in Case 4(ii). The situations in which $(A \cap K, B \cap K) = (\{k\}, \{h, i, j\})$ are handled analogously after exchanging A with B.

The next step is to identify which pairs (A', B') contributing to equation (3) for the H graphs are produced by the above construction. We refer to

G case	m:m	H case	A'
(a)	2:2	(α)	$\begin{cases} (A \setminus \{v_*\}) \cup \{hi, jk\} \\ (A \setminus \{v_*\}) \cup \{hj, ik\} \end{cases}$
(b1)	2:2	(β1)&($\beta 2'$)	$\begin{cases} (A \setminus \{v_*\}) \cup \{hi, jk\} \\ (A \setminus \{v_*\}) \cup \{hj, ik\} \end{cases}$
(b2)	1:1	($\beta 2''$)	$(A \setminus \{v_*\}) \cup \{hi, jk\}$
	0:1	(β3)	sign-reversing involution

Table 2: Summary of Case 4(ii).

the edges *ab*, *ac*, *ad*, *bc*, *bd*, and *cd* as *new edges*, so each H graph has two new edges. The case analysis for these pairs (A', B') is:
(i) the two new edges of H are in different sets from the pair (A', B');
(ii) the two new edges of H are in the same set from the pair (A', B').
We describe in detail the situation in case (ii) in which both new edges are in A'. (The other situation is analogous.)
(ii-α) both A' and B' are spanning trees;
(ii-β) both A' and B' are forests with two components:
(ii-β1) neither of the new edges is in the cycle C' of $A' \cup \{e, f\}$;
(ii-β2) exactly one of the new edges is in the cycle C' of $A' \cup \{e, f\}$;
(ii-β3) both of the new edges are in the cycle C' of $A' \cup \{e, f\}$.
(Case (ii-β2) must be further divided, as explained below.)

The inverse of the above construction is described as follows.

In case (i), if the new edges of H are $hi \in A'$ and $jk \in B'$ then put

$$A^\circ = (A' \setminus \{hi\}) \cup \{h, i\} \quad \text{and} \quad B^\circ = (B' \setminus \{jk\}) \cup \{j, k\}.$$

In case (ii) we describe the situation in which the new edges of H are hi and jk, both in A'. The other situation is analogous upon exchanging A' with B'.

In case (ii-α), since A' is a spanning tree there is a unique path in A' with the new edges hi and jk as end-edges. Let i_* and j_* be the end-vertices of this path, and let h_* and k_* be the other ends of the new edges. We associate two pairs with (A', B') in this case, by

$$A^\circ = (A' \setminus \{hi, jk\}) \cup \{h, i, j\} \quad \text{and} \quad B^\circ = B' \cup \{k\}.$$

and

$$A^{\circ\circ} = (A' \setminus \{hi, jk\}) \cup \{i, j, k\} \quad \text{and} \quad B^{\circ\circ} = B' \cup \{h\}.$$

In case (ii-β1), there are one or two paths in $A' \cup \{e,f\}$ with the new edges hi and jk as end-edges. Let i_* and j_* be the end-vertices of such a path. Notice that if two such paths exist then they have the same end-vertices, so that i_* and j_* are defined unambiguously. Let h_* and k_* be the other ends of the new edges, and associate two pairs with (A', B') in this case by the same construction as in case (ii-α).

In case (ii-β2), let hi be the new edge of H in the cycle C' of $A' \cup \{e,f\}$, and let jk be the new edge of H not in C'. We must distinguish between the following two cases:
(ii-$\beta 2'$) the edges hi and jk are in the same component of A';
(ii-$\beta 2''$) the edges hi and jk are in different components of A'.
In case (ii-$\beta 2'$), there is a unique path in A' with the new edges hi and jk as end-edges. Let i_* and j_* be the end-vertices of this path, let h_* and k_* be the other ends of the new edges, and associate two pairs with (A', B') in this case by the same construction as in cases (ii-α) and (ii-β1).
In case (ii-$\beta 2''$) let k_* be the end of jk that is in the connected component of $(A' \cup \{e,f\}) \smallsetminus \{jk\}$ which contains C'. We associate one pair with (A', B') in this case, by

$$A^\circ = (A' \smallsetminus \{hi, jk\}) \cup \{h, i, j\} \quad \text{and} \quad B^\circ = B' \cup \{k\}.$$

In each case except (ii-β3) we associate one or two pairs (A°, B°), $(A^{\circ\circ}, B^{\circ\circ})$ to each pair (A', B') contributing to (3) for the three H graphs. In each case, pairs that are associated with one another are of the same type. There are many details to check, but the constructions described above give correspondences as in Table 2. Thus, the pairs contributing to (3) for G are equinumerous (and of the same types) as the pairs contributing to (3) for the H graphs, except for the pairs in case (ii-β3). Comparing this with equations (6) and (7), we see that L_0 is the number of pairs in case (ii-β3) of types $\mathcal{X}^+\mathcal{X}^-$ or $\mathcal{X}^-\mathcal{X}^+$, and that R_0 is the number of pairs in case (ii-β3) of types $\mathcal{X}^+\mathcal{X}^+$ or $\mathcal{X}^-\mathcal{X}^-$.

The following sign-reversing involution on the set of pairs in case (ii-β3) shows that $\mathrm{L}_0 = \mathrm{R}_0$ and completes the analysis of Case 4, the induction step, and the proof. Consider any pair (A, B) in case (ii-β3), with new edges of H being hi and jk, both in A. (The situation in which both new edges are in B is analogous.) Exactly one of the sets

$$(A \smallsetminus \{hi, jk\}) \cup \{hj, ik\} \quad \text{or} \quad (A \smallsetminus \{hi, jk\}) \cup \{hk, ij\}$$

is also a forest F with two components such that both $F \cup \{e\}$ and $F \cup \{f\}$

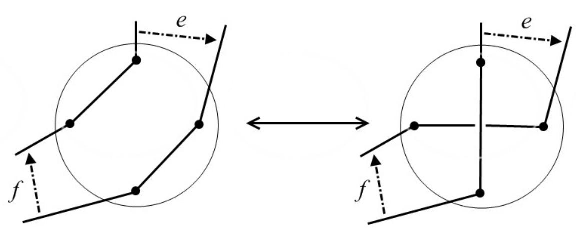

Figure 7: The sign-reversing involution of Case 4(ii-β3).

are spanning trees. Call this set $A^\natural$, and note that $(A^\natural)^\natural = A$. The sign-reversing involution is given by

$$(A, B) \longleftrightarrow (A^\natural, B)$$

as illustrated in Figure 7. Thus, in case (ii-β3) the pairs of types $\mathcal{X}^+\mathcal{X}^-$ or $\mathcal{X}^-\mathcal{X}^+$ are equinumerous with the pairs of types $\mathcal{X}^+\mathcal{X}^+$ or $\mathcal{X}^-\mathcal{X}^-$. That is, $\mathrm{L}_0 = \mathrm{R}_0$. This completes the proof. $\square$

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