# Distance constrained labelings of trees

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**Abstract**

An $H(p,q)$-labeling of a graph $G$ is a vertex mapping $f : V_G \to V_H$ such that the distance (in the graph $H$) of $f(u)$ and $f(v)$ is at least $p$ (at least $q$) if the vertices $u$ and $v$ are adjacent in $G$ (are at distance two in $G$, respectively). This notion generalizes the notions of $L(p,q)$- and $C(p,q)$-labelings of graphs studied as a graph model of the Frequency Assignment Problem. We study the computational complexity of the problem of deciding the existence of such a labeling when the graphs $G$ and $H$ come from restricted graph classes. In this way we are extending known results for linear and cyclic labelings of trees, with a hope that our results would help to open a new angle of view on the still open open problem of $L(p,q)$-labeling of trees for fixed $p > q > 1$ (i.e., when $G$ is a tree and $H$ a path). Our main results are a polynomial time algorithm for $H(p,1)$-labeling of trees for arbitrary $H$, and NP-completeness results for $H(p,q)$-labeling of trees when $H$ is a $q$-caterpillar, and $L(p,q)$-labeling of trees for fixed $q > 1$ and $p$ part of input.

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## 1 Introduction

Motivated by models of wireless communication, the notion of so called distance constrained graph labelings has received a lot of interest in Discrete Mathematics and Theoretical Computer Science in recent years.

In the simplest case of constraints at distance two, the typical task is, given a graph $G$ and parameters $p$ and $q$, to assign integer labels to vertices of $G$ such that labels of adjacent vertices differ by at least $p$, while vertices that share a common neighbor have labels at least $q$ apart. The aim is to minimize the span, i.e., the difference between the smallest and the largest label used.

The notion of proper graph coloring is a special case of this labeling notion — when $(p,q) = (1,0)$. Thus it is generally $\mathsf{NP}$-hard to decide whether such a labeling exists. On the other hand, in some special cases polynomial-time algorithms exist. For example, when $G$ is a tree and $p \geq q = 1$, the algorithm of Chang and Kuo based on dynamic programming finds a labeling of the minimum span [3, 6] in polynomial (but not linear!) time.

Distance constrained labelings can be generalized in several ways. For example, constraints on longer distances can be involved [18, 14], or the constraints on the difference of labels between close vertices can be directly implemented by edge weights — the latter is referred to as the Channel Assignment Problem [22].

Alternatively, more complex metrics on the label set are considered, for example, as a distance between vertices in a graph $H$. Then the labeling becomes a special case of graph homomorphism with distance constraints:

**Definition 1.** Given two positive integers $p$ and $q$, we say that $f$ is an $H(p,q)$-labeling of a graph $G$, if $f$ maps the vertices of $G$ onto vertices of $H$ such that the following holds:

- if $u$ and $v$ are adjacent in $G$, then $dist_H(f(u), f(v)) \geq p$,
- if $u$ and $v$ are nonadjacent but share a common neighbor, then the distance $dist_H(f(u), f(v)) \geq q$.

Observe that if the graph $H$ is a path, the ordinary linear metric is obtained. This has been introduced by Roberts and studied in a number of papers — see, e.g., recent surveys [23, 1]. The cyclic metric (corresponding to the case when $H$ is a cycle) was studied in [17, 20]. The general approach was suggested in [8] and several (both P-time and NP-hardness) results for various fixed graphs $H$ were presented in [7, 10].

Several algorithmic problems can be defined by restricting the graph classes of the input graphs, and/or fixing some values as parameters of the most general problem which we refer to as follows:

| Distance Labeling | DL |
|---|---|
| *Instance:* $G$, $H$, $p$ and $q$ | |
| *Question:* Does $G$ allow an $H(p,q)$-labeling? | |

As already mentioned, the linear metric is often considered, i.e. when $H = P_{\lambda+1}$ is a path of length $\lambda$. In this case we use the traditional notation "$L(p,q)$-labeling of span $\lambda$" for "$P_{\lambda+1}(p,q)$-labeling" and also define the problem explicitly:

| $L(p,q)$-Distance Labeling | $L(p,q)$-DL |
|---|---|
| *Parameters:* $p, q$ | |
| *Instance:* $G$ and $\lambda$ | |
| *Question:* Does $G$ allow an $L(p,q)$-labeling of span $\lambda$? | |

We focus our attention on various parameterized versions of the DL problem. Griggs and Yeh [15] showed NP-hardness of the $L(2,1)$-DL of a general graph, which means that $DL$ is NP-complete for fixed $(p,q) = (2,1)$ and $H$ being restricted to paths. Later Fiala et al. [9] showed that the $L(2,1)$-DL problem remains NP-complete for every fixed $\lambda \geq 4$. Similarly, the labeling problem with the cyclic metric is NP-complete for a fixed span [8], i.e., when $(p,q) = (2,1)$ and $H = C_\lambda$ for an arbitrary $\lambda \geq 6$.

From the very beginning it was noticed that the distance constrained labeling problem is in certain sense more difficult than ordinary coloring. The first polynomial time algorithm for $L(2,1)$-DL for trees came as a little surprise [3]. Attention was then paid to input graphs that are trees and their relatives, either paths and caterpillars as special trees, or graphs of bounded treewidth. Table 1 briefly summarizes the known results on the complexity of the DL problem on these graph classes. Notice in particular that $L(2,1)$-DL belongs to a handful of problems that are solvable in polynomial time on graphs of treewidth one and NP-complete on treewidth two [5].

We start with two observations. For the seventh line, the algorithms of Chang and Kuo [3] can be easily modified to work for arbitrary $p$ and $H$ on the input. It only suffices to modify all tests whether labels of adjacent vertices have difference at least two into testing whether they are mapped onto vertices at distance at least $p$ in $H$.

| **Class of $G$** | **Class of $H$** | $p$ | $q$ | **Complexity** |
|---|---|---|---|---|
| bounded $tw$ | finite class | on input | on input | P [$*$] |
| $tw \leq 2$ | paths | 2 | 1 | NP-c [5] |
| $tw \leq 2$ | cycles | 2 | 1 | NP-c [5] |
| $pw \leq 2$ | all graphs | 2 | 1 | NP-c [5] |
| stars | all graphs | fixed | fixed, $\geq 2$ | NP-c [$*$] |
| trees, $\mathcal{L}$ | paths | fixed | fixed, $\geq 2$ | NP-c [11] |
| trees | all graphs | on input | 1 | P [3, 2, $*$] |
| trees | cycles | on input $\geq q$ | on input $\geq 1$ | P [20, 19] |
| trees | $q$-caterpillars | fixed, $\geq 2q+1$ | fixed, $\geq 2$ | NP-c [$*$] |
| trees | paths | on input | fixed, $\geq 2$ | NP-c [$*$] |
| **trees** | **paths** | **fixed** | **fixed, $\geq 2$** | **Open** |

The symbol $tw$ means treewidth; $pw$ is pathwidth; $\mathcal{L}$ indicates the list version of the DL problem where every vertex has prescribed set (list) of possible labels; P are problems solvable in polynomial time; NP-c are NP-complete problems; the reference [$*$] indicates results of this paper.

Table 1: Summary of the complexity of the DL problem

The first line of the table is a corollary of a strong theorem of Courcelle [4], who proved that properties expressible in MSOL can be recognized in polynomial time on any class of graphs of bounded treewidth. If $H$ belongs to a finite graph class, then the graph property "to allow an $H(p,q)$-labeling" straightforwardly belongs to MSOL. A special case arises, of course, if a single graph $H$ is a fixed parameter of the problem.

In particular, if $\lambda$ is fixed, then $L(p,q)$-DL is polynomially solvable for trees. On the other hand, without this assumption on $\lambda$ the computational complexity of $L(p,q)$-DL on trees remains open so far. It is regarded as one of the most interesting open problems in the area.

## 2 Preliminaries

Due to space restrictions some auxiliary proofs of this and further sections were postponed to the Appendix A.

Throughout the paper the symbol $[a,b]$ means the interval of integers between $a$ and $b$ (including the bounds), i.e., $[a,b] := \{a, a+1, \ldots, b\}$. We also define $[a] := [1,a]$. The relation $i \equiv_q j$ means that $i$ and $j$ are congruent modulo $q$, i.e., $q$ divides $i - j$. For $i \equiv_q j$ we define $[i,j]_{\equiv q} := \{i, i+q, i+2q, \ldots, j-q, j\}$. A set $M$ of integers is *$t$-sparse* if the difference between any two elements of $M$ is at least $t$. We say that a set of integers $M$ is *$\lambda$-symmetric* if for every $x \in M$, it holds that $\lambda - x \in M$.

All graphs are assumed to be finite, undirected, and simple, i.e., without loops or multiple edges. Throughout the paper $V_G$ stands for the set of vertices, and $E_G$ for the set of edges, of a graph $G$.

We use standard terminology: a *path* $P_n$ is a sequence of $n$ consecutively adjacent vertices (its length being $n-1$); a *cycle* is a path where the first and the last vertex are adjacent as well; a graph is *connected* if each pair of vertices is joined by a path; the *distance* between two vertices is the length of a shortest path that connects them; a *tree* is a connected graph without a cycle; and a *leaf* is a vertex of degree one. For precise definitions of these terms see, e.g., the textbooks [21, 16]. A *$q$-caterpillar* is a tree that can be constructed from a path, called the *backbone*, by adding new disjoint paths of length $q$, called *legs*, and merging one end of each leg with some backbone vertex.

As a technical tool for proving NP-hardness results we use the following problem of finding distant representatives:

| | |
|---|---|
| SYSTEM OF $q$-DISTANT REPRESENTATIVES | S$q$-DR |
| *Parameter:* $q$ | |
| *Instance:* A collection of sets $M_i, i \in [m]$ of integers. | |
| *Question:* Is there a collection of elements $u_i \in M_i$, $i \in [m]$ that pairwise differ by at least $q$? | |

It is known that the S1-DR problem allows a polynomial time algorithm (by finding a maximum matching in a bipartite graph), while for all $q \geq 2$ the S$q$-DR problem is NP-complete, even if each set $M$ has at most three elements [12].

We conclude this section with several observations specific to the $L(p, q)$-labelings. For convenience, we represent vertices of $P_{\lambda+1}$ by integers of the interval $[0, \lambda]$. Now an $L(p, q)$-labeling $f$ of a graph $G$ is a mapping $V_G \to [0, \lambda]$ with distance constraints analogous to those of Definition 1.

If $f$ is an $L(p, q)$-labeling of span $\lambda$ then the "reversed" mapping $f'$ defined by $f'(u) := \lambda - f(u)$ is a valid $L(p, q)$-labeling as well. Hence, if we chase for a specific graph construction where only a fixed label is allowed on a certain vertex, we can not avoid symmetry of labelings $f$ and $f'$. For that purpose we need a stronger concept of systems of $q$-distant representatives.

**Lemma 2.** *For any $q \geq 2$ and $t \geq q$, the* S$q$-DR *problem remains* NP-*complete even when restricted to instances whose sets are of size at most 6, $t$-sparse and $\lambda$-symmetric for some $\lambda$.*

# 3 Distance labeling of stars

In the next two sections we consider the variant when both $p$ and $q$ are fixed:

| | |
|---|---|
| $(p, q)$-DISTANCE LABELING | $(p, q)$-DL |
| *Parameters:* $p$ and $q$ | |
| *Instance:* $G$ and $H$ | |
| *Question:* Does $G$ allow an $H(p, q)$-labeling? | |

We show that this problem becomes hard already if $H$ belongs to a very simple class of graphs, namely to the class of stars.

**Theorem 3.** *For any $p \geq 0$ and $q \geq 2$, the $(p, q)$-DL problem is* NP-*complete even when the graph $G$ is required to be a star.*

*Proof.* We reduce the Independent Set (IS) problem, which for a given graph $G$ and an integer $k$ asks whether $G$ has $k$ pairwise nonadjacent vertices. The IS problem is well known to be $\mathsf{NP}$-complete ([13], problem GT20).

Let $H_0$ and $k$ be an instance of IS. We construct a graph $H$ such that the star $K_{1,k}$ has an $H(p,q)$-labeling if and only if $H_0$ has $k$ independent vertices.

The construction of $H$ goes in three steps:

Firstly, if $q = 2$ then we simply let $H_1 := H_0$ and $M := V_{H_0}$. Otherwise, i.e. for $q \geq 2$, we replace each edge of $H_0$ by a path of length $q-1$ to obtain $H_1$. Now let $M$ be the set of the middle points of the replacement paths (i.e., $M$ is of size $|E_{H_0}|$ when $q$ is odd; otherwise $M$ is twice bigger).

For the second step we first prepare a path of length $\max\{0, \lceil \frac{2p-q-1}{2} \rceil\}$ (observe that this path consists of a single vertex when $2p \leq q+1$). We make one its end adjacent to all vertices in the set $M$. We denote the other end of the path by $w$.

Finally, if $q$ is odd, we insert new edges to make adjacent all vertices in $M$, i.e., the set $M$ now induces a clique. This concludes the construction of the graph $H$.

Properties of $H$ can be summarized as follows:

- If two vertices were adjacent in $H_0$, then they have in $H$ distance $q-1$. Analogously, they have distance $q$ if they were non-adjacent.
- Every original vertex is at distance at least $p$ from $w$, and this bound is attained whenever $2p \geq q-1$.
- If $p \leq q$ then every newly added vertex except $w$ is at distance less than $q$ from any other vertex of $H$.
- If $p \geq q$ then every newly added vertex is at distance less than $p$ from $w$.

Straightforwardly, if $H_0$ has an independent set $S$ of size $k$ then we map the center of $K_{1,k}$ onto $w$ and the leaves of $K_{1,k}$ bijectively onto $S$. This yields a valid $H(p,q)$-labeling of $K_{1,k}$.

For the opposite implication assume that $K_{1,k}$ has an $H(p,q)$-labeling $f$. We distinguish three cases:

- When $p < q$, then the images of the leaves of $K_{1,k}$ are pairwise at distance $q$ in $H$. Hence, by the properties of $H$, these are $k$ original vertices that form an independent set in $H_0$.

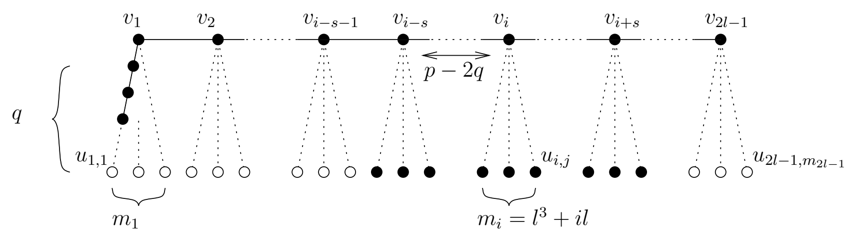


Figure 1: Construction of the target tree $H_l$.
White vertices define $n_i$ as well as a lower bound on $r(u_{i,j})$.

- If $p = q$, then the $q$-distant vertices of $H$ are some nonadjacent original vertices together with the vertex $w$. As the image of the center of $K_{1,k}$ belongs into this set as well, $H_0$ has at least $k$ independent vertices.
- If $p > q$, then $w$ is the image of the center of $K_{1,k}$ (unless $k = 1$, but then the problem is trivial). Analogously as in the previous cases, images of the leaves of $K_{1,k}$ form an independent set of $H_0$.

□

# 4 Distance labeling between two trees

In this section we show that the DL problem is NP-complete for any $q \geq 2$ and $p \geq 2q+1$ even when both graphs $G$ and $H$ are required to be trees. Before we state the theorem, we describe the target graph $H$ and explore its properties.

Let $p$ and $q$ be given, such that $q \geq 2$ and $p \geq 2q+1$. Assume that $l > 2(p-q)$ and for $i \in [l]$ define $m_i := l^3 + il$. For convenience we also let $m_i := m_{2l-i} = l^3 + (2l-i)l$ for $i \in [l+1, 2l-1]$.

We construct a graph $H_l$ as follows: We start with a path of length $2l-2$ on vertices $v_1, \ldots, v_{2l-1}$, called backbone vertices. For each vertex $v_i, i \in [2l-1]$ we prepare $m_i$ paths of length $q$ and unify one end of each of these $m_i$ paths with the vertex $v_i$. By symmetry, every vertex $v_{2l-i}$ has the same number of pending $q$-paths as the vertex $v_i$. Observe that the resulting graph depicted in Fig. 1 is a $q$-caterpillar.

For $i \in [2l-1]$ and $j \in [m_i]$ let $u_{i,j}$ denote the final vertex of the $j$-th path hanging from the vertex $v_i$.

Observe that the total number of leaves in $H_l$ is $2l^4$.

We define variable $s := p - 2q$ to shorten some expressions.

For $i \in [2l-1]$ let $n_i$ be the number of leaves of $H_l$ at distance at least $p-q$ from $v_i$, i.e.,

$$n_i := \sum_{j \in S_i} m_i = \begin{cases} 2l^4 - (s+i)l^3 - (s+2i)il + \frac{ls(s+1)}{2} & \text{if } i \in [s-1], \\ 2l^4 - (2s+1)l^3 - (2s+1)il & \text{if } i \in [s, l-s], \\ 2l^4 - (2s+1)l^3 - (2s+1)il + \\ \quad + (s+i-l)(s+i-l+1)l & \text{if } i \in [l-s+1, l], \end{cases}$$

where $S_i := [2l+1] \setminus [i-s, i+s]$. By symmetry, $n_i := n_{2l-i}$ for $i \in [l+1, 2l-1]$. Observe that the sequence $n_1, \ldots, n_{l-s}$ is decreasing.

For a vertex $u \in V_{H_l}$ we further define $r(u)$ to be the maximum size of a set of vertices of $H_l$ that are pairwise at least $q$ apart, and that are also at distance at least $p$ from $u$. In other words, $r(u)$ is an upper bound on the degree of a vertex which is mapped onto $u$ in an $H_l(p,q)$-labeling.

We claim that for the leaves $u_{i,j}$ with $i \in [l], j \in [m_i]$ the $r(u_{i,j})$ can be bounded by

$$n_i \le r(u_{i,j}) \le n_i + \frac{2l-p}{q},$$

since the desired set can be composed from the $n_i$ leaves that are at distance at least $p-q$ from $v_i$ together with suitable backbone vertices. Consequently, $r(u_{i,j}) < n_{i-1}$ for $i \in [2, l-s]$.

Finally, observe that if $u$ is a non-leaf vertex of $H$, then $r(u) < n_{l-s}$, since with every step away from a leaf decreases the size of the set of $p$ distant vertices by the factor of $\Omega(l^3)$.

By the properties of values $n_i$ and $r(u)$ we get that:

**Lemma 4.** *For given $p, q$ and $l$ such that $p \ge 2q+1$ and $l > 2(p-q)+1$, let $T$ be a tree of three levels such that for every $i \in [l-s]$ and $j \in [m_i]$ the root $y$ of $T$ has two children $x_{i,j}, x_{2l-i,j}$, both of degree $n_i$.*

*Every $H_l(p,q)$-labeling $f$ of $T$ satisfies that $f(y) \in \{u_{l,j} \mid j \in [m_l]\}$, and*

$$\forall i \in [l-s] : \{f(x_{i,j}), f(x_{2l-i,j}) \mid j \in [m_i]\} = \{u_{i,j}, u_{2l-i,j} \mid j \in [m_i]\}.$$

*In addition, $T$ has an $H_l(p,q)$-labeling such that the leaves of $T$ are mapped onto the leaves of $H_l$.*

For $i \in [l-s]$ let $T_i$ denote the tree $T$ rearranged such that its root is one of the children of $x_{i,1}$.

We are ready to prove the main theorem of this section.

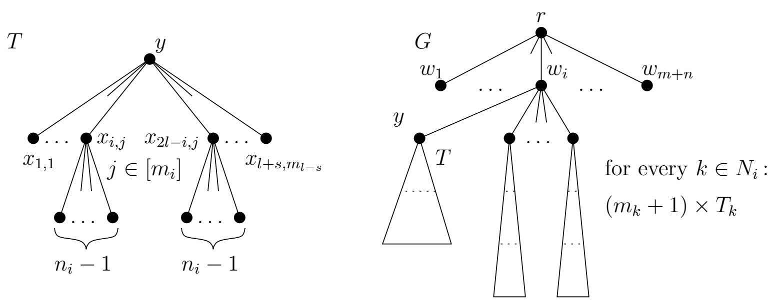


Figure 2: Construction of trees $T$ and $G$.

**Theorem 5.** *For any $q \geq 2$ and $p \geq 2q + 1$, the $(p, q)$-DL problem is NP-complete, even if both graphs on the input are trees.*

*Proof.* We reduce the 3-SAT problem and extend the reduction to the S$q$-DR problem exposed in Lemma 2.

For a formula with $n$ variables we set $\alpha := p-q+1$, $\beta := 2p-3q$, $t := p$, $l := \alpha + (n-1)p + nq + \beta$ and $\lambda := 2l$. According to these parameters we build graph $H_l$ and transform the given formula into a collection of $t$-sparse sets $M_i, i \in [m+n]$.

We construct the tree $G$ of six levels as follows: the root $r$ has children $w_i, i \in [m+n]$, representing sets $M_i$.

For each $i \in [m+n]$ let $N_i$ be the set containing all numbers of $[l-s]$ that are at least $p-q$ apart from any number of $M_i$. Formally, $N_i := [l-s] \setminus \bigcup_{j \in M_i} [j-p+q+1, j+p-q-1]$.

For each $k \in N_i$ we take $m_k + 1$ copies of the tree $T_k$ and add $m_k + 1$ edges to the roots of these trees become children of the vertex $w_i$. Finally, we insert a copy of the tree $T$ and insert a new edge so that the root of this $T$ is also a child of $w_i$.

We repeat the above construction for all $i \in [m+n]$ to obtain the desired graph $G$. (See Fig. 2.)

We claim that if $f$ is an arbitrary $H_l(p,q)$-labeling of $G$ then every vertex $w_i$ is mapped on some vertex $v_j$ with $j \in M_i$.

The child of $w_i$, which is the root $y$, maps on some $u_{l,j}$. Also the children of $y$ map onto all leaves of form $u_{i,j}, i \in [l-p+q], j \in [m_i]$. Hence, the image of $w_i$ is one of the backbone vertices $v_i$ with $i \in [l-p+q] \cup [l+p-q, 2l-1]$.

On the other hand, for any $k \in N_i$ the image of $w_i$ is at least $p$ apart from some $u_{k,l}$ as well as from some $u_{2l-k,l'}$ with $l, l' \in [m_k]$. This follows from the fact that $w_i$ has in $T_k$ more children than there are the leaves under $v_k$ (or under $v_{2l-k}$), so both $k$ and $l-k$ appear as the first index of the leaf which is the image of a child of $w_i$. This proves the claim.

Therefore, the existence of such mapping $f$ yields a valid solution of the original 3-SAT and S$q$-DR problems.

In the opposite direction observe that any valid solution of the S$q$-DR problem transforms naturally to the mapping on vertices $w_i, i \in [m+n]$. We extend this partial mapping onto the remaining vertices of $G$ such that the root $r$ is mapped onto $u_{1,1}$, all vertices $y$ onto $u_{l,1}$. The copies of trees $T$ are labeled as described in Lemma 4.

For every $w_i$ we map its children in copies of $T_k$ onto distinct vertices of the set $\{u_{k,j}, u_{2l-k,j} \mid\ j \in [m_k]\} \setminus \{u_{1,1}, u_{l,1}\}$. Then we extend the labeling onto the entire copy of each $T_k$ like in Lemma 4 without causing any conflicts with other labels. In particular, every child of $w_i$ in $T_k$ is of degree $n_k + 1$ and its children are labeled by leaves of $H_l$, while its parent (the vertex $w_i$) by a backbone vertex. $\square$

## 5 The tree labeling problem with $q$ fixed

In this section we show that the distance labeling problem remains NP-complete even if $G$ is a tree, $H$ is a path, and $q \geq 2$ is fixed. The price we pay for this result is that we have to allow $p$ to be part of the input. Thus we consider this problem:

| $L(\bullet, q)$-DISTANCE LABELING | $L(\bullet, q)$-DL |
|---|---|
| *Parameter:* $q$ | |
| *Instance:* $G, p$ and $\lambda$ | |
| *Question:* Does $G$ allow an $L(p, q)$-labeling of span $\lambda$? | |

**Theorem 6.** *For any $q \geq 2$, the $L(\bullet, q)$-DL problem is NP-complete, even when the input is restricted to the class of trees.*

To prove this theorem, it is necessary to show that certain labels can be forced. Due to space restrictions auxiliary constructions of "forcing" rooted trees $T_j^<$ and $T_j$ were moved to the Appendix B together with Lemmas 9–11, which describe properties of these trees.

*Proof of Theorem 6.* We reduce the 3-SAT problem, following the guidelines of the proof of Lemma 2. For given $q$ and a formula with $n$ variables we set $p := (4n+1)q + \lceil \frac{q}{2} \rceil$, $\lambda := 2p + (16n+1)q$, $t := 3q$, $a_i := p + 4iq$, $b_i := p - r + (4i+1)q$ and construct the $n+m$ many $\lambda$-symmetric sets $M_i$.

We now build our final tree $T$ of eight levels which allows a $L(p,q)$-labeling of span $\lambda$ if and only if the sets $M_i$ have a system of $q$-distant representatives (i.e. the original formula can be satisfied).

The root $r$ of $T$ has $m+n$ children $w_i$, one for each set $M_i$. The construction of the subtree rooted below $w_i$ depends on the content of $M_i$.

For $i \in [n]$ the set $M_{m+i}$ consists of two elements $b_i$ and $\lambda - b_i$, hence we simply add $T_{b_i}$ and unify its root with the vertex $w_{m+i}$.

For $i \in [m]$ the set $M_i$ consists either of four or six elements. Denote them in the increasing order $c, e, e', c'$ or $c, d, e, e', d', c'$ respectively. We further denote by $j$ the largest lower bound on $c$ such that $j \equiv_q p$ and by $k$ the smallest upper bound on $e$, also congruent with $p$ modulo $q$. We set $l := \frac{\lambda - 2p}{q} - 2k - 1$.

We also prepare a set $N_i$ of integers such that:

- $N_i \subset [j-q, k+q]$,
- $c-q, c+q, e-q, e+q$ are elements of $N_i$, and if $|M_i| = 6$ then also $d-q, d+q$,
- $c, d, e \notin N_i$,
- $N_i$ is inclusion-wise maximal,
- all elements of $N_i$ are congruent to 0 or $p$ modulo $q$, and
- elements of $N_i$ are at least $q$ apart.

Observe that since the differences between $c, e$ (and $d$) are at least $3q$ then such a set always exists.

We now take a copy of the tree $T_j^<$ together with $l$ copies of $T_{k+q}^<$, an extra vertex $z_i$ and for each element of $a \in N_i$ two copies of the tree $T_a$.

We merge the root of $T_j^<$ with the vertex $w_i$. Then we rebuild the first $l+1$ trees, so they share the vertex $v_0$ as was described in the statement of Lemma 10. Finally, we connect the vertex $z_i$ by new edges to the vertex $w_i$ and also to all roots of the $2|N|$ trees $T_a$.

We repeat the above construction for all $i \in [n]$ to obtain the final tree $T$ (see Fig. 5 in Appendix B). Observe that as the tree has eight levels. As all degrees are bounded by $12n$, the tree size is $O(n^7)$, so the reduction is polynomial. (By a closer look, it is of size $O(n^5)$.)

Assume that an $L(p,q)$-labeling of span $\lambda$ of $T$ exists. We argue that each $w_i$ receives label from the set $M_i$. The tree $T_j^<$ with trees $T_{k+q}^<$ assure that the label of $w_i$ belongs to the set $[j,k] \cup [\lambda - k, \lambda - j]$ by arguments already used in Lemmas 10 and 11.

The forced labels of the children of $u$ assure that the label of $w_i$ belongs to the set $M_i$, since any other label of $[j,k] \cup [\lambda - k, \lambda - j]$ is at distance less than $q$ from some label from the set $N$ or from a $\lambda$-symmetric label.

As vertices $w_i$ are at distance two, their labels must be at least $q$ apart. Hence, they provide a feasible solution to the S$q$-DR problem and consequently also for the 3-SAT problem.

For the opposite direction we first assign vertices $w_i, i \in [n+m]$ the labels provided by a solution of the S$q$-DR problem. It remains to label the rest of the tree. It is possible to label $r$ by $8(n+1)q + p$, all vertices $z_i$ by $12(n+1)q$ and give all other vertices labels enforced by the constructions of the particular subtrees (consult for details Lemmas 9–11). □

# 6 Conclusion

In this paper we have studied the computational complexity of the $H(p,q)$-labeling problem when both the input graph $G$ and the label space graph $H$ are trees. As the main NP-hardness result we see the result that $L(p,q)$-labeling is NP-complete for trees if $p$ is part of the input and $q \geq 2$ fixed. This could hopefully pave the way to the solution of the $L(p,q)$-labeling of trees (with both $p$ and $q$ fixed), which is the most interesting open problem in the area of computational complexity of distance constrained labeling problems. Another persistent open problem is the complexity of the $L(2,1)$-labeling problem for graphs of bounded path-width.

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# Appendix

## A Proofs from Sections 2 and 4

*Proof of Lemma 2.* We extend the construction from [12] where an instance of the 3-SATISFIABILITY (3-SAT) problem (a well known NP-complete problem, see [13], problem L02) was transformed into an instance of the S$q$-DR problem as follows:

Assume that the given Boolean formula is in the conjunctive normal form and consists of $m$ clauses, each of size at most three, over $n$ variables, each with one positive and two negative occurrences.

- The three literals for a variable $x_i$ are represented by a triple $a_i, b_i, a_i + q$ such that $a_i < b_i < a_i + q$. The number $b_i$ represents the positive literal, while $a_i, a_i + q$ the negative ones.
- Triples representing different variables are at least $t$ apart (e.g., the elements $a_i$ form an arithmetic progression of step $q + t$).
- The sets $M'_i, i \in [m]$ represent clauses and are composed from at most three numbers, each uniquely representing one literal of the clause.

The equivalence between the existence of an satisfying assignment and the existence of a set of $q$-distant representatives is straightforward (for details see [12]).

Without loss of generality assume that there are positive integers $\alpha$ and $\beta$ such that $M'_i \subset [\alpha, \beta]$ for every $i \in [m]$ (we may assume that these bounds are arbitrarily high, but sufficiently apart).

We set $\lambda := 2\beta + t$ and construct the family of sets $M_i, i \in [m + n]$ as follows:

- for $i \in [m] : M_i := \{a, \lambda - a : a \in M'_i\}$,
- for $i \in [n] : M_{m+i} := \{b_i, \lambda - b_i\}$.

If we choose the representatives for the sets $M_{m+i}, i \in [n]$ arbitrarily, and exclude infeasible numbers from the remaining sets, then the remaining task is equivalent to the original instance $M'_i, i \in [m]$ of the S$q$-DR problem. $\square$

*Proof of Lemma 4.* Assume by induction that all vertices $x_{k,j}$ and $x_{2l-k,j}$ with $k < i$ are mapped onto the set $W = \{u_{k,j}, u_{2l-k,j} \mid k < i, j \in [m_k]\}$.

Then vertices $x_{i,j}$, $x_{2l-i,j}$ with $j \in [m_k]$ must be mapped onto vertices that are at least $q$ apart from $W$, i.e. on the backbone vertices or inside a path under some $v_{i'}$ with $i' \in [i, 2l-i]$. Among those vertices only leaves $u_{i,j}$ and $u_{2l-i,j}$ satisfy $r(u_{i,j}) = r(u_{2l-i,j}) \geq n_i$, and can be used as images for $x_{i,j}$, $x_{2l-i,j}$.

When the labels of all $x_{i,j}$ are fixed, the root must be mapped onto a vertex that is at distance at least $p$ from all $u_{i,j}$ with $i \leq l - p + 2q$ or $i \geq l + p - 2q$. The only such vertices are $u_{l,j}$ with $j \in [m_l]$.

If the first two levels of $T$ are partially labeled as described above then the children of $x_{i,j}$ can be labeled by vertices $u_{k,j}$ with $|k - i| > s$, only the label of the root $y$ must be avoided. This provides a valid $H_l(p,q)$-labeling of $T$. □

# B Auxiliary constructions of Section 5

Through this section assume that $p$ and $\lambda$ are sufficiently large, so the derived variables shown later have feasible values. Exact expressions for $p$ and $\lambda$ will be given at the beginning of the proof of Theorem 6.

In the context of fixed $p, q$ and $\lambda$ we define the term *label set* $\Lambda(u)$ of a vertex $u \in V_G$ to be the union of labels of $u$ taken over all $L(p,q)$-labelings of $G$ of span $\lambda$.

We use variables $r, r', s$ and $t$ to abbreviate frequent expressions. Let $r$ denotes the remainder of $p$ divided by $q$. Analogously, $r'$ is the remainder of $\lambda - p$ divided by $q$.

We further set $s := \lfloor \frac{p-2q-r'-1}{q} \rfloor$ and $t := \lfloor \frac{\lambda-p}{q} \rfloor + 1$. Observe that $t$ is the maximum number of $q$-distant numbers on $[p, \lambda]$, i.e., $t$ is an upper bound on the maximum degree of a vertex in a graph that allows an $L(p,q)$-labeling of span $\lambda$.

We first force labels close to 0 or $\lambda$.

**Observation 7.** *Let be given $p, q$ and $\lambda$ such that $s \geq 0$ and $t \geq s + 2$. Then for any $i \in [0, s]$ the center of the star $K_{1,t-i}$ has the label set $[0, iq + r'] \cup [\lambda - iq - r', \lambda]$.*

*Proof.* Suppose on the contrary that the label $x$ of the center of the star belongs to the interval $[iq + r' + 1, \lambda - iq - r' - 1]$.

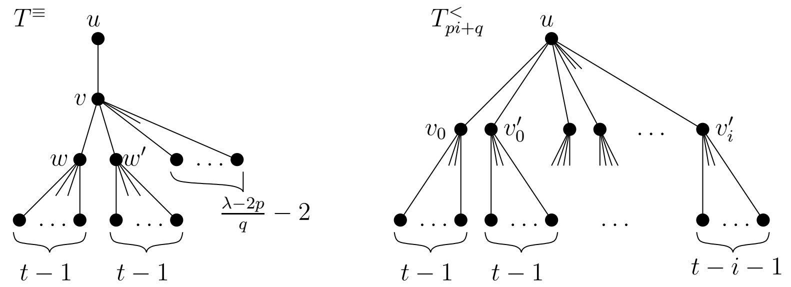


Figure 3: Construction of $T^{\equiv}$ and $T^{<}_{p_i+q}$

If $x < 2p$ then the $t-i$ neighbors should fit into the interval $[x+p, \lambda]$. This interval is of length at most $\lambda - p - iq - r' - 1$ while the shortest interval that can accommodate $t-i$ labels $q$ apart needs length at least $(t-i-1)q = \lambda - p - qi - r'$. The case when $x > \lambda - 2p$ follows by the same arguments.

If $x \in [2p, \lambda - 2p]$, then two intervals for the labels are available of the total length $\lambda - 2p$. On the other hand the minimum necessary length is $(t-i-2)q \geq (t-s-2)q\ \lambda - 2p$, a contradiction as in the previous two cases. □

We now show how to force labels divisible by $q$. (Recall that $[i,j]_{\equiv q} := \{i, i+q, i+2q, \ldots, j-q, j\}$.)

**Lemma 8.** *Let be given $p, q$ and $\lambda$ such that $p > q$ and $\lambda \equiv_q 2p$. We construct the rooted tree $T^{\equiv}$ of four levels as follows: The root $u$ has only one child $v$. This $v$ is of degree $\frac{\lambda-2p}{q}+1$. It has two children $w, w'$, both of degree $t$, while the remaining children of $v$ are leaves. (See Fig. 3.)*

*Then the label set of the root $u$ satisfies*

$$\Lambda(u) = [q, \lambda - 2p]_{\equiv q} \cup [2p, \lambda - q]_{\equiv q}$$

*Proof.* By Observation 7 labels of $w$ and $w'$ belong to one of the two intervals $[0, r']$ and $[\lambda - r', \lambda]$. As they share a common neighbor, we may without loss of generality assume that the label of $w$ is at most $r'$ while the label of $w'$ is at least $\lambda - r'$.

Thus, the label of $v$ belongs to $[p, \lambda - p]$. As it has exactly $\frac{\lambda-2p}{q}+1$ neighbors, the label of $v$ must be of form $p + kq$. Then the labels of its neighbors are uniquely determined as $[0, kq]_{\equiv q} \cup [2p + kq, \lambda]_{\equiv q}$. Now the labels of $w$ and $w'$ are forced to be 0 and $\lambda$. The claim of $\Lambda(u)$ then follows by considering all possible labels of $v$. $\square$

The next lemma forces labels inside the interval $[0, \lambda]$.

**Lemma 9.** *Let be given $p, q$ and $\lambda$, such that $s \geq 0$ and $r \in [q - r, r]$.*

*For an $i \in [0, s]$, let $T^{<}_{pi+q}$ be the rooted tree of three levels, where the degrees of the vertices in the middle level are $t, t, t-1, t-1, \ldots, t-i, t-i$. (See Fig. 3.)*

*Then the label set of the root $u$ satisfies that $\Lambda(u) \subseteq [p + iq, \lambda - p - iq]$ and moreover all labels congruent to 0 or to $p$ modulo $q$ from this interval belong to $\Lambda(u)$ (and analogously for $\lambda$-symmetric labels).*

*Proof.* Denote the vertices of middle level by $v_0, v'_0, v_1, v'_1, \ldots, v_i, v'_i$, in the order as they appear in the lemma statement.

First observe that each of the intervals $[0, r']$ and $[\lambda - r', \lambda]$ contains the label of one of $v_0$ and $v'_0$. After that $[q, q + r']$ and $[\lambda - q - r', \lambda - q]$ host labels of $v_1, v'_1$ and so on.

Whatever choice of the labels for the vertices of the middle level is done, the label of $u$ is forced to belong to the interval $[p + iq, \lambda - p - iq]$.

For the other claim we choose the labels of $v_0, \ldots, v_i$ as $[0, iq]_{\equiv q}$ and the labels of $v'_0, \ldots, v'_i$ $\lambda$-symmetrically.

Let the label of $u$ be divisible by $q$. It remains to select labels of the leaves in $T^{<}_{p+iq}$. For the neighbors of any $v_j$ (including $u$) we choose the set $[p - r + (j+1)q, p - r + tq]_{\equiv q}$. Straightforwardly, the neighbors of $v'_j$ have labels $[0, (t - 1 - j)q]_{\equiv q}$.

For the other case we assume that the label of $u$ is congruent to $p$. Then we give neighbors of a $v_j$ labels $[p + jq, p + (t-1)q]_{\equiv q}$ and the neighbors of $v'_j$ labels $[r, r + (t - j - 1)q]_{\equiv q}$.

A tedious verification shows that the above strategy yields a valid $L(p, q)$-labeling of span $\lambda$ of $T^{<}_{p+iq}$. $\square$

Now we show that certain labels can be forced exactly.

**Lemma 10.** *Let be given $p, q$ and $\lambda$ such that $s \geq 1$, $\lambda \equiv_q 2p$, and $r \in [q - r', r']$.*

*For $i \in [0, s-1]$, we set $j := p + iq$, $k := \frac{\lambda - 2p}{q} - 2i - 1$. Let $T_j$ be the rooted tree of five levels constructed from the disjoint union of a copy of $T^{<}_j$ and $k$ copies of $T^{<}_{j+q}$. We rebuild them to share the vertex $v_0$ as follows:*

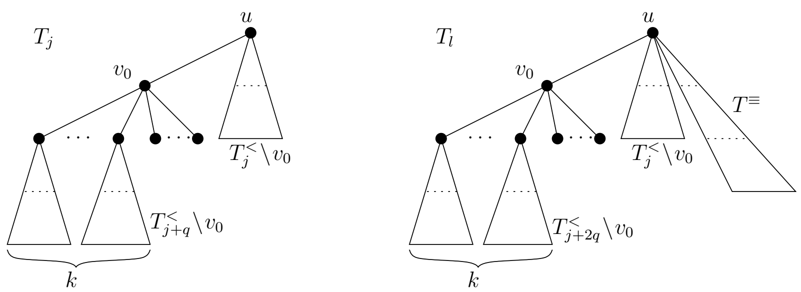


Figure 4: Construction of $T_j$ and $T_l$

*In each copy of $T_{j+q}^{<}$ we identity the vertex $v_0$ of degree $t$ and delete it with all its descendants. The roots of all copies of $T_{j+q}^{<}$ become children of the vertex $v_0$ of $T_j^{<}$. (See Fig. 4.)*

*The root $u$ of $T_j$ is the root of $T_j^{<}$ and we claim that its label set is $\Lambda(u) = \{j, \lambda - j\}$.*

*Proof.* By Lemma 9 the $k$ roots of the copies of $T_{j+q}^{<}$ must accommodate in the interval $[j+q, \lambda - j - q]$. As they must be at least $q$ apart, they form the set $[j+q, \lambda - j - q]_{\equiv q}$.

Then the label of the root $u$ belongs to the intersection of the intervals $[0, j] \cup [\lambda - j, \lambda]$ (as it should be $q$ apart from the $k$ "former" roots) and $[j, \lambda - j]$ (as it is the root of the tree $T_j^{<}$).

On the other hand a feasible labeling of $T_j$ can be composed from labelings of its subtrees (see the proof of Lemma 9 for details). □

**Lemma 11.** *Let be given $p, q$ and $\lambda$ such that $p > q$, $\lambda \equiv_q 2p$, and $r \in [q - r', r']$.*

*For $i \in [0, s-2]$, we set $j := p + iq$, $k := \frac{\lambda - 2p}{q} - 2i - 3$, and $l$ to be the only multiple of $q$ in the interval $[j, j+q]$, i.e. $l := j - r + q$. Let $T_l$ be the rooted tree of five levels constructed from the disjoint union of a copy of $T^{\equiv}$, a copy of $T_j^{<}$ and $k$ copies of $T_{j+2q}^{<}$. We first merge the roots of $T^{\equiv}$ and $T_j^{<}$ into the root $u$ of of $T_l$. Then we rebuild these $k+1$ trees, so they share the vertex $v_0$ as was described in the statement of Lemma 10. (See Fig. 4.)*

*Then the label set of the root $u$ is $\Lambda(u) = \{l, \lambda - l\}$.*

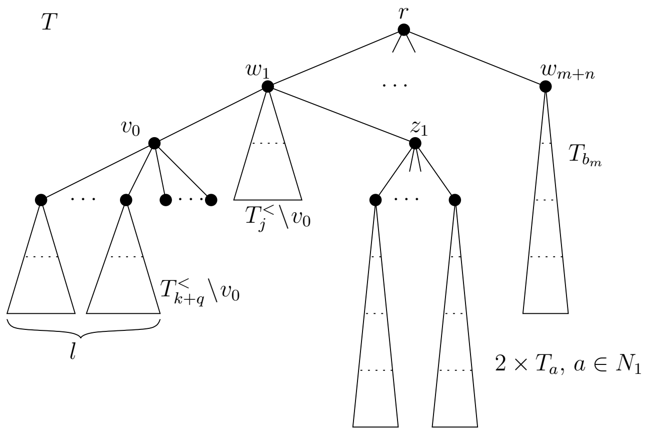


Figure 5: Construction of the final tree $T$

*Proof.* By the same argument as in the proof of Lemma 10 the $k$ roots are labeled by $[j + 2q, \lambda - j - 2q]_{\equiv q}$.

The label of the root $u$ then belongs to the set $[j, j+q] \cup [\lambda - j - q, \lambda - j]$. The presence of the tree $T^{\equiv}$ assures, that the label of $u$ is either $l$ or $\lambda - l$ (observe that $[j, j + q]$ does not intersect the set $[2p, \lambda - q]_{\equiv q}$ of Lemma 8).

For the opposite direction we compose a feasible labeling of $T_l$ from labelings of its subtrees. □