Rank of divisors on tropical curves

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Abstract

We investigate, using purely combinatorial methods, structural and algorithmic properties of linear equivalence classes of divisors on tropical curves. In particular, an elementary proof of the Riemann-Roch theorem for tropical curves, similar to the recent proof of the Riemann-Roch theorem for graphs by Baker and Norine, is presented. In addition, a conjecture of Baker asserting that the rank of a divisor D on a (non-metric) graph is equal to the rank of D on the corresponding metric graph is confirmed, and an algorithm for computing the rank of a divisor on a tropical curve is constructed.

1 Introduction

Tropical geometry investigates properties of tropical varieties, objects which are commonly considered to be combinatorial counterparts of algebraic varieties. There are several survey articles on this recent branch of mathematics [10, 12, 14]. In particular, [4, 9] concentrate on topics which are particulary close to the subject of this paper.

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Tropical varieties share many important features with their algebro-geometric analogues, and allow for a variety of algebraic, combinatorial and geometric techniques to be used. Let us demonstrate this miscellany on a few examples.

- In [2], a version of the Riemann-Roch theorem for graphs was proved by purely combinatorial methods. Shortly afterwards Gathmann and Kerber [5] used the result to prove Riemann-Roch theorem for tropical curves. Their contribution was a method of approximating a tropical curve by graphs.
- Mikhalkin and Zharkov [11] gave (among others) another proof of the Riemann-Roch theorem for tropical curves. Their approach used a combination of algebraic and combinatorial techniques.
- Gathmann and Markwig [6] proved Kontsevich's formula for tropical curves. They followed the original Kontsevich's approach, translating thoroughly the proof into the language of tropical geometry.
- Recently, a machinery which allows one to transfer certain results from Riemann surfaces to tropical curves has been developed in [1]. Note that this method necessarily has some limitations. Indeed, it is known that analogues of some theorems about Riemann surfaces do not hold in the tropical context.

In this paper, we contribute further towards the theory by proving new structural results on divisors on tropical curves. These yield among others an alternative proof of the Riemann-Roch theorem for tropical curves and an algorithm for computing ranks of divisors on tropical curves. In particular, we confirm a conjecture of Baker [1] relating the ranks of a divisor on a graph and on a tropical curve. All the proofs in the paper are purely combinatorial.

1.1 Overview and notation

Throughout the paper, a *graph* G is a finite connected multigraph that can contain loops, i.e., G is a pair consisiting of a set $V(G)$ of *vertices* and a multiset $E(G)$ of *edges*, which are unordered pairs of not necessarily distinct vertices. The degree $\deg_G(v)$ of a vertex v is the number of edges incident with it (counting loops twice). The k-th *subdivision* of a graph G is the graph G^k obtained from G by replacing each edge with a path with k inner vertices.

Graphs have been considered as analogues of Riemann surfaces in several contexts, in particular, in [2, 3] in the context of linear equivalence of divisors. In this paper we further investigate the properties of linear equivalence classes of divisors. We primarily concentrate on metric graphs, but let us start the exposition by first recalling the definitions and results from [2] related to (non-metric) graphs.

A *divisor* D on a graph G is an element of the free abelian group $\mathrm{Div}(G)$ on $V(G)$. We can write each element $D \in \mathrm{Div}(G)$ uniquely as

$$D = \sum_{v \in V(G)} D(v)(v)$$

with $D(v) \in \mathbb{Z}$. We say that D is *effective*, and write $D \geq 0$, if $D(v) \geq 0$ for all $v \in V(G)$. For $D \in \mathrm{Div}(G)$, we define the *degree* of D by the formula

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Analogously, we define

$$\deg^+(D) = \sum_{v \in V(G)} \max\{0, D(v)\}.$$

For a function $f : V(G) \to \mathbb{Z}$, the *divisor associated to* f is given by the formula

$$D_f = \sum_{v \in V(G)} \sum_{e=vw \in E(G)} (f(v) - f(w))\,(v).$$

Divisors associated to integer-valued functions on $V(G)$ are called *principal*. An equivalence relation $\sim$ on $\mathrm{Div}(G)$, is defined as $D \sim D'$, if and only if $D - D'$ is principal. We sometimes write $\sim_G$ instead of $\sim$ when the graph is not clearly understood from the context. For a divisor D, we use $|D|$ to denote the set of effective divisors associated to it, i.e.,

$$|D| = \{E \in \mathrm{Div}(G) \;:\; E \geq 0 \text{ and } E \sim D\}.$$

We refer to $|D|$ as the *(complete) linear system* associated to D. If $D \sim D'$, we call the divisors D and D' *equivalent* (or *linearly equivalent*).

The *rank* of a divisor D on a graph G is defined as

$$r_G(D) = \min_{\substack{E \geq 0 \\ |D-E| = \emptyset}} \deg(E) - 1\,.$$

We frequently omit the subscript G in $r_G(D)$ when the graph G is clear from the context. Also note that $r(D)$ depends only on the linear equivalence class of D. In the classical case, $r(D)$ is usually referred to as the dimension of the linear system $|D|$. In our setting, however, we are not aware of any interpretation of $r(D)$ as the topological dimension of a physical space. Thus, we refer to $r(D)$ as "the rank" rather than "the dimension". See Remark 1.13 of [2] for further discussion about similarities and differences between our definition of $r(D)$ and the classical definition in the Riemann surface case.

The *canonical divisor* on G is the divisor K_G defined as

$$K_G = \sum_{v \in V(G)} (\deg(v) - 2)(v).$$

The *genus* of G is the number $g = |E(G)| - |V(G)| + 1$. In graph theory, g is called the *cyclomatic number* of G.

The following graph-theoretical analogue of the classical Riemann-Roch theorem is one of the main results of [2].

Theorem 1. *If D is a divisor on a loopless graph G of genus g, then*

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

Let us note that while the graph-theoretical results, such as Theorem 1, can be viewed as simply being analogous to classical results from algebraic geometry, there exist deep relations between the two contexts, e.g., a connection arising from the specialization of divisors on arithmetic surfaces is explored in [1].

Tropical geometry provides another connection between graph theory and the theory of algebraic curves. The analogue of an algebraic curve in tropical geometry is an *(abstract) tropical curve*, which following Mikhalkin [9], can be considered simply as a metric graph. A *metric graph* Γ is a graph with each edge being assigned a positive length. Each edge of a metric graph is associated with an interval of the length assigned to the edge with the end points of the interval identified with the end vertices of the edge. The points of these intervals are referred to as *points* of Γ. The internal points of the interval are referred to as *internal* points of the edge and they form the *interior* of the edge. Subintervals of these intervals are then referred to as *segments*.

This geometric representation of Γ equips the metric graph with a topology, in particular, we can speak about open and closed sets. The distance

$\mathrm{dist}_\Gamma(v,w)$ between two points v and w of Γ is measured in the metric space corresponding to the geometric representation of Γ (the subscript Γ is omitted if the metric graph Γ is clear from the context). For an edge e of Γ and two points $x,y\in e$ we use $\mathrm{dist}_e(x,y)$ to denote the distance between x and y measured on the edge e.

The vertices of Γ are called *branching points* and the set of branching vertices of Γ is denoted by $B(\Gamma)$. We assume that the degree of every vertex of Γ is distinct from two unless Γ is a loop in which case $B(\Gamma)$ is formed by the single vertex incident with the loop. Clearly, this assumption does not restrict the generality of metric graphs and tropical curves considered throughout the paper. At several occasions in the paper, we however allow for convenience a metric graph (or a tropical curve) to contain branching vertices of degree two—it will always be clear when this is the case.

A *tropical curve* is a metric graph where some edges incident with vertices of degree one (leaves) have infinite length. Such edges are identified with the interval $\langle 0,\infty\rangle$, such that ∞ is identified with the vertex of degree one, and are called *infinite edges*. The points corresponding to ∞ are referred to as *unbounded ends*. The unbounded ends are also considered to be points of the tropical curve.

The notions of genus, divisor, degree of a divisor and canonical divisor K_Γ readily translate from graphs to metric graphs and tropical curves (with basis of the free abelian group of divisors $\mathrm{Div}(\Gamma)$ being the infinite set of all the points of Γ). In order to define linear equivalence on $\mathrm{Div}(\Gamma)$, the notion of rational function has to be adopted.

A *rational function* on a tropical curve Γ is a continuous function $f:\Gamma\to\mathbb{R}\cup\{\pm\infty\}$ which is a piecewise linear function with integral slopes on every edge. We require that the number of linear parts of a rational function on every edge is finite and the only points v with $f(v)=\pm\infty$ are unbounded ends.

The *order* $\mathrm{ord}_f v$ of a point v of Γ with respect to a rational function f is the sum of outgoing slopes of all the segments of Γ emanating from v. In particular, if v is not a branching point of Γ and the function f does not change its slope at v, $\mathrm{ord}_f v=0$. Hence, there are only finitely many points v with $\mathrm{ord}_f v\neq 0$. Therefore, we can associate a divisor D_f to the rational function f by setting $D_f(v)=\mathrm{ord}_f v$ for every point v of Γ. Observe that $\deg(D_f)$ is equal to zero as each linear part of f with slope s contributes towards the sum defining $\deg(D_f)$ by $+s$ and $-s$ (at its two boundary points). Note that $\mathrm{ord}_f v$ need not be zero for unbounded ends v.

Rational functions on tropical curves lead to a definition of principal divisors on tropical curves. In particular, we say that divisors D and D' on Γ are *equivalent* and write $D \sim D'$ if there exists a rational function f on Γ such that $D = D' + D_f$. With this notion of equivalence the linear system and the rank of a divisor on a tropical curve are defined in the same manner as for finite graphs above, in particular:

$$|D| = \{E \in \mathrm{Div}(\Gamma) \; : \; E \geq 0 \text{ and } E \sim D\},$$

$$r_\Gamma(D) = \min_{\substack{E \geq 0,\, E \in \mathrm{Div}(\Gamma) \\ |D-E| = \emptyset}} \deg(E) - 1 \; .$$

Gathmann and Kerber [5] and, independently, Mikhalkin and Zharkov [11] have proved the following version of the Riemann-Roch theorem for tropical curves.

Theorem 2. *Let D be a divisor on a tropical curve Γ of genus g. Then*

$$r(D) - r(K_\Gamma - D) = \deg(D) + 1 - g.$$

One of the main goals of this paper is to establish closer connection between Theorems 1 and 2. In Section 2 we prove that every equivalence class of divisors on a metric graph contains a unique reduced element (with respect to a chosen base point). We use this structural information and the Riemann-Roch criterion from [2] to derive a new proof of Theorem 2 that closely parallels the proof of Theorem 1 in [2].

In Section 3 we prove the following theorem relating the ranks of divisors on ordinary and metric graphs. Before stating the theorem we need to introduce a definition. We say that a metric graph Γ *corresponds* to the graph G if Γ is obtained from G by setting the length of each edge of G to be equal to one.

Theorem 3. *Let D be a divisor on a graph G and let Γ be the metric graph corresponding to G. It holds, that*

$$r_G(D) = r_\Gamma(D).$$

The sets of effective divisors and principal divisors on Γ are both strictly larger than the respective sets for G. Hence, Theorem 3 is not a priori obvious. Note that Theorem 2 and Theorem 3 together imply Theorem 1. Gathmann and Kerber [5] showed how to deduce Theorem 2 from Theorem 1. Consequently, our arguments complete the circle of ideas showing that Theorems 1 and 2 are equivalent. Theorem 3 also implies a conjecture of Baker that the rank of a divisor on a graph G is the same as its rank on the graph G^k (see [1]). We finish the paper by considering algorithmic applications of the results established in Sections 2 and 3, and design an algorithm for computing the rank of divisors on tropical curves.

1.2 The Riemann-Roch criterion

In this section, we recall an abstract criterion from [2] giving necessary and sufficient conditions for the Riemann-Roch formula to hold. We will show in Section 2 that divisors on tropical curves satisfy this criterion, thereby proving Theorem 2.

The setting for the results of this section is as follows. Let X be a non-empty set, and let $\mathrm{Div}(X)$ be the free abelian group on X. Elements of $\mathrm{Div}(X)$ are called *divisors* on X, divisors E with $E \geq 0$ are called *effective*. Let $\sim$ be an equivalence relation on $\mathrm{Div}(X)$ satisfying the following two properties:

(E1) If $D \sim D'$, then $\deg(D) = \deg(D')$.

(E2) If $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_1' \sim D_2 + D_2'$.

As before, given $D \in \mathrm{Div}(X)$, define

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0 \text{ and } E \sim D\}$$

and

$$r(D) = \min_{\substack{E \geq 0 \\ |D-E| = \emptyset}} \deg(E) - 1 \ .$$

For a nonnegative integer g, let us define the set of *non-special divisors*

$$\mathcal{N} = \{D \in \mathrm{Div}(X) \ : \ \deg(D) = g-1 \text{ and } |D| = \emptyset\} \ .$$

Note that for the Riemann surfaces the notion of non-special divisor is slightly different. In particular, classically, non-special divisors do not necessarily have rank of the genus decreased by one.

Finally, let K be an element of $\mathrm{Div}(X)$ having degree $2g-2$. The following theorem from [2] gives necessary and sufficient conditions for the Riemann-Roch formula to hold for elements of $\mathrm{Div}(X)/\sim$.

Theorem 4. *Define* $\epsilon : \mathrm{Div}(X) \to \mathbb{Z}/2\mathbb{Z}$ *by declaring that* $\epsilon(D) = 0$ *if* $|D| \neq \emptyset$ *and* $\epsilon(D) = 1$ *if* $|D| = \emptyset$*. Then the Riemann-Roch formula*

$$r(D) - r(K - D) = \deg(D) + 1 - g \tag{1}$$

holds for all $D \in \mathrm{Div}(X)$ *if and only if the following two properties are satisfied:*

(RR1) *For every* $D \in \mathrm{Div}(X)$*, there exists* $\nu \in \mathcal{N}$ *such that* $\epsilon(D)+\epsilon(\nu-D) = 1$*.*

(RR2) *For every* $D \in \mathrm{Div}(X)$ *with* $\deg(D) = g-1$*, we have* $\epsilon(D)+\epsilon(K-D) = 0$*.*

In addition to Theorem 4, we will later use the following lemmas from [2] that also hold in the abstract setting.

Lemma 5. *For all* $D, D' \in \mathrm{Div}(X)$ *such that* $r(D), r(D') \geq 0$*, we have* $r(D + D') \geq r(D) + r(D')$*.*

Lemma 6. *If (RR1) holds, then for every* $D \in \mathrm{Div}(X)$ *we have*

$$r(D) = \left(\min_{\substack{D' \sim D \\ \nu \in \mathcal{N}}} \deg^{+}(D' - \nu) \right) - 1 \ . \tag{2}$$

1.3 Reducing tropical curves to metric graphs

Let us finish the introductory part of the paper by reducing the study of divisors on tropical curves to the corresponding situation on metric graphs. Let Γ be a tropical curve, and let Γ' be the metric graph obtained from Γ by removing interiors of infinite edges and their unbounded ends. There exists a natural retraction map $\psi_\Gamma : \Gamma \to \Gamma'$ that maps deleted points of infinite edges of Γ to the ends of those edges that belong to Γ' and acts as identity on the points of Γ'. This map induces a map from $\mathrm{Div}(\Gamma)$ to $\mathrm{Div}(\Gamma')$, which is denoted by ψ_Γ. The following theorem combines the results of Lemma 3.4, Remark 3.5, Lemma 3.6 and Remark 3.7 of [5].

Theorem 7. *Let* Γ *be a tropical curve, and let* Γ' *and* ψ_Γ *be defined as above. Let* $D \in \mathrm{Div}(\Gamma)$*, and set* $D' = \psi_\Gamma(D)$*. We have* $D \sim_\Gamma D'$, $\deg(D) = \deg(D')$, $r_\Gamma(D) = r_{\Gamma'}(D')$*, and* D *is effective if and only if* D' *is. In addition, it holds that* $K_{\Gamma'} = \psi_\Gamma(K_\Gamma)$*.*

It follows from Theorem 7 that Theorem 2 restricted to metric graphs implies Theorem 2 in full generality. It also follows that given an algorithm to compute the rank of divisors on metric graphs one can readily design an algorithm to compute rank of divisors on tropical curves. Based on these observations we concentrate our further investigations on metric graphs.

2 Non-special divisors

As in Subsection 1.2, we define the set of *non-special divisors* on a metric graph Γ to be

$$\mathcal{N} = \{D \in \mathrm{Div}(\Gamma) \ : \ \deg(D) = g-1 \text{ and } |D| = \emptyset\}$$

where g is the genus of Γ. In this section, we present an important class of non-special divisors and prove that every non-special divisor is equivalent to a divisor in this class. Using this structural information we prove that divisors on a metric graph satisfy conditions (RR1) and (RR2) from Theorem 4, and therefore we give an alternative (purely combinatorial) proof of Theorem 2.

2.1 Reduced divisors

As our first step towards the goal of this section, we define a v_0-reduced divisor on a metric graph Γ. We prove that, for a fixed point $v_0 \in \Gamma$, every divisor is equivalent to a unique v_0-reduced divisor.

For a closed connected subset X of a metric graph Γ and a point $v \in \partial X$ we define *the number of edges leaving X at v* to be the maximum size of a collection of internally disjoint segments in $\Gamma \setminus (X - \{v\})$ with an end in v. A boundary point v of a closed connected subset X of a metric graph Γ is *saturated* with respect to $D \in \mathrm{Div}(\Gamma)$ and the set X if the number of edges leaving X at v is at most $D(v)$, and is *non-saturated*, otherwise. If the divisor D and/or the set X are clear from the context, we omit them when talking about saturated and non-saturated points. A divisor D on a metric graph Γ is said to be *v_0-reduced* with respect to a point v_0 if D is non-negative on Γ except possibly for the point v_0 and every closed connected set X of points of Γ with $v_0 \notin X$ contains a non-saturated point $v \in \partial X$.

The notion of v_0-reduced divisors on metric graphs is analogous to the one used in [2] for graphs. Let us recall the definition. For a graph G, a set $A \subseteq V(G)$ and $v \in A$, $\mathrm{outdeg}_A(v)$ denotes the number of edges of G having

v as one endpoint and whose other endpoint lies in $V(G) - A$. We say that a divisor $D \in \mathrm{Div}(G)$ is *v_0-reduced* for some $v_0 \in V(G)$ if $f(v) \geq 0$ for all $v \in V(G) - \{v_0\}$, and for every non-empty set $A \subseteq V(G) - \{v_0\}$, there exists a vertex $v \in A$ such that $D(v) < \mathrm{outdeg}_A(v)$. The following simple proposition helps us to establish the analogy between reduced divisors on metric and non-metric graphs, and to motivate the definition of reduced divisors for metric graphs above.

Proposition 8. *Let G be a graph, let $D \in \mathrm{Div}(G)$, and let $v_0 \in V(G)$. Then D is v_0-reduced as a divisor on G if and only if D is v_0-reduced as a divisor on the metric graph Γ corresponding to G.*

Proof. We assume that D is non-negative on $V(G) - \{v_0\}$, as otherwise D is not v_0-reduced on both G and Γ. If D is not v_0-reduced on G, then there exists $A \subseteq V(G) - \{v_0\}$ such that $D(v) \geq \mathrm{outdeg}_A(v)$ for every $v \in A$. Let X be a closed connected subset of Γ, consisting of A and all the points of the edges joining the vertices of A. Then every point of ∂X is saturated with respect to X and D, and, thus, D is not v_0-reduced on Γ.

Conversely, let X be a closed connected subset of Γ, such that $v_0 \notin X$, and every point of ∂X is saturated with respect to X and D. Consequently, $\partial X \subseteq \{v \in \Gamma \mid D(v) > 0\} \subseteq V(G)$. In particular, the number of edges leaving X at every point $v \in \partial X$ is not greater than $\mathrm{outdeg}_{\partial X}(v)$. It follows that if D is not v_0-reduced on Γ, then D is not v_0-reduced on G. ☐

Reduced divisors on graphs correspond to G-parking functions defined in [13]. Hence, we essentially provide a generalization of the notion of G-parking functions for metric graphs.

We now present a simple example of the behavior of reduced divisors. Let Γ be a metric graph consisting of two branching points v_0 and v_1 joined by k edges. One can routinely verify that a divisor $D \in \mathrm{Div}(\Gamma)$ is v_0-reduced if and only if the following conditions hold:

- $D(v) \geq 0$ for every $v \in \Gamma - \{v_0\}$,
- $D(v) \leq 1$ for every $v \in \Gamma - \{v_0, v_1\}$,
- supp (D) has at most one point in the interior of any edge of Γ, and
- $\sum_{v \in \text{ supp } D - \{v_0\}} D(v) < k$.

Consider a divisor $D = k(v_1)$ which is not v_0-reduced: the only boundary point of singleton set $\{v_1\}$ is saturated. If d_0 is the length of the shortest edge in Γ, then the unique v_0-reduced divisor equivalent to D has the form $D' = \sum_{e \in E(\Gamma)}(x_e)$ where x_e is the point on the edge e of Γ with $\mathrm{dist}_e(v_1, x_e) = d_0$. In particular, the unique v_0-reduced divisor equivalent to D depends non-trivially on the metric of Γ.

Before we actually prove the existence and uniqueness of v_0-reduced divisors, we need to introduce some additional notation. If D is a divisor on a metric graph Γ and v_0 a point of Γ, we define O_D to be the largest connected subset of Γ such that $v_0 \in O_D$ and $(\mathrm{supp}\, D \cup B(\Gamma)) \cap O_D \subseteq \{v_0\}$. Such a set O_D can be obtained from Γ by deleting all points of supp D and $B(\Gamma)$, except for v_0, and taking the connected component of v_0 in what remains. In particular, O_D is uniquely determined, and is always an open set. Hence, its complement $\overline{O_D}$ is closed (which includes the case that $\overline{O_D}$ is empty).

We next introduce the notion of a v_0-extremal rational function. A rational function f on a metric graph Γ is said to be a *basic v_0-extremal function* if there exist closed connected disjoint sets $X_{\min}$ and $X_{\max}$ with the following properties:

- v_0 is contained in $X_{\min}$,
- f is constant on $X_{\min}$ and on $X_{\max}$,
- every branching point is contained in $X_{\min} \cup X_{\max}$,
- if $v \in \partial X_{\min}$, then $\mathrm{ord}_f v$ is equal to the number of edges leaving $X_{\min}$ at v,
- if $v \in \partial X_{\max}$, then $-\mathrm{ord}_f v$ is equal to the number of edges leaving $X_{\max}$ at v, and
- $\mathrm{ord}_f v = 0$ for $v \notin \partial X_{\min} \cup \partial X_{\max}$.

Note that $X_{\min}$ and $X_{\max}$ are exactly the subsets of Γ on which f achieves its minimum and maximum, respectively. Also, observe that for given sets $X_{\min}$ and $X_{\max}$ there exists at most one rational function f satisfying the above conditions. We say that a rational function f on a metric graph Γ is a *v_0-extremal rational function* if it can be represented as a sum of finitely many basic v_0-extremal rational functions. Observe that if f is a v_0-extremal rational function, then $(D + D_f)(v_0) \geq D(v_0)$ for every divisor D.

We show in the next lemma that adding a divisor corresponding to a v_0-extremal rational function preserves saturated points on the boundary of O_D.

Lemma 9. *Let $D \in \mathrm{Div}(\Gamma)$, let $v_0 \in \Gamma$, let f be a v_0-extremal rational function on Γ and let $D' = D + D_f$. If $D(v_0) = D'(v_0)$ and D is non-negative on ∂O_D, then D' is non-negative on $\partial O_{D'}$ and every non-saturated point of $\overline{O_{D'}}$ is also a non-saturated point of $\overline{O_D}$.*

Proof. It is enough to prove the lemma when f is a basic v_0-extremal rational function. Let $X_{\min}$ and $X_{\max}$ be as in the definition of a basic extremal rational function.

Let $X^o_{\min}$ be the interior of the set $X_{\min}$. As $D(v_0) = D'(v_0)$ and $v_0 \in X_{\min}$, we have $v_0 \in X^o_{\min}$. The definition of $O_{D'}$ yields that $O_{D'} = O_D \cap X^o_{\min}$. Therefore, every point $v \in \partial O_{D'}$ is contained in ∂O_D or in $\partial X_{\min}$. If $v \notin \partial O_D$, then v is not a branching point and $D(v) = 0$. Since $v \in \partial X_{\min}$ and $\mathrm{ord}_f v = 1$, it holds that $D'(v) = 1$. This implies that v is a saturated point of $\overline{O_{D'}}$.

We focus on $v \in \partial O_D \cap \partial O_{D'}$ in the rest. As $O_{D'} \subseteq O_D$, the number of edges leaving $\overline{O_{D'}}$ at v is smaller or equal to the number of edges leaving $\overline{O_D}$ at v. We have $D'(v) \geq D(v)$, as $v \in X_{\min}$. Thus if v is not saturated in $\overline{O_{D'}}$, then v is also not saturated in $\overline{O_D}$. Finally, we have seen that $D'(v) \geq 0$ for every $v \in \partial O_{D'}$. ☐

We are now ready to prove the existence and the uniqueness of a v_0-reduced divisor equivalent to a given divisor on a metric graph. Let us remark that a similar statement for graphs can be found in [2].

Theorem 10. *Let Γ be a metric graph, let v_0 be a point of Γ, and let D be a divisor on Γ. There exists a unique v_0-reduced divisor D_0 equivalent to D.*

Proof. Let us first establish that a v_0-reduced divisor equivalent to a given divisor D, if it exists, is unique. Assume, for the sake of contradiction, that there are two different v_0-reduced divisors D_1 and D_2 equivalent to D. Consider the rational function f such that $D_1 = D_2 + D_f$, and let X be the set of points of Γ where the function f attains its minimum. Observe that the set X is closed and $X \neq \Gamma$. We can assume, by switching D_1 and D_2 if necessary, that $v_0 \notin X$.

Since X is the set of all points where the function f attains its minimum, the order $\mathrm{ord}_f v$ at every point $v \in \partial X$ is at least the number of edges leaving X at v. Since the divisor D_2 is v_0-reduced, $D_2(v)$ is non-negative for every point $v \neq v_0$, in particular, D_2 is non-negative for every point $v \in \partial X$. Hence, $D_1(v) = (D_2 + D_f)(v) \geq \mathrm{ord}_f v$ for every point of $v \in \partial X$. We conclude that every boundary point of every connected component of X is saturated with respect to D_1 which contradicts our assumption that D_1 is v_0-reduced.

We now turn to proving the existence of a v_0-reduced divisor equivalent to a given divisor D. As the first step, we construct a divisor D' equivalent to D that is non-negative on Γ except possibly for the point v_0. Let $v_1, \ldots, v_k$ be the points of Γ different from v_0 where the divisor D is negative. For each $1 \leq i \leq k$, consider a rational function $f_i(v) = -\min\{\mathrm{dist}(v_0, v), \mathrm{dist}(v_0, v_i)\}$. Observe that $\mathrm{ord}_{f_i} v$ is non-negative for all points $v \neq v_0$ and $\mathrm{ord}_{f_i} v_i \geq 1$. Hence, $D' = D - \sum_{i=1}^{k} D(v_i) D_{f_i}$ is a divisor equivalent to D and non-negative on Γ except possibly for the point v_0.

The rest of the proof is devoted to establishing the following claim:

If D' is a divisor non-negative on Γ except possibly for v_0 then there exists a v_0-extremal rational function f such that $D' + D_f$ is v_0-reduced.

The above claim clearly implies the theorem.

The proof of the claim proceeds by induction on $b(\Gamma)$ where $b(\Gamma)$ is equal to $|B(\Gamma) \cup \{v_0\}|$. Before we start with the actual proof of the claim, let us observe that we can assume without loss of generality that v_0 is not a cut-point of Γ, i.e., $\Gamma \setminus \{v_0\}$ is connected.

Suppose that v_0 is a cut-point. A *block of* Γ *corresponding to* v_0 is a metric graph Γ' obtained from Γ by deleting all but one connected components of $\Gamma \setminus \{v_0\}$. First note that a divisor is v_0-reduced in Γ if and only if it is v_0-reduced in each block Γ' corresponding to v_0 and a rational function is v_0-extremal if and only if it is v_0-extremal in each block. As $b(\Gamma') \leq b(\Gamma)$ for every block Γ' corresponding to v_0, the existence of a v_0-reduced divisor can be established in each block separately using the arguments that follow. For each block Γ'_i, we obtain a v_0-extremal rational function f'_i, such that $D' + D_{f'_i}$ is v_0-reduced on Γ'_i. Observe, that a v_0-extremal function f'_i on Γ'_i can be extended to a v_0-extremal function on Γ by setting it to be constant and equal to $f'_i(v_0)$ outside Γ'_i. We extend f'_i and still call the resulting function f'_i. The function $f = \sum_i f'_i$ is v_0-extremal, and the divisor $D' + D_f$ is v_0-reduced, since it is v_0-reduced in each block of Γ corresponding to v_0.

We now continue with the proof of the claim, assuming v_0 not to be a cut-point. Both in the base case of the induction and the induction step, we modify the divisor D' to an equivalent divisor D'' as described further. Choose D'' to be a divisor with the following properties:

1. $D'' = D' + D_f$ for a v_0-extremal rational function f,
2. D'' is non-negative on Γ with a possible exception of v_0,
3. $D''(v_0)$ is among the divisors satisfying the first two conditions maximal, and
4. the number of non-saturated points of $\partial O_{D''}$ is minimal among all divisors obeying the first three conditions.

Observe that the divisor D'' that satisfies the above four conditions always exists though it need not be unique.

If $\overline{O_{D''}}$ is empty (which can happen only if Γ is a loop), then D'' is a v_0-reduced divisor (and thus we have proven the claim). In the rest, we assume that $\overline{O_{D''}}$ is non-empty. We claim that $\partial O_{D''}$ contains a point that is not saturated. Otherwise, consider the following rational function f:

$$f(v) = \begin{cases} d - \min\{d, \min\limits_{v' \in \partial O_{D''}} \mathrm{dist}(v, v')\} & \text{if } v \in O_{D''}, \text{ and} \\ d & \text{otherwise,} \end{cases}$$

where d is the minimal distance between v_0 and a point of $\partial O_{D''}$, i.e.,

$$d = \min_{v \in \partial O_{D''}} \mathrm{dist}(v_0, v) \ .$$

Observe that f is v_0-extremal, $\mathrm{ord}_f v_0 > 0$, $\mathrm{ord}_f v$ is non-negative for every point, except for those of $\partial O_{D''}$, and $-\mathrm{ord}_f v$ is equal to the number of edges leaving $O_{D''}$ for every $v \in \partial O_{D''}$. Hence, $D'' + D_f$ is non-negative on Γ except possibly for v_0 and $(D'' + D_f)(v_0) > D''(v_0)$, contradicting our choice of D''. Thus, we have established the existence of a non-saturated point in $\partial O_{D''}$.

The proof now proceeds differently for $b(\Gamma) = 1$ (the base case of the induction argument) and $b(\Gamma) > 1$ (the induction step). If $b(\Gamma) = 1$, then Γ is comprised of one or more loops at v_0. Since v_0 is not a cut-point of Γ, Γ consists of a single loop.

As $O_{D''}$ is an interval, $\overline{O_{D''}}$ is also an interval. Assume first that the set $\overline{O_{D''}}$ is a non-trivial interval, i.e., $\overline{O_{D''}}$ is not comprised of a single point. Let v' and v'' be the two boundary points of $\overline{O_{D''}}$. By the choice of $O_{D''}$, both $D''(v')$ and $D''(v'')$ are positive. Consequently, $\partial O_{D''}$ does not contain a non-saturated point which is impossible. Hence, $\overline{O_{D''}}$ must be comprised of a single point (as it is non-empty as established earlier).

Let v be the only point of $\overline{O_{D''}}$. The point v must be non-saturated (as $\partial O_{D''}$ contains a non-saturated point). By the choice of $O_{D''}$, $D''(v)$ is positive. Hence, $D''(v) = 1$ and $D''(v') = 0$ for $v' \neq v_0, v$. Clearly, D'' is a v_0-reduced divisor on Γ.

We now consider the case that $b(\Gamma) > 1$. Let b_0 be a non-saturated point in $\partial O_{D''}$. By the choice of $O_{D''}$, either b_0 is a branching point or $D''(b_0) > 0$ (or both). If $D''(b_0) > 0$ and b_0 is non-branching, then b_0 is saturated. We infer that b_0 must be a branching point. Let E be the union of all the segments of Γ with ends v_0 and b_0 whose interiors are in $O_{D''}$. We now modify the metric graph Γ by deleting the interior of E and identifying the points v_0 and b_0 to the point v_0^*. Let Γ^* be the resulting metric graph. Note that v_0^* can be a cut-point in Γ^* and the modification procedure can create loops. Let D^* be the divisor with the value $D''(v_0) + D''(b_0)$ on the point v_0^* and equal to D'' on all the points of Γ^* except for v_0^*. As $b(\Gamma^*) < b(\Gamma)$, the induction implies the existence of a v_0^*-reduced divisor D^{**} such that $D^{**} = D^* + D_{f^*}$ for some a v_0^*-extremal rational function f^*. Extend f^* to a rational function f on Γ by setting f to be equal to $f(v_0^*)$ on E. Observe that f is v_0-extremal on Γ and set D''' to $D'' + D_f$.

We show that D''' is a v_0-reduced divisor on Γ which would finish the proof of the claim and thus the proof of the theorem.

It is straightforward to check that D''' is non-negative on Γ except possibly for v_0: this follows from the fact D^{**} is a v_0^*-reduced divisor on Γ^* and D''' is equal to D'' (and thus to 0) in the interior of E. Hence, it remains to verify that every closed connected set X, such that $v_0 \notin X$, contains a non-saturated point $v \in \partial X$.

By Lemma 9 and the choice of D'', the point b_0 is contained in $\partial O_{D'''}$ and it is still non-saturated. Let X be a closed connected set avoiding v_0.

Suppose first that there exists a point $v \in O_{D'''} \cap \partial X$. Then $D'''(v) = 0$, and therefore v is non-saturated with respect to D''' and X. Thus without loss of generality we assume $X \cap O(D''') = \emptyset$. If $b_0 \in X$, then b_0 is a non-saturated point of ∂X. Therefore we can further assume that $X \cap E = \emptyset$.

Consider now the set X^* corresponding to X in Γ^*. Since D^* is v_0^*-reduced, there is a point $v \in \partial X^*$ that is non-saturated. However, the point v is also contained in ∂X and since $D'''(v) = D^{**}(v)$ and the number of edges leaving X at v in Γ is the same as the number of edges leaving X^* at v in Γ^*, the point v is also non-saturated in Γ. The proof of the claim (and thus the proof of the whole theorem) is now finished. $\square$

2.2 The Riemann-Roch theorem for metric graphs

Theorem 10, in particular, allows us to infer information about the structure of non-special divisors on a metric graph. We now present a class of non-special divisors that is of primary interest to us in our later considerations.

Let P be an ordered sequence of finitely many points of Γ. We say that the set of points in P is is the *support* of P and denote it by supp P. The sequence P can also be viewed as a linear order $<_P$ on supp P. If $B(\Gamma) \subseteq$ supp P, then P is a *permutation* of points of Γ. The set of all permutations of points of Γ is denoted by $\mathcal{P}(\Gamma)$

We now define a divisor ν_P corresponding to a permutation P. A segment L of Γ is a P*-segment* if both ends of L belong to supp P, and the interior of L is disjoint from supp P. For $v \in$ supp P, let $S_P(v)$ denote the set of P-segments of Γ with one end at v and the other end preceding v in the order determined by P. Finally, let

$$\nu_P = \sum_{v \in \text{ supp } P} (|S_p(v)| - 1)(v).$$

It is easy to verify that $\deg(\nu_P) = g - 1$. We start our investigation of divisors corresponding to permutations by proving two simple propositions.

Proposition 11. *Let P be a permutation of points of a metric graph Γ. For every point v of Γ that is not contained in* supp P*, there exists a permutation P' such that* supp P' = supp $P \cup \{v\}$ *and $\nu_P = \nu_{P'}$.*

Proof. Such a permutation P' can be obtained by inserting the point v in the sequence P between the boundary points of the (unique) segment containing v. □

Proposition 12. *If P is permutation of points of a metric graph Γ, then $\nu_P \in \mathcal{N}$.*

Proof. It suffices to show that $|\nu_P| = \emptyset$. Let $D = \nu_P + D_f \in \text{Div}(\Gamma)$ for some rational function f on Γ. Let X be the set of points of Γ at which f achieves maximum. By Proposition 11, we can assume without loss of generality that $\partial X \subseteq$ supp P. Let v be the minimum point of ∂X in the order determined by P and k the number of edges leaving X at v. It holds that $\nu_P(v) < k \leq -\text{ord}_f v$. Consequently, $D(v) < 0$. We conclude that every divisor equivalent to ν_P is ineffective, as desired. □

As a corollary of Theorem 10, we can now show that every divisor is either equivalent to an effective divisor, or is equivalent to a divisor dominated by ν_P for some permutation P, and not both.

Corollary 13. *Let Γ be a metric graph. For every $D \in \mathrm{Div}(G)$, exactly one of the following holds*

1. *$r(D) \geq 0$; or*
2. *$r(\nu_P - D) \geq 0$ for some permutation P.*

Proof. Fix arbitrarily a point v_0 of Γ. By Theorem 10, there exists a v_0-reduced divisor D_0 equivalent to D. If $D_0(v_0) \geq 0$ then D_0 is effective, and (1) holds. Therefore we may assume that $D_0(v_0) < 0$. Let $Q = B(\Gamma) \cup \operatorname{supp} D_0$. Clearly, the set Q is finite. Our next goal is to order the points of the set Q in such a way that the resulting order determines a permutation P satisfying (2).

Set $q_1 = v_0$. Assume we have already defined the points $q_1, \ldots, q_k$ (for some $k \in \mathbb{N}$) and that $Q \setminus \{q_1, \ldots, q_k\} \neq \emptyset$. Let Y_k be obtained from Γ by removing all the points of $Q \setminus \{q_1, \ldots, q_k\}$ and taking the connected component of v_0 in what remains. Since Y_k is an open set, its complement X_k is a closed set. Observe that $\partial X_k \subseteq Q \setminus \{q_1, \ldots, q_k\}$. Hence, ∂X_k contains a point $v \in Q$ that is not saturated with respect to the component of X_k that it belongs to. We define q_{k+1} to be this point v.

Let P be the resulting ordering of Q. The value of $\nu_P(q_1)$ is equal to -1, and the value of $\nu_P(q_{k+1})$ is equal to the number of edges leaving X_k at q_{k+1} decreased by one. Since q_{k+1} is not saturated with respect to the set X_k, and $\operatorname{supp} D_0 \subseteq Q$, we infer that $D_0(v) \leq \nu_P(v)$ for every $v \in B(\Gamma) \cup \operatorname{supp} D_0$. We conclude that $\nu_P - D_0 \geq 0$ and (2) holds.

If conditions (1) and (2) held simultaneously, Lemma 5 would imply $r(\nu_P) \geq r(D) + r(\nu_P - D) \geq 0$ for permutation P satisfying (2), in contradiction with Proposition 12. $\square$

We immediately get that non-special divisors are equivalent to divisors corresponding to the permutation of points.

Corollary 14. *If ν is a non-special divisor on a metric graph Γ of genus g, then $\nu \sim \nu_P$ for some permutation P of a finite set of points of Γ.*

Proof. By Corollary 13, there exists a divisor D_0 equivalent to ν and a permutation P of a finite set of points of Γ such that $D_0 \leq \nu_P$. Since $\deg(\nu_P) = \deg(\nu) = \deg(D_0) = g - 1$, it must hold that $D_0 = \nu_P$. In particular, $\nu_P \sim \nu$. $\square$

Corollary 15. *Divisors on a metric graph* Γ *satisfy conditions (RR1) and (RR2) from Theorem 4.*

Proof. The condition (RR1) immediately follows from Corollaries 13 and Corollary 14.

To prove (RR2), it suffices to show that, for every $D \in \mathcal{N}$, we have $K_\Gamma - D \in \mathcal{N}$. By Corollary 14, it suffices to show that $K_\Gamma - \nu_P \in \mathcal{N}$ for every permutation P of points of Γ. Let $\bar{P}$ be the *reverse* of P (i.e. $\operatorname{supp} P = \operatorname{supp} \overline{P}$, $v <_P w \Leftrightarrow w <_{\bar{P}} v$). Then, for every point $v \in \operatorname{supp} P$, it holds that $\nu_P(v) + \nu_{\bar{P}}(v) = \deg(v) - 2 = K_\Gamma(v)$. Proposition 12 now yields $K_\Gamma - \nu_P = \nu_{\bar{P}} \in \mathcal{N}$, as desired. $\square$

Corollary 15 implies Theorem 2, as we have noted previously.

We finish this section with establishing a formula for rank of divisors on metric graphs that will be central in our later analysis of the rank.

Corollary 16. *If* D *is a divisor on a metric graph* Γ*, then the following formula holds:*

$$r(D) = \min_{\substack{D' \sim D \\ P \in \mathcal{P}(\Gamma)}} \deg^+(D' - \nu_P) - 1.$$

Proof. Lemma 6 ca be applied by Corollary 15, and we infer that

$$r(D) = \min_{\substack{D' \sim D \\ \nu \in \mathcal{N}}} \deg^+(D' - \nu) - 1 \ .$$

By Proposition 12, the minimum in the statement of the corollary is taken over smaller set of parameters than the minimum in the above formula. Hence, it is enough to show that there exist $D'' \sim D$ and $P \in \mathcal{P}(\Gamma)$ such that $r(D) = \deg^+(D'' - \nu_P) - 1$. Let $D' \sim D$ and $\nu \in \mathcal{N}$ be chosen so that $r(D) = \deg^+(D' - \nu) - 1$.

By Corollary 14, we have $\nu \sim \nu_P$ for some permutation P of points of Γ. Setting $D'' = D' + (\nu_P - \nu)$ yields

$$r(D) = \deg^+(D' - \nu) - 1 = \deg^+(D'' - \nu_P) - 1,$$

as desired. $\square$

Note that the result analogous to Corollary 16 also holds for non-metric graphs, as shown in [2]. Let $\mathcal{P}(G)$ denote the set of all permutations of $V(G)$. As in the case of metric graphs, we can define the divisor ν_P corresponding to $P \in \mathcal{P}(G)$ by setting $\nu_P(v)$ to be equal to the number of edges from v

to vertices in $V(G)$ preceding v, decreased by one. The next formula for the rank of a divisor on a finite graph G was established by Baker and Norine [2].

Lemma 17. *The following formula holds for the rank of every divisor D on a graph G:*

$$r(D) = \min_{\substack{D' \sim D \\ P \in \mathcal{P}(G)}} \deg^+(D' - \nu_P) - 1 \ .$$

3 Rank of divisors on metric graphs

In this section we show that the divisor and the permutation in Corollary 16 can be assumed to have a very special structure. We establish a series of lemmas strengthening our assumptions on this structure. It will then follow from our results that the rank of a divisor on a graph and on the corresponding metric graph are the same.

Lemma 18. *Let D be a divisor on a metric graph Γ. Suppose there exists a permutation P of points of Γ such that $r(D) = \deg^+(D - \nu_P) - 1$. Then there also exists a permutation P' of the points of $B(\Gamma) \cup \operatorname{supp} D$ such that $r(D) = \deg^+(D - \nu_{P'}) - 1$.*

Proof. By Proposition 11, we can assume that the support of P contains all the points of $B(\Gamma) \cup \operatorname{supp} D$. Choose among all permutations P' satisfying $r(D) = \deg^+(D - \nu_{P'}) - 1$ and $B(\Gamma) \cup \operatorname{supp} D \subseteq \operatorname{supp} P'$ a permutation such that $|\operatorname{supp} P' \setminus (B(\Gamma) \cup \operatorname{supp} D)|$ is minimal.

If $\operatorname{supp} P' = B(\Gamma) \cup \operatorname{supp} D$, then the lemma holds. Assume that there exists a point $v_0 \in \operatorname{supp} P' \setminus (B(\Gamma) \cup \operatorname{supp} D)$. Let $v_1, v_2 \in \operatorname{supp} P'$ be such that the segments in Γ with ends v_0 and v_i, for $i = 1, 2$, contain no other points of $\operatorname{supp} P'$. For simplicity, we consider only the case when $v_1 \neq v_2$, but our arguments readily translate to the case when $v_1 = v_2$. If $v_1 \neq v_2$, we can assume by symmetry that $v_1 <_{P'} v_2$.

Consider now the permutation P'' obtained from P' by removing the point v_0. We shall distinguish three cases based on the mutual order of v_0, v_1 and v_2 in P', and conclude in each of the cases that $\deg^+(D - \nu_{P''}) \leq \deg^+(D - \nu_{P'})$. This, together with the fact that $\operatorname{supp} P'' \subsetneq \operatorname{supp} P'$ will contradict the choice of P'.

If $v_0 <_{P'} v_1$ and $v_0 <_{P'} v_2$, then $\nu_{P'}(v_0) = -1$. Observe that $\nu_{P''}(v_1) =$

$\nu_{P'}(v_1) - 1$, $\nu_{P''}(v_0) = 0$, and $\nu_{P''}(v) = \nu_{P'}(v)$ for $v \neq v_0, v_1$. We infer that

$$\begin{aligned}&\deg^+(D - \nu_{P'}) - \deg^+(D - \nu_{P''}) = \\ &\quad = 1 + \max\{D(v_1) - \nu_{P'}(v_1), 0\} - \max\{D(v_1) - \nu_{P'}(v_1) + 1, 0\} \geq 0 \ .\end{aligned}$$

Therefore $\deg^+(D - \nu_{P''}) \leq \deg^+(D - \nu_{P'})$.

If $v_1 <_{P'} v_0 <_{P'} v_2$, then $\nu_{P'} = \nu_{P''}$ and again $\deg^+(D - \nu_{P''}) = \deg^+(D - \nu_{P'})$.

It remains to consider the case $v_1 <_{P'} v_0$ and $v_2 <_{P'} v_0$. Observe that $\nu_{P'}(v_0) = 1$, $\nu_{P''}(v_0) = 0$, $\nu_{P''}(v_2) = \nu_{P'}(v_2) + 1$, and $\nu_{P''}(v) = \nu_{P'}(v)$ for $v \neq v_0, v_2$. We conclude that

$$\begin{aligned}&\deg^+(D - \nu_{P'}) - \deg^+(D - \nu_{P''}) = \\ &\quad = \max\{D(v_2) - \nu_{P'}(v_2), 0\} - \max\{D(v_2) - \nu_{P'}(v_2) - 1, 0\} \geq 0 \ .\end{aligned}$$

Consequently, $\deg^+(D - \nu_{P''}) \leq \deg^+(D - \nu_{P'})$. □

Next, we show that the divisor $D' \sim D$ that minimizes

$$\min_{P \in \mathcal{P}(\Gamma)} \deg^+(D' - \nu_P)$$

can be assumed to be non-negative everywhere except for $B(\Gamma)$.

Lemma 19. *Let D be a divisor on a metric graph Γ. There exist a divisor D' equivalent to D and a permutation P of the points of $B(\Gamma) \cup \operatorname{supp} D'$ such that $r(D) = \deg^+(D' - \nu_P) - 1$, and D' is non-negative on the interior of every edge of Γ.*

Proof. By Corollary 16 and Lemma 18, there exist a divisor D_0 equivalent to D and a permutation P_0 of the points of $B(\Gamma) \cup \operatorname{supp} D_0$ such that $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$. Among all such divisors let us consider the divisor D_0 such that the sum

$$S = \sum_{v \,\in\, \operatorname{supp} D_0 \setminus B(\Gamma)} \min\{0, D_0(v)\}$$

is maximal. If $S = 0$, then the divisor D_0 is non-negative on the interior of every edge of Γ, and there is nothing to prove. Hence, we assume $S < 0$ in the rest, i.e., there exists an edge e with an internal point where D_0 is negative.

Let $v_1, \dots, v_k$ be the longest sequence of points of supp D_0 in the interior of e, such that $D_0(v_i) < 0$ for $i = 1, \dots, k$ and the points are consecutive points, i.e., there is no point of supp D_0 on the segment between v_i and v_{i+1}, $i = 1, \dots, k-1$. Let w_1 be the point of $B(\Gamma) \cup \text{supp } D_0$ such that the segment between v_1 and w_1 contains no point of $B(\Gamma) \cup \text{supp } D_0$ and $w_1 \neq v_2$, and let w_2 be the point of $B(\Gamma) \cup \text{supp } D_0$ such that the segment between v_k and w_2 contains no point of $B(\Gamma) \cup \text{supp } D_0$ and $v_{k-1} \neq w_2$ (if $k = 1$ and e is not a loop, we require $w_1 \neq w_2$).

We now modify the divisor D_0 and the permutation P_0. By symmetry, we can assume that $\text{dist}(w_1, v_1) \leq \text{dist}(w_2, v_k)$. Let $d_0 = \text{dist}(w_1, v_1)$, let L be the segment from w_1 to w_2 that contains $v_1, \dots, v_k$ and let v'_k be the point of L at distance d_0 from w_2. Consider the rational function f equal to 0 on L between v_1 and v'_k and $f(v) = \min\{d_0, \text{dist}(v, v_1), \text{dist}(v, v'_k)\}$ elsewhere. Observe that $\text{ord}_f w_1 = \text{ord}_f w_2 = -1$ if $w_1 \neq w_2$, $\text{ord}_f w_1 = -2$ if $w_1 = w_2$, $\text{ord}_f v_1 = \text{ord}_f v'_k = 1$ if $v_1 \neq v'_k$, $\text{ord}_f v_1 = 2$ if $v_1 = v'_k$, and $\text{ord}_f v = 0$ if $v \neq w_1, w_2, v_1, v'_k$. In addition, observe that if $w_1 = w_2$, then L is a loop in Γ, and $w_1 = w_2$ is a branching point of Γ.

Let $D'_0 = D_0 + D_f$. We first show that the sum

$$S' = \sum_{v \,\in\, \text{supp } D'_0 \setminus B(\Gamma)} \min\{0, D'_0(v)\}$$

is strictly larger than S. The value of D'_0 is smaller than the value of D_0 only at w_1 and w_2. If w_1 is a branching point, then the change of the value of the divisor at w_1 does not affect the sum. Otherwise, the points w_1 and w_2 are distinct (as we have observed earlier), and $D_0(w_1) \geq 1$ by the choice of w_1. Hence, $D'_0(w_1) \geq 0$ and the sum is not affected by the corresponding summand. Analogous statements are true for the point w_2. We infer from $\text{ord}_f v_1 > 0$ that $D'_0(v_1) > D_0(v_1)$. Since $D_0(v_1) < 0$, this change increases the sum by one. Finally, the change at v'_k either increases the sum by one (if $D_0(v'_k) < 0$) or does not affect the sum (if $D_0(v'_k) \geq 0$) at all. We conclude that $S' \geq S + 1$.

We next modify the permutation P_0 to P'_0 in such a way that $\deg^+(D_0 - \nu_{P_0}) = \deg^+(D'_0 - \nu_{P'_0})$. Without loss of generality, we assume that $v'_k \in \text{supp } P_0$ (cf. Proposition 11). The permutation P'_0 is obtained from P_0 as follows: all the points of supp P_0 distinct from $v_1, \dots, v_k$ and v'_k form the initial part of the permutation in the same order as in P_0, and the points $v_1, v_2, \dots, v_k, v'_k$ then follow (in this order).

Let $W = \{w_1, w_2, v_1, \dots, v_k, v'_k\}$. For simplicity, let us assume that the points w_1 and w_2 are distinct, as well as the points v_1, v_k and v'_k. It is easy to verify that all our arguments translate to the setting when some of these points coincide. Since $D_0(v) = D_0'(v)$ and $\nu_{P_0}(v) = \nu_{P_0'}(v)$ for all points $v \notin W$, the following holds:

$$\begin{aligned}&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) = \\ &\quad = \sum_{v \in W} \left(\max\{0, (D_0 - \nu_{P_0})(v)\} - \max\{0, (D_0' - \nu_{P_0'})(v)\}\right) \ .\end{aligned}$$

By the choice of the points $v_1, \ldots, v_k$, we have $D_0(v_i) \leq -1$ and therefore $(D_0 - \nu_{P_0})(v) \leq 0$ for $v \in W \setminus \{v_k', w_1, w_2\}$. Note also that $\nu_{P_0'}(v_i) = 0$ and $\nu_{P_0'}(v_k') = 1$. Finally, note that $D_0'(v_i) \leq D_0(v_i) + 1 \leq 0$, unless $v_i = v_k'$, and $D_0'(v_k') \leq 1$. As a result, we have $(D_0' - \nu_{P_0'})(v) \leq 0$ for $v \in W \setminus \{w_1, w_2\}$. Consequently, we obtain the following:

$$\begin{aligned}&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) = \\ &\quad = \max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \\ &\qquad + \max\{0, (D_0 - \nu_{P_0})(w_2)\} - \max\{0, (D_0' - \nu_{P_0'})(w_2)\} \ .\end{aligned}$$

Since $\mathrm{ord}_f w_1 = -1$, we have $D_0'(w_1) = D_0(w_1) - 1$. On the other hand, the value $\nu_{P_0'}(w_1)$ is either equal to $\nu_{P_0}(w_1)$, or to $\nu_{P_0}(w_1) - 1$ (the latter is the case if $w_1 >_{P_0} v_1$). We conclude that $(D_0' - \nu_{P_0'})(w_1)$ is equal to either $(D_0 - \nu_{P_0})(w_1)$ or $(D_0 - \nu_{P_0})(w_1) - 1$. Hence,

$$\max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \geq 0 \ .$$

An entirely analogous argument yields that

$$\max\{0, (D_0 - \nu_{P_0})(w_2)\} - \max\{0, (D_0' - \nu_{P_0'})(w_2)\} \geq 0 \ .$$

Consequently, we obtain that

$$\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) \geq 0 \ .$$

Since $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$, and D_0' is equivalent to D, the inequality above must be the equality, and thus $r(D) = \deg^+(D_0' - \nu_{P_0'}) - 1$. Since the permutation P_0' can be chosen in such a way that $\mathrm{supp}\, P_0' = B(\Gamma) \cup \mathrm{supp}\, D_0'$ by Lemma 18, and $S' > S$, the existence of D_0' contradicts the choice of D_0. $\square$

Next, we show that the divisor D' can be assumed to be zero outside $B(\Gamma)$, except possibly for a single point on each edge, where its value could be equal to one.

Lemma 20. *Let D be a divisor on a metric graph Γ. There exist a divisor D' equivalent to D and a permutation P of the points of $B(\Gamma) \cup \operatorname{supp} D'$ such that $r(D) = \deg^+(D' - \nu_P) - 1$, and every edge e of Γ contains at most one point v where D' is non-zero, and if such a point v exists, then $D'(v) = 1$.*

Proof. By Lemma 19, there exist a divisor D_0 equivalent to D and a permutation P_0 of the points of $B(\Gamma) \cup \operatorname{supp} D_0$ such that $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$, and D_0 is non-negative in the interior of every edge of Γ. Among all such divisors, consider the divisor D_0 such that the sum

$$S = \sum_{v \in \operatorname{supp} D_0 \setminus B(\Gamma)} D_0(v)$$

is minimal. If every edge e contains at most one point v where D_0 is non-zero, and $D_0(v) = 1$ at such a point v, then the lemma holds. We assume that D_0 does not have this property for a contradiction.

Choose an edge e such that the sum of the values of D_0 in the interior of e is at least two. Let w_1 and w_2 be the end points of e and $v_1, \dots, v_k$ all the points of supp D_0 inside e ordered from w_1 to w_2. In the rest we assume that $w_1 \neq w_2$ and $v_1 \neq v_k$. As in the proof of the previous lemma, our arguments readily translate to the setting when some of these points are the same, but this assumption helps us to avoid technical complications during the presentation of the proof. Let us note, in order to assist the reader with the verification of the remaining cases, that if $v_1 = v_k$, then $D_0(v_1) \geq 2$.

By symmetry, we can assume that $\operatorname{dist}(w_1, v_1) \leq \operatorname{dist}(w_2, v_k)$. Let $d_0 = \operatorname{dist}(w_1, v_1)$ and let w_2' be the point on the segment between v_k and w_2 at distance d_0 from v_k. For the sake of simplicity, we assume that $w_2 \neq w_2'$; again, our arguments readily translate to the setting when $w_2 = w_2'$. Consider the rational function f equal to 0 on the points outside the edge e and on the segment between w_2 and w_2' and $f(v) = \min\{\operatorname{dist}(v, w_1), \operatorname{dist}(v, w_2'), d_0\}$ elsewhere. Observe that $\operatorname{ord}_f w_1 = \operatorname{ord}_f w_2' = 1$, $\operatorname{ord}_f v_1 = \operatorname{ord}_f v_k = -1$, and $\operatorname{ord}_f v = 0$ if $v \neq w_1, w_2', v_1, v_k$.

Let $D_0' = D_0 + D_f$. Since $D_0'(v_1) = D_0(v_1) - 1 \geq 0$, $D_0'(v_k) = D_0(v_k) - 1 \geq 0$ and $D_0'(w_2') = 1$, the sum

$$S' = \sum_{v \in \operatorname{supp} D_0' \setminus B(\Gamma)} D_0'(v)$$

is equal to $S-1$, and D'_0 is non-negative in the interior of all the edges of Γ.

Next, we construct a permutation P'_0 such that such that

$$r(D) = \deg^+(D'_0 - \nu_{P'_0}) - 1.$$

First, insert w'_2 into P_0 between v_k and w_2, preserving the order of v_k and w_2 (this did not change ν_{P_0}, see Proposition 11). The permutation P'_0 is obtained from P_0 as follows: the points $v_1, \ldots, v_k$ form the initial part of P'_0 in the same order as they appear in P_0, and the remaining points form the final part of P'_0, again in the same order as they appear in P_0.

It is easy to verify that $\nu_{P_0}(v) = \nu_{P'_0}(v)$ for all points $v \notin \{w_1, w'_2, v_1, v_k\}$. Hence,

$$\begin{aligned}
&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D'_0 - \nu_{P'_0}) = \\
&\quad = \max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(w_1)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(w'_2)\} - \max\{0, (D'_0 - \nu_{P'_0})(w'_2)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(v_1)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(v_k)\} - \max\{0, (D'_0 - \nu_{P'_0})(v_k)\} .
\end{aligned}$$

Let us first consider the points v_1 and w_1. We distinguish two cases based on the mutual order of v_1 and w_1 in P_0.

The case we consider first is that $v_1 <_{P_0} w_1$. We have $\nu_{P_0}(v_1) = \nu_{P'_0}(v_1) \leq 0$ and $\nu_{P_0}(w_1) = \nu_{P'_0}(w_1) \geq 0$. As $D'_0(v_1) = D_0(v_1) - 1 \geq 0$, we have that

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(v_1)\} = 1 .$$

As $D'_0(w_1) = D_0(w_1) + 1$, we have that

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(v_1)\} \geq -1 .$$

We conclude that

$$\begin{aligned}
&\max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(w_1)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D'_0 - \nu_{P'_0})(v_1)\} \geq 0 .
\end{aligned}$$

Let us deal with the other case when $v_1 >_{P_0} w_1$. Since $\nu_{P_0}(v_1) = \nu_{P'_0}(v_1) + 1$ and $D'_0(v_1) = D_0(v_1) - 1$, we have

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} = \max\{0, (D'_0 - \nu_{P'_0})(v_1)\} .$$

Similarly, since $\nu_{P_0}(w_1) = \nu_{P_0'}(w_1) - 1$ and $D_0'(w_1) = D_0(w_1) + 1$, we have

$$\max\{0, (D_0 - \nu_{P_0})(w_1)\} = \max\{0, (D_0' - \nu_{P_0'})(w_1)\} .$$

Therefore, in this case we also obtain that

$$\max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \\ + \max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D_0' - \nu_{P_0'})(v_1)\} = 0 .$$

A symmetric argument yields that

$$\max\{0, (D_0 - \nu_{P_0})(w_2')\} - \max\{0, (D_0' - \nu_{P_0'})(w_2')\} \\ + \max\{0, (D_0 - \nu_{P_0})(v_k)\} - \max\{0, (D_0' - \nu_{P_0'})(v_k)\} = 0 .$$

Hence,

$$\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) = 0 .$$

Since $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$, we have that $r(D) = \deg^+(D_0' - \nu_{P_0'})$. Since the permutation P_0' can be chosen in such a way that supp $P_0' = B(\Gamma) \cup \text{supp } D_0'$ by Lemma 18 and $S' < S$, the existence of D_0' and P_0' contradict the choice of D_0 and P_0. ☐

Finally, we show that, in addition to the conditions given in Lemma 20, the divisor D' can be assumed to be zero inside edges of a spanning tree of Γ.

Lemma 21. *Let D be a divisor on a metric graph Γ. There exists a divisor D' equivalent to D, a spanning tree T of Γ, and a permutation $P \in \mathcal{P}(\Gamma)$ such that $r(D) = \deg^+(D' - \nu_P) - 1$, D' is zero in the interior of every edge of T, and every edge $e \notin T$ contains at most one interior point v where $D'(v) \neq 0$, and, if such a point v exists, then $D'(v) = 1$.*

Proof. Let D' be a divisor equivalent to D, and let P be a permutation of the points of $B(\Gamma) \cup \text{supp } D'$ as in Lemma 20. We first show that the permutation P can be assumed to be such that all the non-branching points of supp P follow the branching points in the order determined by P.

Consider the permutation P' obtained from P by moving a point $v \in \text{supp } D' \setminus B(\Gamma)$ to the end of the permutation. We claim that $r(D) = \deg^+(D' - \nu_{P'}) - 1$.

By Corollary 16, it suffices to show that $\deg^+(D' - \nu_{P'}) \leq \deg^+(D' - \nu_P)$. Let w_1 and w_2 be the end points of the edge containing v. We consider in detail the case when $v <_P w_1$ and $v <_P w_2$; the other cases are analogous. As $D'(v) = 1$ and $\nu_P(v) = -1$, it holds that $D'(v) - \nu_P(v) = 2$ and $D'(v) - \nu_{P'}(v) = 0$. Similarly, $\nu_{P'}(w_i) = \nu_P(w_i) - 1$, and thus $D'(w_i) - \nu_{P'}(w_i) = D'(w_i) - \nu_P(w_i) + 1$ for $i = 1, 2$. We conclude that

$$\begin{aligned}
&\deg^+(D' - \nu_P) - \deg^+(D' - \nu_{P'}) = \\
&\quad = \max\{0, D'(w_1) - \nu_P(w_1)\} - \max\{0, D'(w_1) - \nu_{P'}(w_1)\} \\
&\qquad + \max\{0, D'(w_2) - \nu_P(w_2)\} - \max\{0, D'(w_2) - \nu_{P'}(w_2)\} \\
&\qquad + \max\{0, D'(v) - \nu_P(v)\} - \max\{0, D'(v) - \nu_{P'}(v)\} \\
&\quad \geq (D'(w_1) - \nu_P(w_1)) - (D'(w_1) - \nu_{P'}(w_1)) \\
&\qquad + (D'(w_2) - \nu_P(w_2)) - (D'(w_2) - \nu_{P'}(w_2)) + 2 \geq 0\ .
\end{aligned}$$

The claim now follows. Hence, we can assume without loss of generality that all the points of $\operatorname{supp} D' \setminus B(\Gamma)$ follow the points of $B(\Gamma)$ in the order determined by P, i.e., $\nu_P(v) = 1$ for $v \in \operatorname{supp} D' \setminus B(\Gamma)$.

Let us now color the edges of Γ with red and blue, so that the red edges contain in their interior a point v in Γ with $D'(v) = 1$ and the blue edges do not. Let $V_1, \ldots, V_k$ be the components of Γ formed by blue edges. Choose among all divisors D' equivalent to D, and permutations P, satisfying the conditions of Lemma 20, the divisor D' such that the number k of the components $V_1, \ldots, V_k$ is the smallest possible. If $k = 1$, there exists a spanning tree of Γ formed by the blue edges, and there is nothing to prove.

Assume now that $k \geq 2$ for the divisor D' which minimizes k. Recall that $v <_P v'$ for every $v \in B(\Gamma)$ and $v' \in \operatorname{supp} D' \setminus B(\Gamma)$. Let us call the red edges between a points of $V_1 \cap B(\Gamma)$ and points of $B(\Gamma) \setminus V_1$ *orange* edges. We can assume that the points of $V_1 \cap B(\Gamma)$ follow all the other points of $B(\Gamma)$ in the order determined by P, as every orange edge contains a point in $\operatorname{supp} D' \setminus B(\Gamma)$. For an orange edge e incident with a branching point v_1 of V_1, let $d(e)$ be the distance between v_1 and the point of $\operatorname{supp} D'$ in the interior of e. Let d_0 be the minimum $d(e)$ taken over all orange edges e.

Consider the following rational function f: $f(v) = 0$ for points v on edges between two branching points of V_1,

$$f(v) = \min\{d_0, \max\{0, \operatorname{dist}(v_1, v) + d_0 - d(e)\}\}$$

for points v on orange edges incident with any branching point v_1 of V_1, and $f(v) = d_0$ for the remaining points of Γ. Set $D'' = D' + D_f$. Clearly, D'' is a divisor equivalent to D that is non-zero on at most one point in the interior of every edge of Γ and is equal to one at such a point. Moreover, since the blue edges remain blue and the orange edges e with $d(e) = d_0$ become blue, the number of components formed by blue edges in D'' is smaller than this number in D'.

We now find a permutation P' of the points of $B(\Gamma) \cup \text{supp } D''$ such that $\deg^+(D'' - \nu_{P'}) = \deg^+(D' - \nu_P)$. The existence of such a permutation P' would contradict our choice of D'. The permutation P' is defined as follows: The branching points of Γ are ordered as in P, and they precede all the points of $\text{supp } D'' \setminus B(\Gamma)$. The points of $\text{supp } D'' \setminus B(\Gamma)$ are ordered arbitrarily. If v is a point of $B(\Gamma) \cup \text{supp } D''$ that is not contained inside an orange edge, and that is not a branching point of V_1, then $D''(v) = D'(v)$ and $\nu_{P'}(v) = \nu_P(v)$. Hence, such points do not affect $\deg^+(D'' - \nu_{P'})$.

We now consider a branching point v of V_1. Let ℓ be the number of edges incident with v that are orange with respect to D' and blue with respect to D''. Clearly, ℓ is the number of orange edges e incident with v such that $d(e) = d_0$. By the choice of f, $D''(v) = D'(v) + \ell$. In addition, since the other branching points, incident with such edges, precede v in the order determined by P', $\nu_{P'}(v) = \nu_P(v) + \ell$. Hence, $(D'' - \nu_{P'})(v) = (D' - \nu_P)(v)$.

It remains to consider internal points of orange edges. Let v be such a point. If v is contained in the interior of an orange edge e with respect to D', then $D'(v) = 1$ and $\nu_P(v) = 1$, i.e., $(D' - \nu_P)(v) = 0$. If v is contained in the interior of an orange edge e with respect to D'', then $D''(v) = 1$ and $\nu_{P'}(v) = 1$, i.e., $(D'' - \nu_{P'})(v) = 0$. We conclude that such points do not affect $\deg^+(D'' - \nu_{P'})$ at all. Consequently, $\deg^+(D'' - \nu_{P'}) = \deg^+(D' - \nu_P)$, as desired. □

We are now ready to prove Theorem 3 stated in Subsection 1.1.

Proof of Theorem 3. By Lemma 17, there exist a divisor $D' \in \text{Div}(G)$, $D' \sim_G D$, and $P \in \mathcal{P}(G)$ such that $r_G(D) = \deg^+(D' - \nu_P) - 1$. Since $D' \sim_\Gamma D$, we have $r_\Gamma(D) \le r_G(D)$ by Corollary 16. In the rest of the proof we prove the opposite inequality.

Let T be a spanning tree of Γ, D' be a divisor equivalent to D, and let P be a permutation of the points of $B(\Gamma) \cup \text{supp } D'$ as in Lemma 21. In particular, $r_\Gamma(D) = \deg^+(D' - \nu_P) - 1$ and $D'(v) = 1$ for every $v \in \text{supp } D' \setminus B(\Gamma)$.

Let us now color the edges of G with red and blue as in Lemma 21: red edges contain a point v in Γ with $D'(v) = 1$, and blue edges do not. Let f be the rational function such that $D' = D + D_f$. We will now establish two auxiliary claims about red and blue edges.

Claim 1. *Let v_1 and v_2 be two distinct vertices of Γ. The difference $f(v_1) - f(v_2)$ is an integer.*

Let us traverse from v_1 to v_2 using the edges of the tree T. As the difference of the values of f between the two end points of each edge of the path is an integer, $f(v_1) - f(v_2)$ must also be an integer.

Claim 2. *Every edge of Γ is blue.*

Assume that there is a red edge v_1v_2. By symmetry, we can assume that $f(v_1) \leq f(v_2)$. Let d be the distance between the single point of supp D' inside the edge v_1v_2 in Γ to v_1. Note that $0 < d < 1$. It is easy to infer that $f(v_2) - f(v_1) = k - d$ for some integer k. By Claim 1, the difference $f(v_2) - f(v_1)$ should be an integer which contradicts our assumption that the edge v_1v_2 is red.

Since all the edges of Γ are blue by Claim 2, supp $D' \subseteq B(\Gamma)$. Hence, supp $P = B(\Gamma)$, i.e., P can be viewed a permutation of $V(G)$, and the divisor D' can be viewed as a divisor on G, as supp $D' \subseteq B(\Gamma)$. We conclude $r_\Gamma(D) = r_G(D)$. □

As corollary of Theorem 3 we can prove that the rank of a divisor on a graph is preserved under subdivision. We say that a bijection φ between the points of a metric graph Γ and the points of a metric graph Γ' is a *homothety* if there exists a real number $\alpha > 0$ such that $\mathrm{dist}_\Gamma(v, w) = \alpha \cdot \mathrm{dist}_{\Gamma'}(\varphi(v), \varphi(w))$ for every two points v and w of Γ. Note that composition of a rational function with a homothety is a rational function, and thus a homothety preserves the rank of divisors.

Corollary 22. *Let D be a divisor on a graph G and let G^k be the graph obtained from G by subdividing each edge of G exactly k times. The ranks of D in G and in G^k are the same.*

Proof. Let Γ be the metric graph corresponding to G. Observe that there exists a homothety from Γ to the metric graph Γ' corresponding to G^k. Since the rank of D in G is equal to the rank of D in Γ by Theorem 3 and the rank of D in G^k is equal to the rank of D in Γ' by the same theorem, the ranks of D in G and in G^k are the same. □

4 An algorithm for computing the rank

We now present the main algorithmic result of this paper. We describe an algorithm which, given a metric graph Γ and a divisor D on it, computes its rank. It not a priori clear that such an algorithm has to exist (for example, a famous result of Matiyasevich [8] states that there exists no universal algorithm for solving Diophantine equations). If the lengths of all the edges of D and all the distances of non-zero values of D to the branching points are rational, then the problem is solvable on a Turing machine. However, this need not be the case in general. As the input can contain irrational numbers, we assume real arithmetic operations with infinite precision to be allowed in our computational model. The bound on the running time of our algorithm can easily be read from its construction; it is a simple function depending on the number of edges, number of vertices of Γ, the ratio between the longest and the shortest edge in Γ, and the values of D. The running time is not more than exponential in any of these parameters.

Theorem 23. *There exists an algorithm that for a divisor D on a metric graph Γ computes the rank of D.*

Proof. The algorithm utilizes Lemma 21. We will show that there are only finitely many divisors equivalent to a given divisor D that satisfy the conditions in the statement of Lemma 21.

We write ℓ_e for the length of an edge e. Without loss of generality, we can assume that supp $D \subseteq B(\Gamma)$ (introduce new branching points incident with only two edges if needed). We can also assume that the length of each edge of Γ is at least one. Let n be the number of branching points of Γ, m the number of edges of Γ, $M = \max_{v \in \operatorname{supp} D} |D(v)|$, and ℓ the maximal length of an edge of Γ. We assume that $n \geq 2$ (and thus $m \geq 1$) since otherwise Γ is formed by a single point w_0 and $r(D) = \max\{D(w_0), -1\}$. Similarly, we can also assume that $M \geq 1$ since otherwise D is equal to zero at all points and thus $r(D) = 0$. Finally, let $U = 3(nM+m)(m+1)^{n-1}+1$.

We first describe the algorithm and then verify its correctness. Fix an arbitrary vertex $w \in B(\Gamma)$. The algorithm ranges through all spanning trees T of Γ (here, T is the set of edges of the tree, i.e., $|T| = n-1$) and all functions $F : T \to \{-U, -U+1, \ldots, U-1, U\}$.

The algorithm then constructs all rational functions f on Γ such that for every branching point $v \in B(\Gamma)$ we have

$$f(v) = \sum_{i=1}^{k} F(e_i)\ell_{e_i} ,$$

where $e_1, e_2, \ldots, e_k$ are the edges of T on the path from w to v, f is linear on every edge of T, and $\mathrm{ord}_f v \neq 0$ for at most one point v on every edge not in T (and $\mathrm{ord}_f v = 1$ for such a point v if it exists).

Let us observe that there are only finitely many rational functions f satisfying the above constraints. Indeed, the function f is uniquely defined on edges of T as it should be linear on such edges. Consider now an edge e between branching points v_1 and v_2 that is not contained in T. By symmetry, we can assume that $f(v_1) \leq f(v_2)$. If f is not linear on e, then e contains a point v with $\mathrm{ord}_f v = 1$, and f is linear on e everywhere except for v. We write d_v for the distance of v from v_2 on e. It is easy to infer that $f(v_2) - f(v_1) - d_v$ must be an integral multiple of ℓ_e. Hence, there are at most $(f(v_2) - f(v_1))/\ell_e$ choices for such a vertex v in the interior of e and each such choice uniquely determines the behavior of f on e (note, that it can also be impossible to extend f to e at all). If $(f(v_2) - f(v_1))/\ell_e$ is an integer, the function f can be linear on e (which is another possibility of the behavior of f on e). We conclude that there are only finitely many rational functions f that satisfy conditions described in the previous paragraph.

The algorithm now computes the divisor $D' = D + D_f$, and then ranges through all permutations P of the points $B(\Gamma) \cup \mathrm{supp}\ D'$. For each such permutation, the value of $\deg^+(D' - \nu_P) - 1$ is computed and the minimum of all such values over all the choices of T, F (and thus f) and P is output as the rank of D. Since the numbers of choices of T, F and P are finite, the algorithm eventually finishes and outputs the rank of D.

We have to verify that the above algorithm is correct. By Corollary 16, the output value is greater than or equal to the rank of D. Hence, we have to show that the algorithm at some point of its execution considers $D' \in \mathrm{Div}(\Gamma)$ and $P \in \mathcal{P}(\Gamma)$ such that $\deg^+(D' - \nu_P) - 1 = r(D)$. Consider now the divisor D' and the permutation P as in Lemma 21. Since $\mathrm{supp}\ P = B(\Gamma) \cup \mathrm{supp}\ D'$, and the algorithm ranges through all permutation P of $B(\Gamma) \cup \mathrm{supp}\ D'$ for every constructed divisor D', it is enough to show that the algorithm constructs a rational function f such that $D' = D + D_f$.

Consider the step when the algorithm ranges through T as in Lemma 21 and let f_0 be the rational function given by the lemma. We can assume without loss of generality that $f_0(w) = 0$.

We establish that there exists a function $F : T \to \{-U, \dots, U\}$ such that f_0 can be constructed (as described above) from F. The existence of such a function F will yield the correctness of the presented algorithm. In order to establish the existence of F, it is enough to show that absolute value of the slope of f_0 is bounded by U on every edge of T . This will be achieved in two steps. In the first step, we find a bound for $|\mathrm{ord}_{f_0} v|$ for any $v \in B(\Gamma)$, and, in the second step, we use this bound to get a restriction on the slope of f_0 on every edge of Γ.

It can be inferred from the definition of the rank that $r(D) \leq \deg(D)$. Hence, $r(D) \leq nM$. We now show that $|D'(v)| \leq 2(nM + m)$ for every $v \in B(\Gamma)$. If there exists a branching point v_0 with $D'(v_0) > 2(nM + m)$, then

$$nM \geq r(D) = \deg^{+}(D' - \nu_P) \geq D'(v_0) - \nu_P(v_0) > 2nM + m - m \geq 2nM,$$

which is impossible. On the other hand, if there exists a branching point v_0 with $D'(v_0) \leq -2(nM + m)$, then $D'(v_0) - \nu_P(v_0) \leq -2(nM + m) + 1 < 0$ and thus $\deg^{+}(D' - \nu_P) = \deg^{+}(D'')$, where $D''(v_0) = 0$ and $D''(v) = (D' - \nu_P)(v)$ for $v \neq v_0$. Observe that

$$\begin{aligned} \deg(D'') &\geq 2(nM + m) - \deg(D') - \deg(\nu_P) \\ &\geq 2(nM + m) - nM - (m - n) > nM + m\ . \end{aligned}$$

Since $\deg^{+}(D'') \geq \deg(D'')$, we have

$$\deg^{+}(D' - \nu_P) \geq \deg^{+}(D'') > nM + m \geq r(D),$$

which contradicts our choice of D' and P. We conclude that $|D'(v)| \leq 2(nM+m)$, and that $|\mathrm{ord}_f v| \leq M + |D'(v)| \leq 3(nM+m)$ for every $v \in B(\Gamma)$.

Order the branching points $v_0, \dots, v_{n-1}$ of Γ in such a way that $f_0(v_0) \leq f_0(v_1) \leq \cdots \leq f_0(v_{n-1})$. We prove by induction on $i = 0, \dots, n-1$ that the slope at the vertex v_i of f_0 on edges of Γ from v_i to $v_{i'}$, $i' > i$, does not exceed $3(nM+m)(m+1)^i$. Before we do so, observe that the slopes of edges leaving from v_i to $v_{i'}$ for $i < i'$ are non-negative because $f_0(v_i) \leq f_0(v_{i'})$ and each edge contains at most one point v with $\mathrm{ord}_{f_0} v \neq 0$ and if such v exists, $\mathrm{ord}_{f_0} v = 1$.

Let us first consider the case $i = 0$. The value of $\mathrm{ord}_{f_0} v_0$ is equal to the sum of the slopes of f on edges incident with v_0. All the slopes are positive (since f_0 attains its minimum at v_0) and $\mathrm{ord}_{f_0} v_0 \leq 3(nM + m)$, implying that each such slope is at most $3(nM + m)$.

Assume that $i > 0$. Let s be the maximal slope at the point v_i of an edge from $v_{i'}$ to v_i, $i' < i$. Since the slope of any such edge at $v_{i'}$ is at most $3(nM + m)(m + 1)^{i-1}$ and the edge between $v_{i'}$ and v_i contains at most one internal point v with $\mathrm{ord}_{f_0} v \neq 0$ (and $\mathrm{ord}_f v = 1$ at such a point v), the value of s does not exceed $3(nM + m)(m + 1)^{i-1} + 1$. The slope of an edge from v_i to $v_{i''}$, $i < i''$, is bounded by the sum S of the slopes of edges from any $v_{i'}$ to v_i (with $i' < i$), increased by $\mathrm{ord}_{f_0} v_i$. The sum S is at most $(m - 1) \cdot s \leq (m - 1)(3(nM + m)(m + 1)^{i-1} + 1)$. We recall that $\mathrm{ord}_{f_0} v_i \leq 3(nM + m)$ and conclude that the slope of an edge from v_i to $v_{i'}$, $i < i'$, at the point v_i is at most

$$\begin{aligned}(m-1)(3(nM+m)(m+1)^{i-1}+1)+3(nM+m) &\leq \\ \leq m \cdot 3(nM+m)(m+1)^{i-1}+m-1 &\leq 3(nM+m)(m+1)^{i} \ .\end{aligned}$$

Thus, we have proven that the slope of f_0 on every edge of T is, in absolute value, bounded by $3(nM + m)(m + 1)^{n-1} + 1 = U$. This finishes the proof of the theorem. □

Theorem 7 and Theorem 23 now imply the existence of an algorithm for computing the rank of a divisor on tropical curves.

Corollary 24. *There exists an algorithm that for a divisor D on a tropical curve Γ computes the rank of D.*

The algorithm which we presented is finite, i.e., it terminates for every input, however, its running time is exponential in the size of the input. It seems natural to ask whether it is possible to design a polynomial-time algorithm for computing the rank of divisors. In the case of graphs the question was posed by Hendrik Lenstra [7], and, to the best of our knowledge, is still open. Tardos [15] presented an algorithm which decides whether a divisor D on a graph has a non-negative rank. His algorithm is weakly polynomial, i.e., the running time is bounded by a polynomial in the size of the graph and $\deg^+(D)$ (note that Tardos was using a different language to state the result). It is possible to modify his algorithm in such a way that the running time becomes polynomial in the size of the graph and $\log(\deg^+(D))$, i.e., to obtain a truly polynomial-time algorithm for deciding whether a given divisor on a graph has a non-negative rank. We omit further details.

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