Equitable colourings of graphs with bounded density and given girth*

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We study the equitable chromatic number of graphs of minimum degree at least 2, lower-bounded girth and upper-bounded density. As direct corollaries, we obtain that the equitable chromatic number of a planar graph with minimum degree at least 2 is at most 3 if the graph has girth at least 14, and at most 4 if the graph has girth at least 10. This improves the previously known bounds, obtained by Wu and Wang [Discrete Mathematics, 308(5-6):985–990, 2008].

1 Introduction

Given a graph $G = (V, E)$ and a positive integer m, an *m-colouring* of G is a mapping $c : V \to \{1, 2, \ldots, m\}$. It is *proper* if $c(u) \neq c(v)$ whenever u and v are two adjacent vertices of G. Graph colouring is a prominent topic in graph theory. It was initiated by the 4-colour conjecture, which is now a theorem [2, 3]. Many variants and generalisations have been considered, in particular in relation with practical problems such as channel assignment or scheduling. The notion considered in this paper is one of those variants.

Given an m-colouring c of a graph G, a *colour-class* is a subset $c^{-1}(\{i\})$ for some $i \in \{1, 2, \ldots, m\}$. The colouring c is *equitable* if

- it is proper; and
- the size of any two colour-classes differ by at most 1.

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The *equitable chromatic number* of a graph G is $\chi_e(G)$, the smallest integer k such that G admits an equitable m-colouring whenever $m \geq k$. (Note that a graph admitting an equitable k-colouring does not necessarily admit an equitable m-colouring for all $m > k$.)

Equitable colourings arise naturally when modelling various problems, e.g. scheduling [14], and are also used to study the deviation of some random variables in probability theory [13, 8]. In 1964, Erdős [5] conjectured that the equitable chromatic number of any graph of maximum degree Δ is at most $(\Delta+1)$. This was proved by Hajnal and Szemerédi [7] in 1970. Finding a simpler proof of this theorem has been a long standing question, and several new proofs (along with polynomial time algorithms) have been found recently, by Kierstead and Kostochka [9] and Mydlarz and Szemerédi [12]. Further research in this direction is offered by an intriguing conjecture of Chen, Lih and Wu [4], that a graph of maximum degree Δ has equitable chromatic number at most Δ unless it is K_Δ or $K_{\Delta,\Delta}$. The survey of Lih [11] provides additional background about equitable colourings.

It is natural to search for conditions ensuring that the equitable chromatic number is equal to the *chromatic number* — the smallest integer k for which the graph admits a proper k-colouring. This is why, very recently, Wu and Wang [15] studied the equitable chromatic number of planar graphs with minimum degree at least 2. For such graphs, they proved that $\chi_e(G) \leq 3$ if G has girth at least 26, and $\chi_e(G) \leq 4$ if G has girth at least 14. (The *girth* of a graph is the size of a smallest cycle.) So, for instance, they derive that if a planar graph has minimum degree at least 2, girth at least 26 and is non-bipartite then its equitable chromatic number is equal to its chromatic number. (This is so since, by Grötzsch's Theorem [6], the chromatic number of any planar graph of girth has least 4 is at most 3.)

The two restrictions on the minimum degree and the girth are natural: on one hand, the equitable chromatic number of stars (i.e. trees whose all vertices but at most one are leaves) is unbounded. More precisely, if T_n is a star with root v and leaves $v_1, v_2, \ldots, v_n$, its equitable chromatic number is $1+\lceil \frac{n}{2} \rceil$. Observe also that the equitable chromatic number of planar graphs with minimum degree 2 (but without any restriction on girth) is unbounded: starting from the star T_n aforementioned, we add a new vertex linked to all the vertices of T_n. The equitable chromatic number of the obtained graph is $2 + \lceil \frac{n}{2} \rceil$.

We aim at improving the bounds on girth given by Wu and Wang. To this end, we study a more general class of graphs, namely the class of graphs with bounded density. The average degree of a graph $G = (V, E)$ is

$\mathrm{Ad}(G) := \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G is $\mathrm{Mad}(G)$, the maximum of the values $\mathrm{Ad}(H)$ taken over all subgraphs H of G.

The purpose of this paper is to prove the following two theorems.

Theorem 1. *The equitable chromatic number of every graph of minimum degree at least* 2*, girth at least* 11 *and maximum average degree less than* $\frac{7}{3}$ *is at most* 3.

Theorem 2. *The equitable chromatic number of every graph of minimum degree at least* 2*, girth at least* 10 *and maximum average degree less than* $\frac{5}{2}$ *is at most* 4.

Those two theorems yield corollaries giving a substantial improvement on the bounds given by Wu and Wang [15]. Their derivation from Theorems 1 and 2 is a direct consequence of Euler's formula. More precisely, Euler's formula implies that for any planar graph G of girth $g \geq 3$,

$$\frac{2g}{g-2} < \mathrm{Mad}(G),$$

which in turn allows us to deduce the following two corollaries from Theorems 1 and 2.

Corollary 3. *The equitable chromatic number of every planar graph of minimum degree at least* 2 *and girth at least* 14 *is at most* 3.

Corollary 4. *The equitable chromatic number of every planar graph of minimum degree at least* 2 *and girth at least* 10 *is at most* 4.

Using the discharging method, we prove Theorem 1 in Section 2 and Theorem 2 in Section 3. In both cases, we identify subgraphs that cannot appear in a minimal counter-example (we call them *reducible configurations*). Then we use a discharging procedure to obtain a contradiction. We finish this section by introducing some definitions and conventions.

A vertex of degree d is a *d-vertex*. It is *big* if $d \geq 3$. Given a graph G, suppose that $T := v_1v_2\ldots v_t$ is an induced path such that each vertex v_i has degree 2 in G. Assume that b and b' are big vertices such that bv_1 and $b'v_t$ are edges of G. If $b \neq b'$ then T is a *t-thread*, and otherwise T is a *t-loop*. We say that b is *incident* to T, and that it is *t-weakly adjacent* to b'. The vertices b and b' are *t-weak neighbours*.

In all the figures of this paper, a black bullet is a vertex of degree 2, a triangle is a vertex of degree 3, a square a vertex of degree 4, a pentagon a vertex of degree 5 and an hexagon a vertex of degree 6. A white bullet indicates a big vertex. A dashed line between two 2-vertices indicates an induced path of G whose all vertices have degree 2 in G.

We use $1, 2, \ldots, m$ as the integers modulo m. Given an equitable m-colouring of a graph G, we always assume that the number of vertices coloured i is at most the number of vertices coloured $i+1$ for every $i \in \{1, 2, \ldots, m-1\}$. When m is fixed, a colour greater than m is always to be understood modulo m.

2 The Proof of Theorem 1

We proceed by contradiction. Throughout this section, we assume that $G = (V, E)$ is a graph of minimum degree at least 2, girth at least 14 and maximum average degree less than $\frac{7}{3}$ whose equitable chromatic number is greater than 3. Subject to this, we assume that G has as few vertices as possible, i.e. every graph of minimum degree at least 2, girth at least 14, maximum average degree less than $\frac{7}{3}$ and fewer vertices than G has equitable chromatic number at most 3. In particular, the equitable chromatic number of any subgraph of G with less vertices than G and minimum degree at least 2 is at most 3.

In the next subsection, we start proving that some subgraphs (called *configurations*) cannot occur in the graph G (i.e. are *reducible*).

2.1 Reducible Configurations

We use the next two lemmas in the sequel implicitly, without referring to them.

Lemma 5. *The graph G has no t-thread with $t = 3$ or $t \geq 5$.*

Proof. Fix an integer $m \geq 3$. Suppose that b_1 and b_t are two big vertices of G that are t-weakly adjacent through the t-thread $T := u_1 u_2 \ldots u_t$, with $t = 3$ or $t \geq 5$. Let G' be the subgraph obtained from G by removing the vertices of T. As G' has minimum degree at least 2, the minimality of G implies that G' admits an equitable m-colouring c. We extend it to T by colouring the vertex u_i with the colour $i \mod m$. Hence, c is an equitable m-colouring of G unless it is not proper, i.e. unless $c(b_i) = c(u_i)$ for some $i \in \{1, t\}$. Thus, up to switching the colours of u_1 and u_2, and the colours of u_t and u_{t-1}, we obtain an equitable m-colouring of G, a contradiction. $\square$

Lemma 6. *A vertex b of G of degree at least 4 is not incident to a t-loop.*

Proof. Suppose on the contrary that b is incident to a t-loop. So $t \geq 14$. Since b has degree at least 4, we can proceed as in the proof of Lemma 5 with $b_1 := b_t := b$. □

We now make two observations that will be used in the forthcoming reducibility proofs. The first one deals with the extension of a colouring to a 2-thread, and we omit its straightforward proof.

Lemma 7. *Let $b_0u_0u_1b_1$ be a path. Suppose that b_0 is coloured α and b_1 is coloured β. Consider two distinct colours γ and γ'. We can properly colour u_0 and u_1 using γ and γ' unless $\alpha = \beta \in \{\gamma, \gamma'\}$.*

Our second observation deals with the extension of colourings to 4-threads.

Lemma 8. *Let $b_0u_0u_1u_2u_3b_3$ be a path. Fix an integer $m \geq 3$. The integers modulo m are colours. Suppose that b_0 is coloured α and b_3 is coloured β. Consider the colours $\gamma, \gamma+1, \gamma+2$ and $\gamma+3$. We can properly colour the vertices u_i using each of the four previous integers once unless $m = 3$ and $\alpha = \gamma = \gamma+3 = \beta \mod 3$. In particular, the extension is always possible if $m > 3$, or if $\alpha \neq \beta \mod m$, or if $\alpha \neq \gamma \mod 3$.*

Proof. We colour u_i with $\gamma + i$. If the vertices b_0 and u_0 have the same colour, then we switch the colours of u_0 and u_1, i.e. we colour u_0 with $\gamma+1$ and u_1 with γ. Similarly, if b_3 and u_3 have the same colour, we switch the colours of u_3 and u_2. This yields a proper colouring unless $\alpha = \gamma = \gamma+3 = \beta \mod m$, which can occur only if $m = 3$. □

We now start proving reducibility of some configurations around vertices of G. Each proof follows the same pattern, which we explain below. Each configuration L is comprised of some vertices u_i and b_i. We proceed by contradiction, that is we suppose that G contains L as a subgraph, and show that G can then be equitably m-coloured for every $m \geq 3$, a contradiction. To this end, we remove the vertices u_i of L from the graph G. We obtain a subgraph G' of G of minimum degree at least 2. To see this, we note that due to the girth of G, all the vertices b_i of L are pairwise distinct in most cases. Whenever this is not true, we explicitly give an additional argument to prove that we actually obtain a subgraph of minimum degree at least 2. So, by the minimality of G, there exists for every integer $m \geq 3$ an equitable

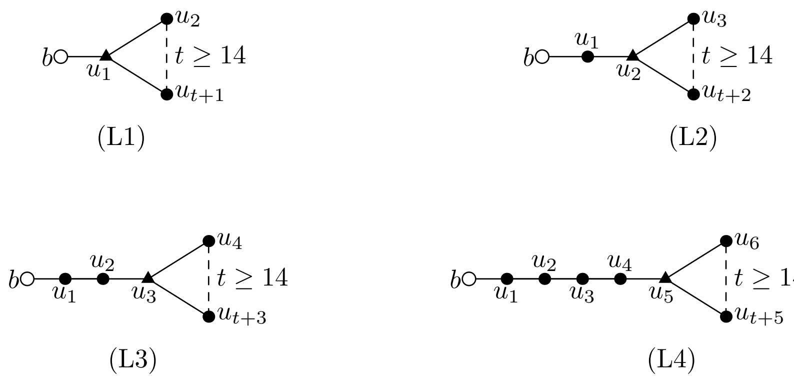


Figure 1: Reducible configurations (L1)–(L4).

m-colouring c of G'. We then extend c to an equitable m-colouring of G. Note that if there are, say, r vertices u_i, and we colour them using once each colour in $\{1, 2, \ldots, r\}$, then the obtained colouring of G is equitable as soon as it is proper (recall that colours greater than m are understood modulo m). Indeed, the size of each colour class has increased by $r' := \left\lfloor \frac{r}{m} \right\rfloor$, except the $r \mod m$ smallest that have increased by $r + 1$. Recalling that we assume for the colouring c of G' that the size of the colour class i is at most the size of the colour class $i + 1$ for each $i \in \{1, 2, \ldots, m - 1\}$, we deduce that as soon as the obtained colouring of G is proper, it is equitable.

2.1.1 Vertices of Degree Three

We give some reducible configurations around a 3-vertex. We start by proving that a 3-vertex cannot be incident to a t-loop. Hence, using Lemma 6, we deduce that G has no t-loop.

Lemma 9. *The configurations of Figure 1 are reducible.*

Proof. For each configuration, we set $c(u_i) := i \mod m$. Hence, the obtained colouring of G is equitable unless it is not proper. Note that the defined colouring is proper unless $c(u_1) = c(b) \mod m$. In this case, we switch the colours of u_1 and u_2, i.e. we colour u_1 with 2 and u_2 with 1. The obtained colouring is then proper, since $c(u_3) = 3 \notin \{1, 2\}$. $\square$

Figure 2: Reducible configurations (L5) and (L6).

The next lemma states that a 3-vertex of G incident to two 2-threads cannot be incident to a 1-thread or a third 2-thread.

Lemma 10. *The configurations of Figure 2 are reducible.*

Proof. (L5). Suppose first that $m \neq 3$. We set $c(u_i) := i \mod m$ for all indices i. We switch the colours of u_1 and u_2 if $c(b_1) = 1 \mod m$, the colours of u_3 and u_4 if $c(b_4) = 4 \mod m$, and the colours of u_5 and u_6 if $c(b_6) = 6 \mod m$. The obtained colouring is the sought one.

Assume now that $m = 3$. We want to colour the 6 vertices u_i using twice each colour. Hence, we may assume that $c(b_1) \neq 1 \mod 3$ and $c(b_4) \neq 2 \mod 3$. We set $c(u_1) := c(u_3) := 1$, $c(u_4) := 2$ and $c(u_2) := 3$. We finish by colouring the 2-thread u_5u_6 with the colours 2 and 3, which is possible by Lemma 7 since $c(u_3) = 1 \notin \{2, 3\}$.

(L6). If one of the vertices b_i, say b_7, is not coloured $7 \mod m$, then we colour u_7 with 7, and we finish colouring the vertices u_i using once each integer in $\{1, 2, \ldots, 6\}$ as for (L5). Thus, we may assume that $c(b_i) = 7 \mod m$ for $i \in \{1, 6, 7\}$. We set $c(u_3) := 5 \mod m$. Then, we extend the colouring to each 2-thread using once each integer in $\{1, 2, \ldots, 7\} \setminus \{5\}$. This is possible by Lemma 7, since $c(u_3) = 5 \neq 7 = c(b_i) \mod m$ for all $i \in \{1, 6, 7\}$. $\square$

The following lemma ensures that a 3-vertex of G incident to a unique 4-thread is either incident to two 1-threads or adjacent to two big vertices.

Lemma 11. *The configurations of Figure 3 are reducible.*

Proof. (L7). We colour u_8 and u_9 with the colours 8 and 9 such that $c(u_9) \neq c(b_9) \mod m$. It remains to properly colour the vertices u_i for $i \in \{1, 2, \ldots, 7\}$ using once each integer in $\{1, 2, \ldots, 7\}$. Observe that we can proceed exactly as for the configuration (L6) of Figure 2.

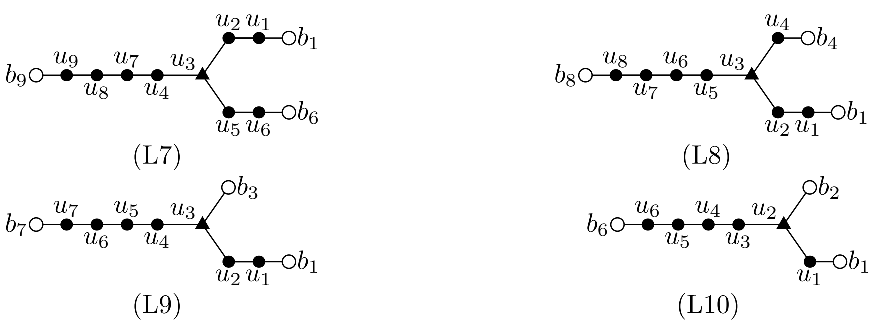


Figure 3: Reducible configurations (L7)–(L10).

(L8). We first properly colour u_7 and u_8 with the colours 7 and 8. Extending this colouring to G using once each integer in $\{1, 2, \ldots, 6\}$ then amounts to what has been done for the configuration (L5) of Figure 2.

(L9). Let $c(u_3) \in \{1, 2, 3\} \setminus \{c(b_3), c(b_7)\}$. Next, let $c(u_1) \in \{1, 2, 3\} \setminus \{c(b_1), c(u_3)\}$ and $c(u_2) \in \{1, 2, 3\} \setminus \{c(u_1), c(u_3)\}$. We now colour the 4-thread $u_4u_5u_6u_7$ by applying Lemma 8 with $\gamma = 4$, which yields the conclusion since $c(u_3) \neq c(b_7) \mod m$.

(L10). Let $c(u_2) \in \{1, 2, 3\} \setminus \{c(b_2)\}$, $c(u_1) \in \{1, 2, 3\} \setminus \{c(b_1), c(u_2)\}$ and $c(u_3) \in \{1, 2, 3\} \setminus \{c(u_1), c(u_2)\}$. Next, we let $c(u_4) \in \{4, 5, 6\} \setminus \{c(u_3)\}$, $c(u_6) \in \{4, 5, 6\} \setminus \{c(b_6), c(u_4)\}$ and $c(u_5) \in \{4, 5, 6\} \setminus \{c(u_4), c(u_6)\}$. $\square$

The next lemma states that a 3-vertex of G is not incident to more than one 4-thread.

Lemma 12. *The configurations of Figure 4 are reducible.*

Proof. (L11). We colour u_{10} and u_{11} with the colours 10 and 11 such that $c(u_{11}) \neq c(b_{11}) \mod m$. It remains to colours the vertices u_i for $i \leq 9$ using once each integer in $\{1, 2, \ldots, 9\}$. We proceed as for the configuration (L7) of Figure 3.

(L12). We colour u_{12} and u_{13} with the colours 12 and 13 such that $c(u_{13}) \neq c(b_{13}) \mod m$. It remains to properly colour the vertices u_i for $i \leq 11$ using once each integer in $\{1, 2, \ldots, 11\}$. Thus we can proceed as for the configuration (L11).

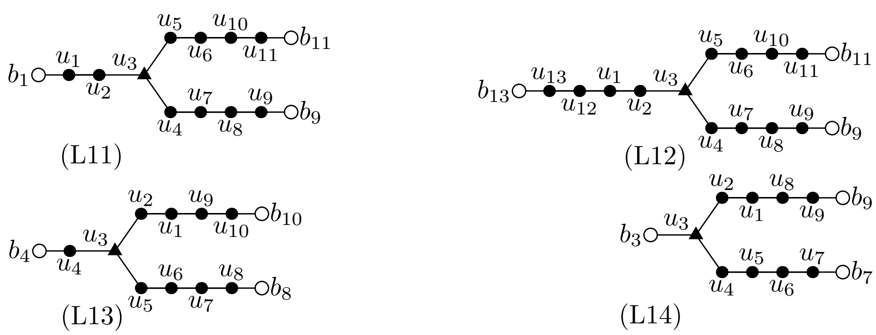


Figure 4: Reducible configurations (L11)–(L14).

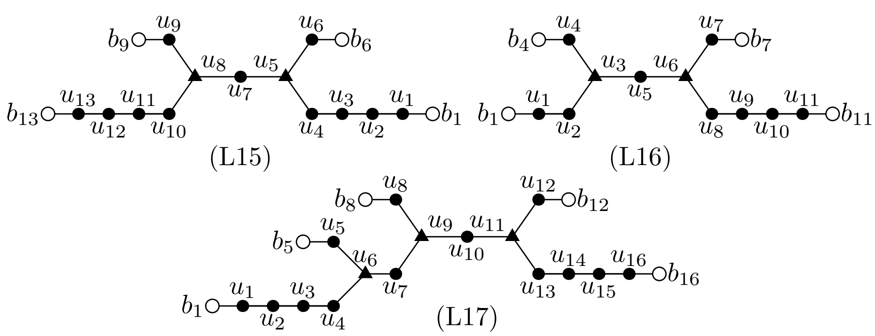


Figure 5: Reducible configurations (L15)–(L17).

(L13). We colour u_9 and u_{10} with the colours 9 and 10 such that $c(u_{10}) \neq c(b_{10}) \mod m$. We then extend the colouring to the remaining vertices u_i as for the configuration (L8) of Figure 3.

(L14). We properly colour u_8 and u_9 with the colours 8 and 9, and then proceed as for the configuration (L9) of Figure 3. $\square$

The next lemma ends our study of 3-vertices of G.

Lemma 13. *The configuration of Figure 5 are reducible.*

Proof. (L15). Note that the girth assumption on G does not prevent the vertices b_1 and b_{13} from being the same one. However, they cannot be the

(L18) (L19)

Figure 6: Reducible configurations (L18) and (L19).

same 3-vertex, otherwise G would have a 3-vertex incident to more than one 4-thread, thereby contradicting Lemma 12. Therefore, we obtain as usual a subgraph of minimum degree at least 2 when we remove all the vertices u_i from G.

We set $c(u_i) := i \mod m$ for all indices i. The obtained colouring is proper up to switching the colours of u_1 and u_2 if $c(b_1) = 1 \mod m$, and of u_i and u_{i-1} if $c(b_i) = i \mod m$ for $i \in \{6, 9, 13\}$.

(L16). We set $c(u_i) := i \mod m$. For each $i \in \{1, 4, 7\}$, we switch the colours of u_i and u_{i+1} if $c(b_i) = i \mod m$. Finally, we switch the colours of u_{11} and u_{10} if $c(b_{11}) = 11 \mod m$.

(L17). Two of the vertices b_i, namely b_1 and b_{16}, could be the same without violating the girth assumption on G. However, as previously, those two vertices cannot be the same 3-vertex without contradicting Lemma 12. Consequently, we obtain a subgraph of minimum degree at least 2 when we remove the vertices u_i from G

We set $c(u_i) := i \mod m$ for all indices i. For each $i \in \{1, 5, 8\}$, we switch the colours of u_i and u_{i+1} if $c(b_i) = i \mod m$. Moreover, for each $i \in \{12, 16\}$, we switch the colours of u_i and u_{i-1} if $c(b_i) = i \mod m$. $\square$

2.1.2 Vertices of Degree Four

We start with 4-vertices incident to two 4-threads.

Lemma 14. *The configurations of Figure 6 are reducible.*

Proof. (L18). Let $\alpha \in \{4, 5, 6\} \setminus \{c(b_{11}), c(b_{13})\}$. We set $c(u_5) := \alpha$. Moreover, we colour u_4 with the smallest integer in $\{4, 5\} \setminus \{\alpha\}$, and u_6 with the colour in $\{4, 5, 6\} \setminus \{c(u_4), \alpha\}$. Note that $c(u_4) \in \{4, 5\}$ and $c(u_6) \in \{5, 6\}$. Next, we set $c(u_3) := 3$ and $c(u_7) := 7$. Then, we colour u_1 and u_2 with the colours 1 and 2 such that $c(u_1) \neq c(b_1) \mod m$. Similarly, we colour u_8

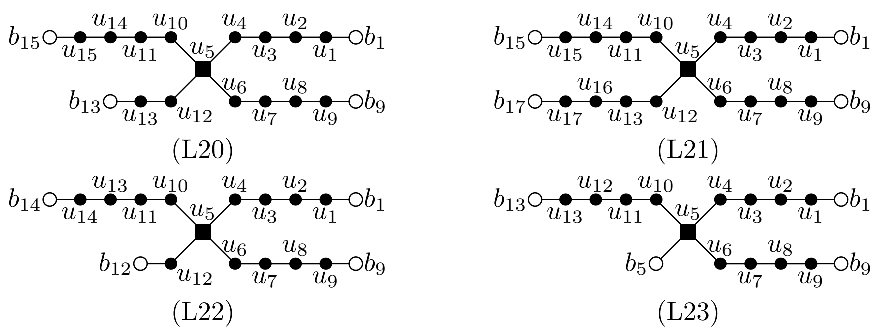


Figure 7: Reducible configurations (L20)–(L23).

and u_9 with the colours 8 and 9 such that $c(u_9) \neq c(b_9) \mod m$. It remains to colour the two 2-threads. We use Lemma 7 twice: for $i \in \{10, 12\}$, we extend the colouring to the 2-thread $u_i u_{i+1}$ using the colours i and $i+1$, which is possible since $c(u_5) \neq c(b_{i+1}) \mod m$.

(L19). Let $\alpha \in \{4,5\} \setminus \{c(b_{11})\}$. We colour u_5 with α and u_4 with the colour in $\{4,5\} \setminus \{\alpha\}$. Next, we set $c(u_3) := 3$ and we properly colour u_1 and u_2 with the colours 1 and 2. We extend this colouring to the 4-thread $u_6u_7u_8u_9$ using each of the integers $6, 7, 8, 9$, which is possible by Lemma 8 since $c(u_5) \neq 6 \mod 3$. We colour u_{12} with a colour of $\{10, 11, 12\} \setminus \{c(b_{12}), c(u_5)\}$. We finish by colouring the 2-thread with the two colours of $\{10, 11, 12\} \setminus \{c(u_{12})\}$, which is possible by Lemma 7 since $c(u_5) \neq c(b_{11}) \mod m$. □

The following lemma ensures that a 4-vertex of G cannot be incident to more than two 4-threads.

Lemma 15. *The configurations of Figure 7 are reducible.*

Proof. (L20). We first colour u_{14} and u_{15} using the colours 14 and 15 such that $c(u_{15}) \neq c(b_{15}) \mod m$. It remains to properly colour the vertices u_i for $i \leq 13$ with the integers in $\{1, 2, \ldots, 13\}$. We proceed as for the configuration (L18) of Figure 6.

(L21). First, we colour u_{16} and u_{17} with the colours 16 and 17 such that $c(u_{17}) \neq c(b_{17}) \mod m$. We extend this colouring to the remaining vertices u_i using each integer in $\{1, 2, \ldots, 15\}$ as for (L20).

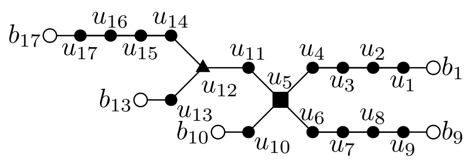


Figure 8: The reducible configuration (L24).

(L22). First, we properly colour the vertices u_{13} and u_{14} with the colours 13 and 14. Then, we colour the remaining vertices u_i using once each integer in $\{1, 2, \ldots, 12\}$ as for the configuration (L19) of Figure 6.

(L23). We set $c(u_i) := i$ for $i \leq 9$. We switch the colours of u_1 and u_2 if $c(b_1) = 1 \mod m$, and the colours of u_9 and u_8 if $c(b_9) = 9 \mod m$. Next, we switch the colours of u_5 and u_6 if $c(b_5) = 5 \mod m$. Now, we extend this colouring to the remaining 4-thread by applying Lemma 8 with $\gamma = 10$. This is possible since neither 5 nor 6 is 10 modulo 3. $\square$

The next lemma ends our study of 4-vertices of G.

Lemma 16. *The configuration of Figure 8 is reducible.*

Proof. (L24). Note that the vertices b_9 and b_{17} need not be distinct. However, they cannot be the same 3-vertex, otherwise G would have a 3-vertex incident to at least two 4-threads, thereby contradicting Lemma 12. So, we actually obtain a subgraph of minimum degree at least 2 when we remove all the vertices u_i from G.

We set $c(u_i) := i \mod m$ for all indices i. We switch the colours of u_1 and u_2, and of u_9 and u_8 if necessary. Similarly, we switch the colours of u_{16} and u_{17} if $c(b_{17}) = 17 \mod m$. Moreover, if $5 = 10 \mod m$ then we switch the colour of u_5 and u_4, while if $5 = 11 \mod m$ we switch the colours of u_5 and u_6. Note that $4 \notin \{10, 11\}$ in the former case, while $6 \notin \{10, 11\}$ in the latter case. Next, we switch the colours of u_{10} and u_{11} if $c(b_{10}) = 10 \mod m$. The obtained colouring is proper unless $c(b_{13}) = 13 \mod m$. Note that $c(u_i) = i \mod m$ for $i \in \{12, 14, 15\}$. Thus, we can safely switch the colours of u_{13} and u_{14} if $c(b_{13}) = 13 \mod m$, which concludes the proof. $\square$

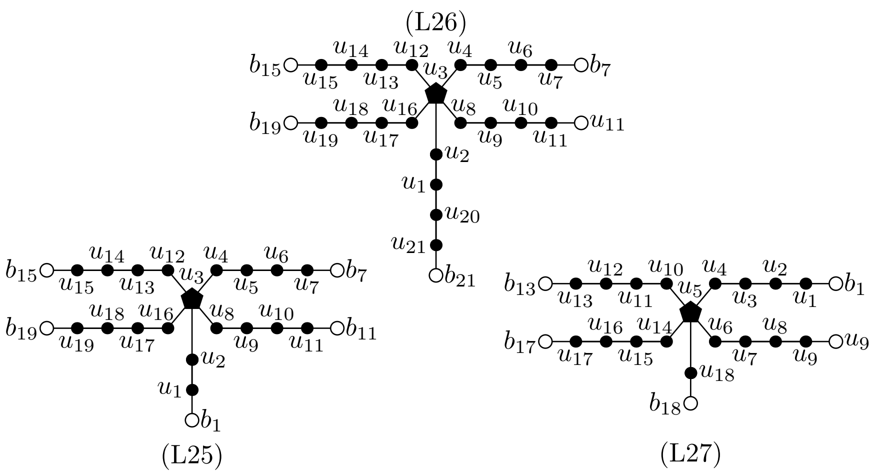


Figure 9: Reducible configurations (L25)–(L27).

2.1.3 Vertices of Degree Five

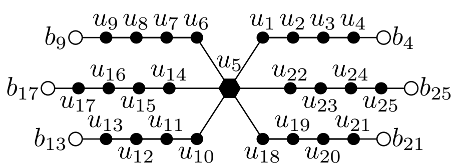
Lemma 17. *The configurations of Figure 9 are reducible.*

Proof. (L25). Suppose first that $m \neq 3$. We properly colour u_1 and u_2 with the colours 1 and 2, and we set $c(u_3) := 3$. For each $i \in \{4, 8, 12, 16\}$, we extend the colouring to the 4-thread $u_i u_{i+1} u_{i+2} u_{i+3}$ by applying Lemma 8 with $\gamma = i$.

Assume now that $m = 3$. We want to properly colour the vertices u_i using 7 times colour 1, and 6 times each of the colours 2 and 3. Thus, without loss of generality, we may assume that $c(b_1) \neq 2 \mod 3$. We set $c(u_3) := c(u_1) := 2$ and $c(u_2) := 1$. Then, we extend the colouring to each of the 4-threads by Lemma 8. More precisely, we apply twice the lemma with γ being 1, and twice with γ being 3.

(L26). We properly colour u_{20} and u_{21} with the colours 20 and 21. Then, we proceed as for (L25).

(L27). If $m \neq 3$, we first properly extend the colouring to u_5 and u_{18} using the colours 1 and 2, and then we use Lemma 8 to extend the colouring to each 4-thread.



(L28)

Figure 10: The reducible configuration (L28).

Assume that $m = 3$. We want to properly colour all the vertices u_i using six times each colour. Hence, we may assume that $c(b_{17}) \neq 2 \mod 3$ and $c(b_{18}) \neq 1 \mod 3$. We set $c(u_{18}) := 1$ and $c(u_5) := 2$. Then, for each $i \in \{1, 6, 10\}$ we colour the 4-thread $u_i u_{i+1} u_{i+2} u_{i+3}$ by applying Lemma 8 with $\gamma = 3$, This is possible since $c(u_5) = 2 \neq 3 \mod 3$. We finish by setting $c(u_{14}) := c(u_{16}) := 1$ and $c(u_{15}) := c(u_{17}) := 2$. $\square$

2.1.4 Vertices of Degree Six

Lemma 18. *The configuration of Figure 10 is reducible.*

Proof. (L28). We set $c(u_5) := 5 \mod m$. Then, using Lemma 8 we successively extend the colouring to each of the 4-threads, provided that either $m \neq 3$, or $m = 3$ and at least one vertex b_i is not coloured 2. If $m = 3$ and all the vertices b_i are coloured 2, we set $c(u_5) := 3$. Then, we apply Lemma 8 to each of the 4-threads, three times with $\gamma = 1$, twice with $\gamma = 2$ and once with $\gamma = 3$, which is possible since $c(u_5) = 3 \neq 2 = c(b_i)$ for all indices i. $\square$

2.2 The Discharging Procedure

Let T be a t-thread for some positive integer t. A vertex v *gives* y *to* T means that v gives $\frac{y}{t}$ units of charge to each 2-vertex of T. A 3-vertex of G is *bad* if it is incident to a 4-thread and two 1-threads. It is *good* if it is not bad.

For every $v \in V$, we define the original charge of v to be $\omega(v) := \deg(v)$. Then, we apply the following discharging rules with $x := \frac{1}{3}$.

(R0) *For every positive integer t, every big vertex gives $t \cdot \frac{x}{2}$ to each of its incident t-threads.*

(R1) *Every big vertex gives $\frac{x}{2}$ to each of its bad* 1*-weak neighbours.*

We show that after applying these rules, the final charge $\omega'(v)$ of every vertex v is at least $\frac{7}{3}$. Thus, we infer that the average degree of G is

$$\sum_{v \in V} \frac{\omega(v)}{|V|} = \sum_{v \in V} \frac{\omega'(v)}{|V|} \geq \frac{7}{3},$$

which contradicts our assumption on the maximum average degree of G.

Recall that G has minimum degree at least 2, no t-thread with $t = 3$ or $t \geq 5$, and no t-loop. Observe that the equitable chromatic number of any cycle is at most 3, thus the maximum degree of G is at least 3. Note that a vertex looses $2x$ units of charge each time it is incident to a 4-thread, and at most x each time it is incident to a t-thread with $t \in \{1, 2\}$.
Let $v \in V$. If $\deg(v) \geq 7$, then the final charge of v is

$$\omega'(v) \geq \deg(v)(1 - 2x) \geq 7 \cdot \frac{1}{3} = \frac{7}{3}.$$

If $\deg(v) = 6$, then v is not incident to six 4-threads by Lemma 18. So, its final charge is

$$\omega'(v) \geq 6 - 5 \cdot 2x - x = \frac{7}{3}.$$

Suppose that $\deg(v) = 5$. The reducibility of the configuration (L26) of Figure 9 ensures that v is not incident to five 4-threads. If v is incident to at most three 4-threads, then

$$\omega'(v) \geq 5 - 3 \cdot 2x - 2 \cdot x = \frac{7}{3}.$$

If v is incident to four 4-threads, then it is adjacent to a big vertex thanks to the reducibility of the configurations (L25) and (L27) of Figure 9. Thus, its final charge is

$$\omega'(v) \geq 5 - 4 \cdot 2x = \frac{7}{3}.$$

Assume that $\deg(v) = 4$. By Lemma 15, the vertex v is incident to at most two 4-threads. If v is incident to at most one 4-thread, then its final charge is $\omega'(v) \geq 4 - 2x - 3 \cdot x = \frac{7}{3}$. So suppose that v is incident to exactly two 4-threads. If v is adjacent to a big vertex, then its final charge is

$$\omega'(v) \geq 4 - 2 \cdot 2x - x = \frac{7}{3}.$$

If v is not adjacent to a big vertex, then we deduce from Lemma 14 that v is incident to two 1-threads. Consequently, we know by Lemma 16 that v has no bad 1-weak neighbour. Therefore, its final charge is

$$\omega'(v) \geq 4 - 2 \cdot 2x - 2 \cdot \frac{x}{2} = \frac{7}{3}.$$

Assume that $\deg(v) = 3$. Suppose first that v is bad. The reducibility of the configuration (L15) of Figure 5 ensures that v has no bad 1-weak neighbour. Thus, v does not send any charge because of Rule R1. Moreover, by Rule R1, the vertex v receives $\frac{x}{2}$ from each of its two 1-weak neighbours. Consequently, its final charge is

$$\omega'(v) \geq 3 - 2x - 2 \cdot \frac{x}{2} + 2 \cdot \frac{x}{2} = \frac{7}{3}.$$

It remains to deal with the case where v is not bad. Suppose that v is incident to a 4-thread. By Lemma 12, the vertex v is incident to exactly one 4-thread. Furthermore, since v is good, we infer from Lemma 11 that v is adjacent to two big vertices. Hence its final charge is $\omega'(v) \geq 3 - 2x = \frac{7}{3}$.

So we assume now that v is not incident to a 4-thread. If v is adjacent to a big vertex, then its final charge is $\omega'(v) \geq 3 - 2 \cdot x = \frac{7}{3}$. If v is not adjacent to a big vertex, we deduce from Lemma 10 that v is incident to at most one 2-thread. Thus, v is either incident to a 2-thread and two 1-threads, or it is incident to three 1-threads. In the former case, the reducibility of the configuration (L16) of Figure 5 ensures that v has no bad 1-weak neighbour. Hence its final charge is

$$\omega'(v) \geq 3 - x - 2 \cdot \frac{x}{2} = \frac{7}{3}.$$

In the latter case, the reducibility of the configuration (L17) of Figure 5 implies that v has at most one bad 1-weak neighbour. Thus, its final charge is

$$\omega'(v) \geq 3 - 3 \cdot \frac{x}{2} - \frac{x}{2} = \frac{7}{3}.$$

To conclude, suppose that $\deg(v) = 2$. So v belongs to a t-thread T for some $t \in \{1, 2, 4\}$. By Rule R0, T receives tx from the big vertices to which it is incident. Thus, each vertex of T receives x, and hence the final charge of v is $2 + x = \frac{7}{3}$.

The proof of Theorem 1 is complete. $\square$

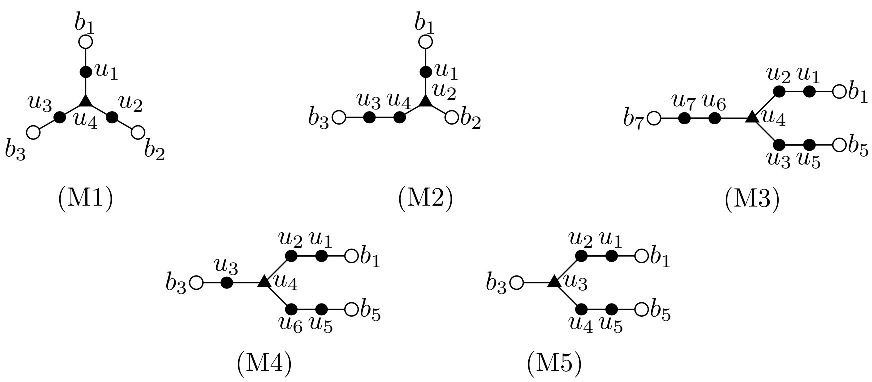


Figure 11: Reducible configurations (M1)–(M5).

3 The Proof of Theorem 2

The proof follows the line of that of Theorem 1. Throughout this section, we assume that $G = (V, E)$ is a counter-example to Theorem 2 with as few vertices as possible. We first exhibit some reducible configurations, and then use a discharging procedure to obtain a contradiction.

For each reducibility proof, we use the same approach as in Subsection 2.1.

3.1 Reducible configurations

We use the following lemma without explicit reference to it. Its proof is the same as that of Lemmas 5, 6, 8 and 9 of Subsection 2.1, and hence we omit it.

Lemma 19. *The graph G has no t-thread with $t \geq 3$ and no t-loop.*

Recall that the girth of G is at least 10, so that in each forthcoming configuration the vertices b_i are pairwise distinct, except for the configuration (M12) of Figure 13.

The next two lemmas exhibit some new reducible configurations concerning 3-vertices of G.

Lemma 20. *The configurations of Figure 11 are reducible.*

(M6) (M7)

Figure 12: Reducible configurations (M6) and (M7).

Proof. (M1) and (M2). We properly colour each vertex u_i one at a time, using once each colour in $\{1, 2, 3, 4\}$. It suffices to finish by the vertex u_4, which is possible since these 4 integers are pairwise distinct modulo m.

(M3). We set $c(u_i) := i \mod m$ for all indices i. The obtained colouring is proper up to switching the colours of u_1 and u_2 if $c(b_1) = 1 \mod m$, the colours of u_7 and u_6 if $c(b_7) = 7 \mod m$, and the colours of u_5 and u_3 if $c(b_5) = 5 \mod m$.

(M4). We set $c(u_i) := i \mod m$ for all indices i. For $i \in \{1, 3, 5\}$, we switch the colours of u_i and u_{i+1} if $c(b_i) = i \mod m$.

(M5). Let $\alpha \in \{3, 4\} \setminus \{c(b_3)\}$, and $\alpha' \in \{3, 4\} \setminus \{\alpha\}$. We set $c(u_3) := \alpha$. Then, we properly colour the 2-thread u_1u_2 with the colours 1 and 2, and the 2-thread u_4u_5 with the colours α' and 5. $\square$

Lemma 21. *The configurations of Figure 12 are reducible.*

Proof. (M6). First, we properly colour the vertices u_1, u_2 and u_3 with the colours $1, 2$ and 3. Next, we properly colour the vertices u_5, u_6 and u_7 with the colours $5, 6$ and 7. Finally, we set $c(u_4) := 4$.

(M7). We first properly colour u_8 and u_9 with the colours 8 and 9. Then, we proceed as for (M6). $\square$

Our third lemma deals with neighbourhoods of 4-vertices of G.

Lemma 22. *The configurations of Figure 13 are reducible.*

Proof. (M8). We set $c(u_7) := 4$. Let $\alpha \in \{1, 2, 3\} \setminus \{c(b_1)\}$, and $\beta \in \{1, 2, 3\} \setminus \{c(b_2), \alpha\}$. We set $c(u_1) := \alpha$ and $c(u_2) := \beta$. Let $\gamma \in \{1, 2, 3\} \setminus \{\alpha, \beta\}$ and $\gamma' \in \{5, 6\} \setminus \{\gamma\}$. We properly colour u_3 and u_4 with γ and γ'. This is possible by Lemma 7 since $\gamma \neq \gamma' \mod m$ and $\gamma \neq c(u_7) = 4 \neq \gamma' \mod m$. Similarly, we properly colour u_5 and u_6 with the two remaining colours.

(M9). Note that if $c(b_i) \neq 8 \mod m$ for some $i \in \{4, 6, 8\}$, then we can colour u_i with 8 and proceed as for (M8). So we may assume that $c(b_i) = 8 \mod m$ for each $i \in \{4, 6, 8\}$. Let $\alpha \in \{3, 4\} \setminus \{8\}$ and $\alpha' \in \{3, 4\} \setminus \{\alpha\}$. We set $c(u_7) := \alpha$, $c(u_4) := 7$ and $c(u_3) := 8$. Next, we set $c(u_5) := 5$ and $c(u_6) := 6$. As $1 \neq \alpha' \mod m$, we can colour u_2 with a colour in $\{1, \alpha'\} \setminus \{c(b_2)\}$. It remains to properly colour the 2-thread u_1u_8 using the two integers in $\{1, 2, \alpha'\} \setminus \{c(u_2)\}$. This is possible by Lemma 7 since $c(u_7) = \alpha \notin \{1, 2, \alpha'\}$.

(M10). If a vertex b_i, say b_9, is not coloured 9, then we colour u_9 with 9 and proceed as for (M9). So we may assume that all the vertices b_i are coloured 9. Let $\alpha \in \{3, 4\} \setminus \{9\}$ and $\alpha' \in \{3, 4\} \setminus \{\alpha\}$. We set $c(u_7) := \alpha$, $c(u_8) := 8$, $c(u_2) := 6$, $c(u_9) := 7$ and $c(u_1) := 9$. It remains to properly colour the two 2-threads u_3u_4 and u_5u_6 using once each integer in $\{1, 2, \alpha', 5\}$. We do so by applying twice lemma 7, which is possible since $c(u_7) = \alpha \notin \{1, 2, \alpha', 5\}$.

(M11). We set $c(u_i) := i$ for $i \leq 3$, and we switch the colours of u_1 and u_2 if $c(b_1) = 1 \mod m$. Next, we properly colour the vertices u_i for $i \in \{4, 5, 6\}$ using each of the integers 4, 5 and 6: we colour first u_4, then u_5 and finally u_6 with the remaining colour. This is possible since $c(u_3) = 3 \notin \{4, 5, 6\}$. Now, we set $c(u_7) := 7$ (note that this is proper since $c(u_6) \neq 7 \mod m$). Finally,

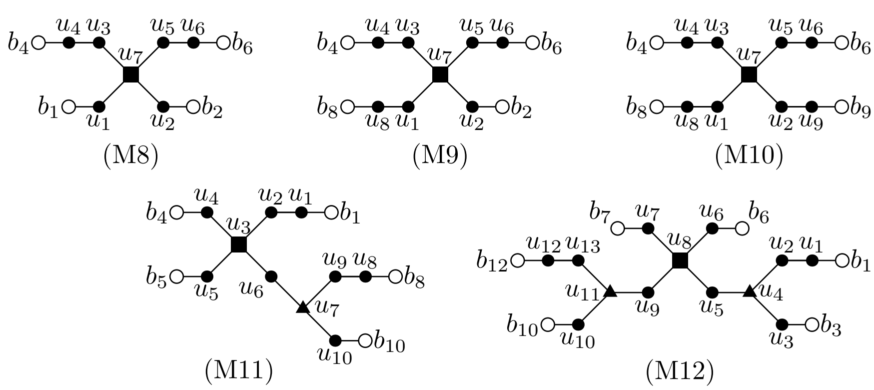


Figure 13: Reducible configurations (M8)–(M12).

we colour the vertices u_8, u_9 and u_{10} using each of the integers $8, 9, 10$. As before, this is possible by colouring u_8 last, since $c(u_7) = 7 \notin \{8, 9, 10\}$.

(M12). The vertices b_1 and b_{12} need not be distinct. However, they cannot be the same 3-vertex, otherwise G would have a 3-vertex incident to at least two 2-threads, thereby contradicting the reducibility of the configuration (M3), (M4) or (M5) of Figure 11. Consequently, the subgraph obtained by removing the vertices u_i from G has minimum degree at least 2.

For each $i \notin \{6, 7, 8\}$, we set $c(u_i) := i \mod m$. For $i \in \{1, 3, 10, 12\}$, we switch the colours of u_i and u_{i+1} if $c(b_i) = i \mod m$. It remains to colour the vertices u_6, u_7 and u_8 using each of the integers $6, 7$ and 8. We properly colour u_6 and u_7 and we finish by u_8, which is possible since $c(u_5) = 5$ and $c(u_9) = 9$ so that neither can be equal to $6, 7$ or 8 modulo m. $\square$

3.2 The Discharging Procedure

Recall that a vertex is *big* if its degree is at least 3. A 2-vertex is *bad* if it belongs to a 2-thread. A 3-vertex is *bad* it it is incident to a 2-thread and two 1-threads. A vertex of degree 2 or 3 is *good* if it is not bad. A 2-vertex is *suspicious* if it is good and adjacent to a bad 3-vertex.

For every $v \in V$, we define the original charge of v to be $\omega(v) := \deg(v)$. Then, we apply the following discharging rules.

(R0) *Every good 3-vertex gives $\frac{1}{2k}$ to each of its k adjacent 2-vertices, for $k \in \{1, 2, 3\}$.*
(R1) *Every bad 3-vertex gives $\frac{1}{2}$ to its adjacent bad 2-vertex.*
(R2) *Every 4-vertex gives $\frac{1}{2}$ to each adjacent bad or suspicious 2-vertex; and $\frac{1}{4}$ to each of its remaining adjacent 2-vertices.*
(R3) *Every vertex of degree at least 5 gives $\frac{1}{2}$ to each of its adjacent 2-vertices.*

We show that after applying these rules, the final charge $\omega'(v)$ of every vertex v is at least $\frac{5}{2}$. Thus, we infer that the average degree of G is

$$\sum_{v \in V} \frac{\omega(v)}{|V|} = \sum_{v \in V} \frac{\omega'(v)}{|V|} \geq \frac{5}{2},$$

which contradicts our assumption on the maximum average degree of G.

Recall that G has minimum degree at least 2, maximum degree at least 3, no t-thread for $t \geq 3$, and no t-loop.

Let $v \in V$. If $\deg(v) \geq 5$, then by Rule R3 the final charge of v is

$$\omega'(v) \geq \deg(v)\left(1 - \frac{1}{2}\right) \geq \frac{5}{2}.$$

Suppose that $\deg(v) = 4$. First suppose that v is adjacent to a bad 2-vertex. As the configuration (M10) of Figure 13 is reducible, the vertex v is adjacent to at most three bad 2-vertices. Hence, if it is adjacent to a big vertex, its final charge is

$$\omega'(v) \geq 4 - 3 \cdot \frac{1}{2} = \frac{5}{2}.$$

Otherwise, the reducibility of the configurations (M8), (M9) and (M11) of Figure 13 implies that v is adjacent to exactly one bad 2-vertex, and zero suspicious vertex. Consequently, its final charge is

$$\omega'(v) \geq 4 - \frac{1}{2} - 3 \cdot \frac{1}{4} > \frac{5}{2}.$$

Now, if v is not adjacent to a bad 2-vertex, then we infer from the reducibility of the configuration (M12) of Figure 13 that either v is adjacent to a big vertex, or it is adjacent to at most one suspicious 2-vertex. Therefore, its final charge is

$$\omega'(v) \geq 4 - 3 \cdot \frac{1}{2} = \frac{5}{2}.$$

If $\deg(v) = 3$, then v sends $\frac{1}{2}$ in total to its neighbours by Rule R0 or R1, so that its final charge is $\frac{5}{2}$, as desired.

Finally, assume that $\deg(v) = 2$. Suppose first that v is bad, and let u be the big neighbour of v. We assert that v receives $\frac{1}{2}$ from u, and hence its final charge is $\frac{5}{2}$. To prove this, we consider several cases regarding the degree and the type of u.

If u has degree at least 4, then it gives $\frac{1}{2}$ to v by Rule R2 or R3.

If u is a bad 3-vertex then it gives $\frac{1}{2}$ to v by Rule R0.

If u is a good 3-vertex, then the reducibility of the configurations (M2), (M3), (M4) and (M5) of Figure 11 along with the definition of a good 3-vertex imply that u is adjacent to two big vertices. Therefore, u gives $\frac{1}{2}$ to v by Rule R0, as asserted.

Assume that v is good. If v is not suspicious, it receives at least $\frac{1}{4}$ from each of its two neighbours according to Rules R0, R2 and R3, thanks to the reducibility of the configuration (M1) of Figure 11. Hence its final charge is at least $\frac{5}{2}$. If v is suspicious and adjacent to a vertex of degree at least 4, then its final charge is $\frac{5}{2}$ by Rules R2 and R3. Otherwise, v is adjacent to a bad 3-vertex and another 3-vertex u. The reducibility of the configuration (M7) of Figure 12 implies that u is good. Hence, we infer from the reducibility of the configurations (M2) and (M4) of Figure 11 that u is not adjacent to a bad 2-vertex. Furthermore, the reducibility of the configurations (M1) of Figure 11 and (M6) of Figure 12 implies that u is adjacent to two big vertices. Therefore, by Rule R0, v receives $\frac{1}{2}$ from u. Thus its final charge is $\frac{5}{2}$, as wanted.
The proof of Theorem 2 is complete. □

4 Concluding Remarks

Our setting allows us to derive results not only for planar graphs with minimum degree at least 2 and lower-bounded girth, but more generally for graphs of minimum degree at least 2 with bounded genus and lower-bounded girth. Such results can be obtained from Theorems 1 and 2 using Euler's generalised formula and a theorem of Alon, Hoory and Linial [1, Theorem 1].

The technique used can probably be pushed further, and in particular for equitable chromatic number 5 or more (those cases though are beyond our original motivation). However, it would be more interesting and challenging to first find a more systematic way of proving reducibility of configurations. In that regards, a useful notion could be the list-version of equitable colouring introduced by Kostochka, Pelsmajer and West [10], which is worth studying.

Finding planar graphs of minimum degree 2, equitable chromatic number 4 and the largest possible girth would also be interesting.

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